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Time-Weighted Balanced Stochastic Model Reduction

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Abstract— **A new relative error model reduction technique for linear time invariant (LTI) systems is proposed in this paper. Both continuous and discrete time systems can be reduced within this framework. The proposed model reduction method is mainly based upon time-weighted balanced truncation and a recently developed inner-outer factorization technique. Compared to the other analogous counterparts, the proposed method shows to provide more accurate results in terms of time weighted norms, when applied to different practical examples. The results are further illustrated by a numerical example.**

I. Introduction

The accurate mathematical modeling of physical and man-made processes leads to models of high complexity. The simulation, analysis, control, design and implementation of the methods and algorithms for the systems of high orders are difficult and costly if at all possible. To cope with these problems, over the past few decades, there has been increasing interest in the methods which reduce the order of dynamical systems while preserving the input-output behavior and important features [1],[7],[14].

Two main categories of order reduction techniques are: Singular Value Decomposition (SVD) based methods and the moment matching based techniques. The SVD-based methods have a guaranteed upper bound for the approximation error and they usually preserve the stability of the original model in the reduction process. The moment matching based methods are usually computationally more efficient, but they do not preserve the stability of the original systems and have no guaranteed error bound [1],[14]. From another viewpoint, model reduction methods can be categorized into two categories: the relative error model reduction methods and the absolute error model reduction techniques. In the family of relative error model reduction methods, the order of dynamical systems are reduced while the relative error of the approximation is kept small and in the second category the model is reduced while the absolute approximation error is kept small. The Balanced Stochastic Truncation (BST) method is the method from the relative error family [10]. This method provides the reduced order approximation with smooth approximation error and preserves phase information. This is the main advantage of this method in comparison with absolute error methods like the Balanced Truncation (BT) [4]. For modeling, simulation and control, it often happens that we are more interested in a specific time /frequency range and the behavior of the systems is not very important outside of this range. This problem leads to so-called time /frequency weighted balancing methods. In these methods, some weights are given to the system in model reduction to compute the reduced order model to reach a small error. The frequency weighted balanced truncation was first proposed in [6]. This method has been explored extensively in the literature and has been improved and extended from different viewpoints [2]. However, despite some researches which have addressed the time domain counterpart of this method, which is 'Time Weighted Balanced Truncation (TWBT)' [8], there are problems which remain open in this context. In this paper we focus our attention on the extension of the method to the family of the relative error model reduction methods. In this paper, based on the BST and TWBT a new method for relative error model reduction is proposed. The method is "Time-Weighted Balanced Stochastic Truncation (TWBST)". This technique shows to provide more accurate results in terms of weighted norms and also keeps good properties of the ordinary BST. The method would be reduced to a novel reduction method which is 'Hilbert-Schmidt-Hankel BST' for a particular choice of the time weight.

The paper is organized as follows. In section II, we introduce some definitions, notations and concepts related to BST. Section III presents TWBT algorithm and its properties. In section IV, the TWBST method is presented. The method is applied to a practical CD player benchmark example in section V and the results are shown. Finally, the last section concludes the paper. The notation used in this paper is as follows: M^* denotes transpose of matrix if $M \in \mathbb{R}^{n\times m}$ and complex conjugate transpose if $M \in \mathbb{C}^{n\times m}$. The standard notation $>$, $\geq (<, \leq)$ is used to denote the positive (negative) definite and semidefinite ordering of matrices.

II. Model Reduction via Balanced Stochastic Truncation (BST)

Consider $G(s)$ a MIMO square transfer matrix with a minimal sate space realization $G := (A,B,C,D)$ and of order n. If $\det(D) \neq 0$ it is possible to compute the left spectral factor $\psi(s)$ of $G(s)G^*(-s)$ satisfying:

$$\psi^*(-s)\psi(s) = G(s)G^*(-s). \tag{1}$$

The state space realization of G is called a balanced stochastic realization if:

$$P^G = Q^\psi = diag(\sigma_1, ..., \sigma_n), \tag{2}$$

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where P^G is the controllability gramian of $G(s)$, the matrix Q^{ψ} is the observability gramian of $\psi(s)$ and σ_i is the i^{th} Hankel singular value of the stable part of the so-called "phase matrix" $F(s)=(\psi^*(-s))^{-1}G(s)$. The singular values in (2) are in decreasing order [3],[9],[11],[15].

The reduced model is achieved by omitting the states related to the lowest set of singular values. The reduced model is stable and satisfies the relative error bound [3],[9],[12]:

$$\left\|G^{-1}(G-G_r)\right\|_\infty \le \prod_{i=r+1}^{n}\frac{1+\sigma_i}{1-\sigma_i}-1. \tag{3}$$

III. Time-Weighted Balanced Truncation Model Reduction Method

Over the last two decades, the balanced model reduction method received a lot of attention and has been improved and extended from different viewpoints. The time-weighted balanced truncation is among the methods which improves the accuracy of the approximation and reduces the error by a piecewise polynomial function [8],[16]. In the sequel this method for both continues and discrete time systems are presented.

This method was first proposed in [8]. For the linear time-invariant continuous-time systems with state space realization (A,B,C,D) the reachability and observability gramians are given by:

$$P=\int_0^\infty e^{A\tau}BB^*e^{A^*\tau}d\tau \ ,$$
$$Q=\int_0^\infty e^{A^*\tau}C^*Ce^{A\tau}d\tau \ ,$$

which are the solutions of the Lyapunov equations:

$$\begin{aligned} &AP+PA^*+BB^*=0,\\ &A^*Q+QA+C^*C=0. \end{aligned} \tag{4}$$

These gramians generalized to expressions of the type

$$P_f:=\int_0^\infty f(t)e^{A\tau}BB^*e^{A^*\tau}d\tau \ ,$$
$$Q_f:=\int_0^\infty f(t)e^{A^*\tau}C^*Ce^{A\tau}d\tau,$$

where $f(t)$ is any desired weighting function.

In particular:

$$\begin{aligned} &P_i:=\int_0^\infty t^i e^{A\tau}BB^*e^{A^*\tau}d\tau,\\ &Q_i:=\int_0^\infty t^i e^{A^*\tau}C^*Ce^{A\tau}d\tau. \end{aligned} \tag{5}$$

A recursive algorithm for computation of these gramians was proposed by Schelfnout, and Moor [8]. It was shown that gramians for the linear time-invariant continuous-time systems with the definition:

$$\begin{aligned} &P_{r+1}:=\frac{1}{r!}\int_0^\infty t^r e^{At}BB^*e^{A^*t}dt, r=0,1,2,...\\ &Q_{r+1}:=\frac{1}{r!}\int_0^\infty t^r e^{A^*t}C^*Ce^{At}dt, r=0,1,2,.... \end{aligned} \tag{6}$$

The weighted gramians can be obtained recursively using the Lyapunov equations bellow:

$$\begin{aligned} &AP_{r+1}+P_{r+1}A^*=-P_r, r=0,1,2,...\,,\\ &A^*Q_{r+1}+Q_{r+1}A=-Q_r, r=0,1,2,...\,, \end{aligned} \tag{7}$$

where:

$$P_o=BB^*,$$
$$Q_o=C^*C.$$

Similarly the weighted gramians for a linear time-invariant discrete-time system are defined as follows:

$$\begin{aligned} &P_{r+1}:=\sum_{k=o}^{\infty}\frac{(k+r)!}{k!}A^kBB^*\left(A^*\right)^k, \quad r=0,1,2,...\,,\\ &Q_{r+1}:=\sum_{k=o}^{\infty}\frac{(k+r)!}{k!}\left(A^*\right)^kC^*CA^k, \quad r=0,1,2,...\,, \end{aligned} \tag{8}$$

which are the solutions of the Lyapunov equations:

$$\begin{aligned} &AP_{r+1}A^*-P_{r+1}=-P_r, \quad r=0,1,2,...\,,\\ &A^*Q_{r+1}A-Q_{r+1}=-Q_r, \quad r=0,1,2,...\,, \end{aligned} \tag{9}$$

where:

$$P_o=BB^*,$$
$$Q_o=C^*C.$$

The rest of the method is very similar to the ordinary balanced truncation. At this point an appropriate similarity transformation which transforms the system into time weighted balanced structure needs to be computed. In the time-weighted balanced realization:

$$\begin{aligned} &T^{-1}P_{r+1}(T^{-1})^*=T^*Q_{r+1}T\\ &=diag(\sigma_1,\sigma_2,...,\sigma_k,\sigma_{k+1},...,\sigma_n), \end{aligned} \tag{10}$$

where:

$$\sigma_1\ge\sigma_2\ge...\ge\sigma_k\ge\sigma_{k+1}\ge...\ge\sigma_n\ge 0.$$

The transformed system needs to be partitioned:

$$G=\begin{bmatrix} T^{-1}AT & T^{-1}B\\ CT & D\end{bmatrix}=\begin{bmatrix} A_{11} & A_{12} & B_1\\ A_{21} & A_{22} & B_2\\ C_1 & C_2 & D\end{bmatrix},$$

where the dimension of the A_{11} is equal to the dimension of $diag(\sigma_1, \sigma_2, ..., \sigma_k, \sigma_{k+1}, ..., \sigma_n)$.

The realization $\left(T^{-1}AT, T^{-1}B, CT\right)$ is known as the time weighted balanced realization and the reduced order model which is of order k is given by:

$$G_1 = \begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix}.$$

The reduced model is stable and satisfies the relative error bound [8],[16]:

$$\left\|G(s) - G_r(s)\right\|_\infty \le \alpha \sum_{i=k+1}^{n} \sigma_i \ , \tag{11}$$

where $\alpha = 2\sigma_{\max}(R^{-1}B)\sigma_{\max}(CS^{-1})$ and R and S are Cholesky factors:

$$P_{r+1} = RR^*, Q_{r+1} = S^*S.$$

In the case which $r = 1$, the method would be an absolute error Hilbert-Schmidt-Hankel model reduction.

IV. Time-Weighted Balanced Stochastic Truncation

In this section, a new method for model reduction is proposed. The stability and computational issues of the algorithm are also discussed.

A. TWBST Algorithm

TWBST keeps the advantages of BST plus reducing an error norm weighted by a piecewise polynomial function in the time domain. Numerical results in the next section show the accuracy enhancement of the proposed method. In TWBST algorithm similar to BST, at first we should find the left spectral factor $\psi(s)$ of $G(s)G^*(-s)$ satisfying (1). The inner-outer factorization is applied to compute the left spectral factor of G to factorize the state space realization:

$$N = (A, P^G C^* + BD^*, -B^*(P^G)^{-1}, D^*),$$

in the form $N_i(s)\psi(s)$ where $N_i(s)$ is the inner factor and $\psi(s)$ is the outer and the left spectral factor [4],[5],[9].

The next step is to compute the time weighted controllability gramian of the system G and the time weighted observability gramian of the left spectral factor $\psi(s)$. The system will be transformed into the time weighted balanced stochastic realization. In time-weighted balanced stochastic realization, the time weighted controllability gramian of the system G and the time weighted observability gramian of the left spectral factor $\psi(s)$ are equal and diagonal with decreasing diagonal elements i. e.

$$P^G_{r+1} = Q^\psi_{r+1} = diag(\sigma_1, ..., \sigma_n). \tag{12}$$

The reduced model is obtained by the truncation of the states which have the least effect on the input-output behavior of the original system (those which related to the lowest set of the singular values). Fig. 1 shows the overall algorithm of TWBST method. When $r = 1$, the method is reduced to a relative error Hilbert-Schmidt-Hankel model reduction.

B. On Stability and Numerical Implementation

Balanced transformation can be numerically ill-conditioned when dealing with systems having some nearly uncontrollable or some nearly unobservable modes. Difficulties associated with computation of the required balanced transformation in [3] drew some attention toward devising alternative numerical methods [13]. Balancing can be badly conditioned even when some states are significantly more controllable than observable or vice versa. In this case, it is suggested to reduce the system in the gramian based framework without balancing. The Schur method and square root algorithms provide projection matrices to apply balanced reduction without balanced transformation [1],[13]. This method can be easily extended to other gramian based method. In TWBST, we can use the same algorithm by plugging time weighted controllability gramian of the system G and the time weighted observability gramian of the left spectral factor $\psi(s)$. Fig. 2 shows the overall algorithm of TBST method by using square root algorithm.

It is important to keep the main features and key properties of the original systems in the reduction process. One of these important properties is the stability of the original system. In the following proposition we show that the proposed framework for model reduction is a stability preserving method. In other words, the proposed method always reduces a stable system into a stable system.

Proposition 1. Let the system with minimal realization $G := (A, B, C, D)$ be stable and the system is reduced by TWBST, the reduced order model is quadratically stable.

Proof:

In the proposed method using square root algorithm, we have:

$$W^*V = I_k\ ,\quad V, W \in \mathbb{R}^{n\times k}, k < n,$$

$$\hat{\Sigma} : \begin{cases} \hat{A} = W^*AV, \\ \hat{B} = W^*B, \\ \hat{C} = CV\ , \\ \hat{D} = D, \end{cases} \tag{13}$$

which is a projected system (reduced order model). The outcome of the proposed square root algorithm for projection [1]: $P^G_{r+1}W = V\Sigma_1$ and $Q^{\psi}_{r+1}V = W\Sigma_1$, where $\Sigma_1 \in \mathbb{R}^{k\times k}$ is diagonal and positive definite. Since P^G_{r+1} is the time controllability gramian of the system G, we have: $AP^G_{r+1} + P^G_{r+1}\ A^* = -P^G_r < 0$, which implies:

$$W^*(AP^G_{r+1} + P^G_{r+1}\ A^*)\, W < 0,$$

on the other hand,

$$W^*(AP^G_{r+1} + P^G_{r+1}\ A^*)\, W = W^* AP^G_{r+1} W + W^* P^G_{r+1}\ A^*\ W$$
$$= W^* AV\Sigma_1 + \Sigma_1^* V^* A^*\ W = \hat{A}\Sigma_1 + \Sigma_1 \hat{A}^*.$$

Hence:

$$\hat{A}\Sigma_1 + \Sigma_1\hat{A}^* < 0\ , \tag{14}$$

where $\Sigma_1 \in \mathbb{R}^{k\times k}$ is positive definite.

Hence, the reduced order model is guaranteed to be quadratically stable with Lyapunov function $V(x)$ for which we have:

$$V(x) = x^*\Sigma_1 x > 0,$$
$$\dot{V}(x) = x^*(\hat{A}\Sigma_1 + \Sigma_1\hat{A}^*)x < 0$$

□

Inputs: System matrices (A,B,C,D)

Outputs: Reduced system matrices (A_r, B_r, C_r, D_r)

1- Form:
$N = (A, P^G C^* + BD^*, -B^*(P^G)^{-1}, D^*)$
2- Apply inner-outer factorization and find the left spectral factor $\psi(s)$
3- Compute the time weighted controllability gramian of (A,B,C,D) system
4- Compute the time weighted observability gramian of the left spectral factor $\psi(s)$
5- Find the similarity transform T for time weighted stochastic balancing and balance the system.
6- Eliminates the states relates to the lowest set of singular value and find (A_r, B_r, C_r, D_r)

Fig. 1. TWBST model reduction algorithm.

Inputs: System matrices (A,B,C,D) and time interval $[t_1,t_2]$

Outputs: Reduced system matrices (A_r, B_r, C_r, D_r)

1- Form:
$N = (A, P^G C^* + BD^*, -B^*(P^G)^{-1}, D^*)$
2- Apply inner-outer factorization and find the left spectral factor $\psi(s)$
3- Compute the time domain controllability gramian of (A,B,C,D) system within a time interval $[t_1,t_2]$
4- Compute the time domain observability gramian of the left spectral factor $\psi(s)$ within the time interval $[t_1,t_2]$
5- Find the projection matrices V and W via square root or Schur algorithm.
6- Compute (A_r, B_r, C_r, D_r)

Fig. 2. TWBST model reduction algorithm using square root algorithm.

V. NUMERICAL EXAMPLE

In this section the proposed method is applied to a practical CD player benchmark example and the results are discussed. The model is of order 120. The CD player model is reduced by applying both BST and TWBST. The relative errors for reduction of the system to the 31st order model by BST and by TBST with $r=1$ are shown in Fig. 3. The step responses are shown in Fig. 4. As it can be seen, TWBST provides more accurate results compared to BST.

The CD player model is also reduced to 73th order model by applying BST and to 70th order model by TWBST. The relative errors for reduction of the system with $r=1$ are shown in Fig. 5. The step responses are shown in Fig. 6. As it can be seen, TWBST provides more accurate results compared to BST while it reduces the system even more.

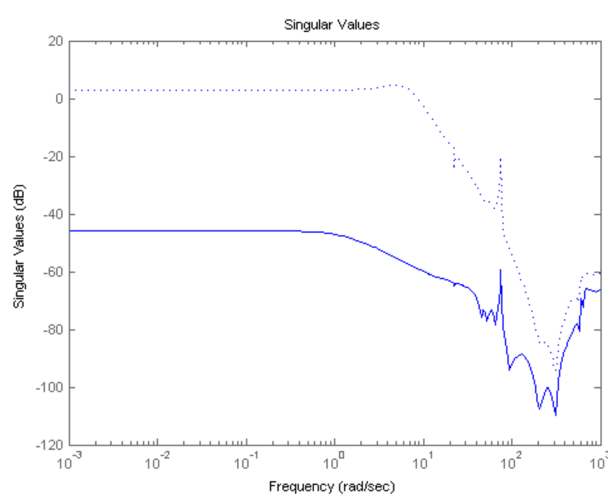

Fig. 3. TWBST model reduction relative error (solid lines) and BST model reduction relative error (dotted).

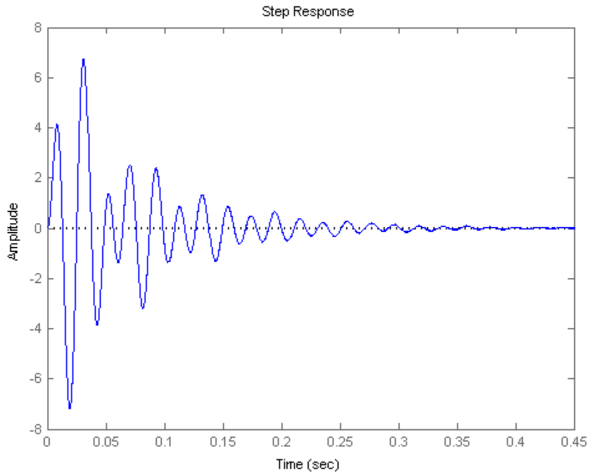

Fig. 4. Step response of the original system (solid lines), TWBST reduced order model (dotted) and BST reduced order model.

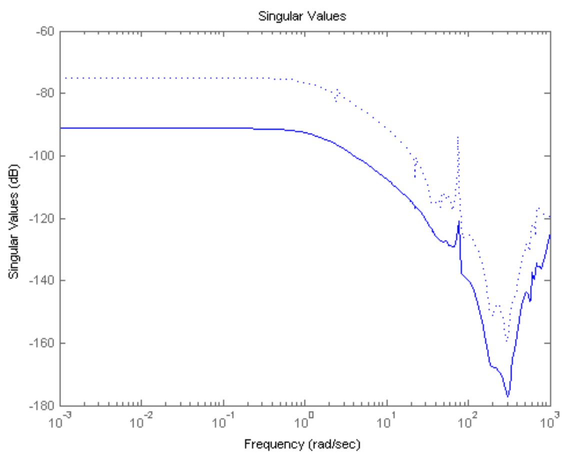

Fig. 5. TWBST model reduction relative error (solid lines) and BST model reduction relative error (dotted).

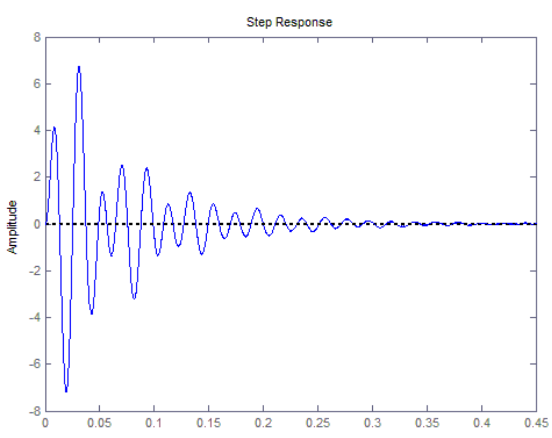

Fig. 6. Step response of the original system (solid lines), TWBST reduced order model of order 73 (dotted) and BST reduced order model of order 70.

VI. CONCLUSION

A new relative error model reduction technique based on time-weighted stochastic balancing is proposed in this paper. Inner-Outer factorization is used as an accurate and an efficient numerical approach in the numerical algorithm of the method. The proposed method is proven to preserve stability and it shows advantages in terms of accuracy and efficiency which are suitable for the practical relative error model reduction.

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