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Published in:
Proceedings of the 45th IEEE Conference on Decision and Control

Publication date:
2006

Document Version
Publisher's PDF, also known as Version of record

Towards Modelling of Hybrid Systems

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***Abstract*— The article is an attempt to use methods of category theory and topology for analysis of hybrid systems. We use the notion of a directed topological space, c.f. [1]; it is a topological space together with a set of privileged paths. Dynamical systems are examples of directed topological spaces. A hybrid system consists of a number of dynamical systems that are glued together according to information encoded in the discrete part of the system. Motivated by [2] we develop a definition of a hybrid system as a functor from the category generated by a transition system to the category of directed topological spaces. Its directed homotopy colimit (geometric realization) is a *single* directed topological space. The behavior of hybrid systems can be then understood in terms of the behavior of dynamical systems through the directed homotopy colimit.**

I. Introduction

We consider a transition system - two tuple (V, E), where V is the set of discrete states and $E \subseteq V \times V$ is the set of edges. The classical definition of a hybrid system associates to each state $v \in V$ a dynamical system, whereas the edges describe how the system "switches" between these dynamical systems.

By a model of a system we understand a mathematical tool for predicting, with certain accuracy, its future behavior. The models of hybrid systems, see an overview article [3], depend to a great extend on a type of enquiry we wish to undertake. There are models with very little structure, c.f. [2], who take merely the underlying topological spaces into account. They can be applied for majority of hybrid systems. However, answers they provide are of very general nature, e.g. Is the given system Zeno? Adding the structure of continuous dynamics can help answering more specific questions of reachability and safety, c.f. [4]. Finally, for control synthesis it might be advantageous to consider the continuous dynamics restricted to piecewise affine dynamical systems, c.f. [5] and [6].

This article is an extension of studies of topology of hybrid systems in [7] and [2]. We want to represent a hybrid system as a single space, then to study a space of trajectories (execution paths).

We define a hybrid system in the following construction. In Section II we associate to a discrete system a directed graph G and corresponding diagram category $\int G$. To work with the space of trajectories we use the notion of a directed topological space of [8]. This is the matter of Sections III - the definition and basic properties, and VI - abstractions. A directed topological space (X, dX) is a topological space X equipped with a directed structure dX, that is a set of continuous maps $\alpha : I \to X$, defined on the standard interval $I = [0, 1]$. Directed topological spaces with maps preserving the directed structures form a category $\mathrm{d}\mathbf{Top}$. Dynamical systems, Section IV, and section cones, Section V, are examples of the directed topological spaces.

A topological hybrid system is a functor $T : \int G \to \mathrm{d}\mathbf{Top}$ taking each object g of $\int G$ to a directed topological space. The category $\int G$ describes the discrete transitions, whereas $H(g),\ g \in \int G$ plays the role of continuous dynamics. Any directed homotopy colimit exists in $\mathrm{d}\mathbf{Top}$. The result, c.f. Section VII, is a single directed topological space. We conclude that the behaviour of a hybrid system can be understood in terms of the behaviour of dynamical systems through the directed homotopy colimit.

All maps in the article are continuous. Paths are maps from the standard interval I to a topological space.

II. Categorical Transition Systems

We adopt the definition of a directed graph from [9].

Definition 1: A *directed graph (d-graph)* G is a pair of sets $G = \{G_0, G_1\}$, where G_0 is a set of vertices and G_1 is the set of edges, along with two functions

$$G_1 \overset{\delta^1}{\underset{\delta^0}{\rightrightarrows}} G_0.$$

The maps $\delta^0,\ \delta^1$ in the definition are used to distinguish the direction in a d-graph.

Definition 1 is equivalent to the standard definition of the transition system (V, E), where V is the set of vertices and $E \subset V \times V$ is the set of edges. We make the standard assumption that $V = \pi_0(E) \cup \pi_1(E)$. The transition system (V, E) defines the directed graph $\{V, E\}$ with $\forall e \in E, \delta^\alpha(e) = \pi_\alpha(e)$, where π_0 is the projection on the first and π_1 on the second factor. Conversely, the directed graph $\{G_0, G_1\}$ with two face maps $\delta^\alpha : G_1 \to G_0,\ \alpha = 0, 1$ defines the transition system (G_0, E) with $E = \{(\delta^0 e, \delta^1 e) |\ e \in G_1\}$.

Example 1: We consider the following transition system

$$\begin{tikzcd} a \arrow[r] & b \arrow[loop right] \end{tikzcd} \quad . \tag{1}$$

It corresponds to the directed graph

$$\begin{tikzcd} G_1 \arrow[d, shift right, "\delta^0"'] \arrow[d, shift left, "\delta^1"] \\ G_0 \end{tikzcd} \qquad \begin{tikzcd} & p \arrow[dl, "\delta^0"'] \arrow[dr, "\delta^1"] & & q \arrow[dl, shift right, "\delta^0"'] \arrow[dl, shift left, "\delta^1"] \\ a & & b. & \end{tikzcd} \tag{2}$$

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Let G and K be two d-graphs, then a morphism μ in a category $\mathbf{Graph}$ of d-graphs is a pair $\{\mu_\alpha : G_\alpha \to K_\alpha | \ \alpha \in \{0,1\}\}$ of functions for which the diagram

$$\begin{tikzcd} G_1 \arrow[r, "\mu_1"] \arrow[d, "\delta^\alpha"] & K_1 \arrow[d, "\delta^\alpha"] \\ G_0 \arrow[r, "\mu_0"] & K_0, \end{tikzcd}$$

for $\alpha = 0, 1$ commutes.

A *labelled discrete system* over a finite set Σ of labels is a directed graph G along with a morphism

$$f : G \to L,$$

where $L = \{\{*\}, \Sigma\}$.

For every d-graph $G \in \mathbf{Graph}$ we associate a *diagram category* $\int G$. The objects of $\int G$ are pairs (α, x) with $x \in G_\alpha$ for $\alpha \in \{0,1\}$. The morphisms are identity morphisms and morphisms $\delta_p^\alpha : (1,p) \to (0,a)$ if $\delta^\alpha(p) = a$. If the d-graph G is as in diagram (2) then $\int G$ is

$$\begin{tikzcd} & (1,p) \arrow[loop above] \arrow[dl, "\delta_p^0"'] \arrow[dr, "\delta_p^1"] & & (1,q) \arrow[loop above] \arrow[dl, shift right, "\delta_q^0"'] \arrow[dl, shift left, "\delta_q^1"] \\ (0,a) \arrow[loop below] & & (0,b) \arrow[loop below] & \end{tikzcd} \tag{3}$$

When it is clear from the context that the domain of $\delta_p^\alpha, \ \alpha = 0, 1$ is p, we suppress the notion and write δ^α instead.

Remark 1: The categorical discrete system can be generalized to the category of (pre-) cubical sets; that is a family of sets $\{D_n | \ n \geq 0\}$ with face maps $\delta_i^\alpha : D_n \to D_{n-1}$ $(1 \leq i \leq n, \ \alpha = 0, 1)$. This observation can be used to extend a definition of hybrid systems of this paper to one including concurrency in discrete transitions. For more information on concurrency and cubical complexes see [10] and [11].

III. Directed Topological Spaces

We bring in a notion of a directed topological space as introduced in [1].

Definition 2 (1.1 in [1]): A directed topological space, or $d-space$ $X = (X, dX)$ is a topological space equipped with a set dX of paths $\alpha : I \to X$, called *directed paths* or *d-paths*, satisfying axioms

1) (*constant path*) every constant path $I \to X$ is directed,
2) (*reparametrisation*) dX is closed under composition with weakly increasing maps $I \to I$,
3) (*concatenation*) dX is closed under concatenation, i.e. if $\alpha, \beta \in dX$ and $\alpha(1) = \beta(0)$ then the concatenation γ

$$\gamma(t) = \begin{cases} \alpha(2t) & \text{if} \quad 0 \geq t \geq \frac{1}{2} \\ \beta(2t-1) & \text{if} \quad \frac{1}{2} \geq t \geq 1 \end{cases}$$

is also a d-path.

The set dX will be called the *d-structure* of X.

Fig. 1. The circle S^1 with two d-structures: $\uparrow S^1$ (left) and $\uparrow O^1$ (right).

We shall study maps between d-spaces that preserve the directed paths. We say that a map $f : X \to Y$ preserves the directed paths if $\alpha \in dX$ implies that $f \circ \alpha \in dY$.

Definition 3: A *directed map*, or *d-map* $f : X \to Y$ is a map between d-spaces (X, dX) and (Y, dY) that preserves the directed paths. Their category will be denoted by $\mathrm{d}\mathbf{Top}$.

Example 2: The directed real line $\uparrow \mathbb{R}$ is the Euclidean line $\mathbb{R}$ with directed paths given by increasing maps $I \to \mathbb{R}$. Then the standard directed interval $\uparrow I$ has the subspace structure of the directed line.

The n-dimensional directed real space $\uparrow \mathbb{R}^n$ is $\mathbb{R}^n$ equipped with paths increasing in each coordinate, similarly $\uparrow I^n$ has a subspace structure of $\uparrow \mathbb{R}^n$. On a circle $\uparrow S^1$ we define directed paths in anticlockwise direction. This comes from the quotient $\uparrow I/\partial I$ that identifies the endpoints of I. Since the concatenation of d-paths is a d-path we see that directed paths in $\uparrow S^1$ and $\uparrow I/\partial I$ agree. This is not the only d-structure on S^1. We can consider the circle embedded in $\mathbb{R}\times \uparrow \mathbb{R}$, then all d-paths move upwards. The circle with this d-structure is denoted by $\uparrow O^1$, c.f. Fig. 1.

The classical definition of a hybrid system associates to each discrete state a dynamical system. In the next sections we show that a dynamical system and its robust version, a section cone, give rise to d-spaces.

IV. Dynamical Systems

We consider a compact smooth manifold, M and a set of smooth vector fields $\mathfrak{X}(M)$ defined on M ($\mathfrak{X}(M)$ has a structure of a complete metric space, c.f. Theorem 4.4, Ch. 2, [12]).

An *integral curve* of a vector field $\xi \in \mathfrak{X}(M)$ through a point $x \in M$ is a smooth map $\phi_x^\xi : (-\vartheta, \epsilon) \to M$, with real numbers $\epsilon, \vartheta > 0$, such that $\phi_x^\xi(0) = p$ and $\frac{d}{dt}\phi_x^\xi(t) = \xi(\phi_x^\xi(t))$ for all $t \in (-\vartheta, \epsilon)$. The image of an integral curve is an orbit. The set of singularities of a vector field ξ is denoted by $\mathcal{C}r(\xi) = \{p \in M | \ \xi(p) = 0\}$. The theorems on existence, uniqueness and differentiability of solutions of ordinary differential equations in $\mathbb{R}^n$ extend to vector fields on M. Since in our setup M is compact each integral curve is defined for all $t \in \mathbb{R}$ and we have a smooth map - the flow, c.f. Theorem 5.2.1, [13]

$$\phi : \mathbb{R} \times M \to M, \ \phi(t, x) = \phi_x(t).$$

Definition 4: $\gamma : I \to M$ is an *integral arc* of a vector field ξ if there exists an injective, increasing map $\alpha : I \to \mathbb{R} \cup \{\pm\infty\}$ and an $x \in M$ such that $\phi^{\xi}_{x} \circ \alpha(t) = \gamma(t)$ for $t \in I$.

We define the d-structure dM as the smallest d-structure on M containing all integral arcs on M. This shows that a dynamical system can be treated as a d-space.

V. Section Cones

To allow more flexibility corresponding to non-determinacy we introduce a section cone.

Definition 5 (Section Cone, c.f. [14]): Let M be a smooth manifold. A *section cone* $\mathcal{K}$ on M is a subset of $\mathfrak{X}(M)$ satisfying the following two conditions:

1) For every pair $\xi, \eta \in \mathcal{K}$, $\mathcal{C}r(\xi) = \mathcal{C}r(\eta)$.
2) If ξ and η are in $\mathcal{K}$ and $\alpha, \beta > 0$ then $\alpha\xi + \beta\eta \in \mathcal{K}$.

The first condition says that all vector fields in a section cone have the same singularities. Also if the zero section 0_M is in $\mathcal{K}$ then $\mathcal{K} = \{0_M\}$. The second condition imposes convexity on the subset $\mathcal{K}$. Particularly, if $\xi \in \mathcal{K}$ then $\alpha\xi \in \mathcal{K}$ for $\alpha > 0$.

Condition 1. allows to speak about singular points of a section cone.

Definition 6 (c.f. [14]): A point p is a *singular point* of a section cone $\mathcal{K}$ if $p \in \mathcal{C}r(\xi)$ for some, thus for all, $\xi \in \mathcal{K}$. We denote the set of singular points of $\mathcal{K}$ by $\mathcal{C}r(\mathcal{K})$.

Example 3: Let g be a Riemannian metric on M, i.e. a pointwise inner product on $T_p(M)$ varying smoothly in p. We pick $\eta \in \mathfrak{X}(M)$ and define the set $\mathcal{K}(\eta) \subset \mathfrak{X}(M)$ by

$$\begin{aligned}\mathcal{K}(\eta) = \{\alpha(\eta + \xi) \in \mathfrak{X}^r(M) |\ \xi \in \mathfrak{X}^r(M), \alpha > 0, \\ g(\xi, \eta) = 0, g(\eta, \eta) \geq g(\xi, \xi)\}.\end{aligned}$$

Note that for $\eta + \xi \in \mathcal{K}(\eta)$ we have $\eta(p) = 0$ for some $p \in M$ if and only if $(\eta + \xi)(p) = 0$. Furthermore, if $\vartheta_i = \alpha_i(\eta + \xi_i) \in \mathcal{K}(\eta)$ for $\alpha_i > 0$ and $i \in \{1, 2\}$ then

$$\|\alpha_1\xi_1 + \alpha_2\xi_2\|^2 \leq (\alpha_1 + \alpha_2)^2\|\eta\|^2, \quad \text{where } \|\cdot\|^2 \equiv g(\cdot, \cdot).$$

Hence $\vartheta_1 + \vartheta_2 = (\alpha_1 + \alpha_2)\eta + \alpha_1\xi_1 + \alpha_2\xi_2 \in \mathcal{K}(\eta)$, and $\mathcal{K}(\eta)$ is a section cone.

We shall use the notation $\mathcal{K}(p) \equiv \{s(p) |\ s \in \mathcal{K}\} \subset T_p(M)$. In particular, $p \in \mathrm{Cr}(\mathcal{K})$ if and only if $\mathcal{K}(p) = \{0\}$.

Let us make precise the notion of a cone in a vector space.

Definition 7 ([15]): Let V be a real vector space. A *cone* K in V is a subset of V satisfying

1) If $\alpha, \beta \geq 0$ and $x,\ y \in K$, then $\alpha x + \beta y \in K$;
2) $K \cap (-K) = \{0\}$.

At every point $p \in M$, a section cone localizes to a cone in the tangent space $T_p(M)$.

Proposition 1 (c.f. [14]): Let $\mathcal{K}$ be a section cone. If $\xi, \eta \in \mathcal{K}$ and $\xi(p) = -\eta(p)$ for some $p \in M$ then $p \in \mathcal{C}r(\xi)$. As a consequence, for each $x \in M$, the set $\mathcal{K}(x) \cup \{0\}$ is a cone in the vector space $T_x(M)$.

Given a section cone, we define a di-path as a concatenation of finite number of integral arcs corresponding to the vector fields belonging to this cone.

Definition 8: Suppose $\mathcal{K}$ is a section cone. We call a piecewise smooth path $\sigma : I \to M$ a ***d-path*** of $\mathcal{K}$ if it is a constant path or there exists a finite set of integral arcs $\{\gamma_1, ..., \gamma_k\}$, for $i = 1, ..., k$ where γ_i is an integral arc of the vector field ξ_i satisfying

1) $\{\xi_1,...,\xi_k\} \subset \mathcal{K}$ and
2) $\sigma = \gamma_1 * ... * \gamma_k$, in particular $\gamma_i(1) = \gamma_{i+1}(0)$.

The totality of all d-paths defines a d-structure on M (generated by the section cone $\mathcal{K}$).

VI. Abstractions in $\mathrm{d}\mathbf{Top}$

The forgetful functor $U : \mathrm{d}\mathbf{Top} \to \mathbf{Top}$, c.f. [1], has the left adjoint c_0 taking $X \in \mathbf{Top}$ to $c_0(X) = (X, |X|)$ with $|X|$ the d-structure of constant paths on X. Its right adjoint is C^0, which takes X to $C_0(X) = (X, \mathbf{Top}(I, X))$. Since all left adjoint functors preserve all colimits, while right adjoints preserve limits, it is concluded that $\mathrm{d}\mathbf{Top}$ is complete and cocomplete. The limits and colimits in $\mathrm{d}\mathbf{Top}$ are constructed as in $\mathbf{Top}$ and equipped with the initial or final d-structure for the structural maps. For instance, a path $I \to \prod X_i$ is directed if and only if all its components $I \to X_i$ are directed. Also a path $I \to \sum X_i$ is directed if and only if it is directed for some X_i.

With this in mind we are able to adapt the notion of bisimulation for the directed spaces.

A. Bisimulation

Definition 9: A d-map $f : Y \to X$ is a *bisimulation map* provided f has a path lifting property:
For each $y \in Y$ and each $\sigma \in dX$ with $f(y) = \sigma(0)$ there is there is $\theta \in dY$ such that $\sigma = f \circ \theta$ and $\theta(0) = y$ that is the following diagram

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ I \arrow[r, "\sigma"'] \arrow[ur, dashed, "\theta"] & X \end{tikzcd}$$

commutes.

We adapt the definition of a bisimulation from [16] and [4].

Two d-spaces X and Y are bisimilar if there exists a third d-space Z and a span bisimulation maps $f : Z \to X$ and $g : Z \to Y$

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y. \end{tikzcd} \quad (4)$$

The bisimulation relation is reflexive and symmetric. The transitivity follows from the existence of the pullback in $\mathrm{d}\mathbf{Top}$.

The next proposition shows that the notion of bisimulation does not work for d-spaces. It shows that any d-space X is bisimilar to a space consisting of a singleton $\{*\}$. Hence by transitivity of the bisimulation relation any two d-spaces are bisimilar.

Proposition 2: Any directed space X is bisimilar to a one point space.

Proof 1: Let the d-space Z in the diagram (4) be

$$Z = X \times \{*\}.$$

Let f be the projection on the first factor, and g the projection on the second factor. Notice that the projections are d-maps with the path lifting property, since for any $\sigma \in dX$, $\theta : I \to X \times \{*\}$ defined by $\theta(t) = (\sigma(t), *)$ is in dZ. $\square$

In Section VII we show that the geometric realization of a hybrid system is a directed space. In order to analyze it we still need an appropriate notion of an abstraction.

B. Directed Homotopy Equivalence

We introduce another equivalence relation in $\mathrm{d}\mathbf{Top}$. First we formulate a definition of a directed homotopy.

Definition 10 (c.f. [1]): Let $f, g : X \to Y$ be d-maps. A *directed homotopy* $\phi : f \to g : X \to Y$ is a d-map $\phi : X \times \uparrow I \to Y$ with $\phi(x, 0) = f(x)$ and $\phi(x, 1) = g(x)$ for $x \in X$.

We define the *d-homotopy preorder* $\preceq$.

Definition 11: We say that $f \preceq g$ if the exists a d-homotopy $\phi : f \to g$.

The d-homotopy preorder is reflexive and transitive but non-symmetric. It is consistent with composition, since for $f, g : X \to Y$ and $f', g' : Y \to Z$, we have $f \preceq g$ and $f' \preceq g'$ imply $f' \circ f \preceq g' \circ g$.

Definition 12 (c.f. [1]): The relation $\simeq$ is the equivalence relation generated by $\preceq$. In other words we say that $f \simeq g$ if there exists a finite sequence $f \preceq f_1 \succeq f_2 \preceq f_3 \succeq ...g$ of d-maps between the same objects.

Recall that for a given category C a function R which assigns to each pair of objects a, b of C a binary relation $R_{a,b}$ on the hom-set C(a,b) is a *congruence* on C (c.f. [9]) if

1) for each pair a, b of objects, $R_{a,b}$ is a reflexive, symmetric, and transitive relation on $C(a, b)$;
2) if $f, f' : a \to b$ have $f R_{a,b} f'$, then for all $g : a' \to a$ and all $h : b \to b'$ we have $(h \circ f \circ g) R_{a',b'} (h \circ f' \circ g)$.

We conclude that $\simeq$ is a congruence on $\mathrm{d}\mathbf{Top}$. It means that there exist a category $\mathrm{d}\mathbf{Top}/\simeq$ and a functor $Q : \mathrm{d}\mathbf{Top} \to \mathrm{d}\mathbf{Top}/\simeq$ (a bijection on objects), such that if $f \simeq g$ in $\mathrm{d}\mathbf{Top}$ then $Qf = Qg$.

Definition 13 (c.f. [1]): A *d-homotopy equivalence* is a d-map $f : X \to Y$ having a d-homotopy inverse $g : Y \to X$, that is $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$. We write $X \simeq Y$, and say that X and Y are *d-homotopy equivalent* or have the same *d-homotopy type*.

A d-space is d-contractible if it is d-homotopy equivalent to a point. A d-subspace $u : X \subset Y$ is a directed deformation retract of Y if there is a d-map $p : Y \to X$ such that $p \circ u = \mathrm{Id}_X$ and $u \circ p \simeq \mathrm{Id}_Y$.

Example 4: The half line $\uparrow]-\infty, 0]$ is a "past" deformation retract of the directed line $\uparrow \mathbb{R}$ by the d-homotopy $\phi(x, t) = \min(x, tx)$. Also the half line $\uparrow]-\infty, 0]$ is "future" d-contractible to 0 by the d-homotopy $\psi(x, t) = (1 - t)x$. Thus, we conclude that the d-line $\uparrow \mathbb{R}$ is d-contractible.

On the other hand $\uparrow S^1$ and $\uparrow O^1 \subset \mathbb{R} \times \uparrow \mathbb{R}$ are not d-homotopy equivalent. A directed map $f : \uparrow S^1 \to \uparrow O^1$ must stay on the right or left side of $\uparrow O^1$. This implies that f is d-homotopic to a trivial map $f' : \uparrow S^1 \to \{*\} \subset \uparrow O^1$, but $\uparrow S^1$ is not contractible.

The geometric realization of a hybrid system studied in Subsection VII-B is a single directed topological space. A notion of Zeno behavior has been introduced in the literature on hybrid systems. It describes a situation where a hybrid system undergoes an infinite number of discrete transitions in a finite interval of time. At this stage we shall remark that if the examined hybrid system - a d-space - is d-contractible there will be no Zeno behavior.

C. Fundamental Category

In this section we bring in still another abstraction: the *fundamental category* $\uparrow \Pi(X)$ of the d-space X, c.f. [1].

Let $\sigma, \theta \in dX$ be paths from x to x', i.e. $\sigma(0) = \theta(0) = x$ and $\sigma(1) = \theta(1) = x'$. If there is a d-homotopy $\phi : \sigma \to \theta$ such that $\phi(0, t) = x$ and $\phi(1, t) = x'$ for $0 \leq t \leq 1$ we write $\sigma \preceq_2 \theta$. A *2-homotopy class of paths* $[\sigma]$ is a class of the equivalence relation $\simeq_2$ spanned by the preorder $\preceq_2$.

The fundamental category $\uparrow \Pi(X)$ of a d-space X has the objects the points of X and morphisms $[\sigma]$ the 2-homotopy classes of paths form x to x'. It is worthwhile mentioning that Grandis in [1] has formulated and proved the Seifert-Van Kampen theorem for fundamental categories.

The fundamental category is a huge gadget, however in many instances, e.g. reachability or safety analysis, we are

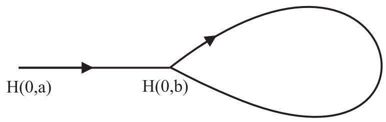

Fig. 2. The arrows depict direction of the directed paths.

only interested in the paths of the hybrid system from a point x (the initial position) to a point x' (the destination), then our abstraction is the 2-homotopy classes of paths from x to x'.

VII. Topological Hybrid Systems

We are in position to define a topological hybrid system.

Definition 14: A *topological hybrid system* is a pair $H \equiv (G, H)$, where $G \in \mathbf{Graph}$ and $H : \int G \to \mathrm{d}\mathbf{Top}$ is a functor.

We shall study a geometric realization (d-homotopy colimit) of a topological hybrid system, in which the transitions of the d-graph define a scheme of gluing the "continuous systems" together. The idea comes from [2]. In Subsection VII-B we provide a formal definition of this construction. For the time being we consider an example, which illustrates how the discrete systems can be seen as a topological space with d-structure.

Example 5: Construct a functor H from the category $\int G$ in the diagram (3) to $\mathrm{d}\mathbf{Top}$ which takes an object of $\int G$ to a singleton, d-space consisting of a single point. We use H to define the following d-space:

$$(H(0,a) \sqcup H(0,b) \sqcup H(1,p) \times \uparrow I \sqcup H(1,q) \times \uparrow I)/\sim, \quad (5)$$

where the relation $\sim$ identifies $(H(1,p),0) \sim H(0,a)$, $(H(1,p),1) \sim H(0,b)$, $(H(1,q),0) \sim H(0,b), (H(1,q),1) \sim H(0,b)$. The result of this geometric realization is given in Fig. 2. Notice the similarity with the diagram (1).

A. D-homotopy Pushout

Homotopy pushouts of d-spaces can be constructed from the directed cylinder.

Let $f : X \to Y$ and $g : X \to Z$ be two d-maps. The *directed homotopy pushout from f to g*, c.f. [8], is a four-tuple $(A; u, v; \lambda)$,

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y \arrow[dr, "u"'] \arrow[rr, phantom, "\overset{\lambda}{\Rightarrow}"] & & Z \arrow[dl, "v"] \\ & A, & \end{tikzcd} \quad (6)$$

where $\lambda : u \circ f \to v \circ g : X \to A$ is a homotopy satisfying the following universal property: For every $(A'; u', v'; \lambda')$ with $\lambda' : u' \circ f \to v' \circ g : X \to A'$ there is precisely one d-map $h : A \to A'$ such that $u' = h \circ u$, $v' = h \circ v$ and $\lambda' = h \circ \lambda$.

We show the existence. Consider the following diagram

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & \\ X \arrow[dr, "\partial_0"'] \arrow[rr, phantom, "\overset{\eta}{\Rightarrow}"] & & X \arrow[dl, "\partial_1"] \\ & \uparrow IX, & \end{tikzcd} \quad (7)$$

where $\partial_\alpha(x) = (x, \alpha),\ \alpha = 0, 1$. The homotopy $\eta : \partial_0 \to \partial_1 : X \to \uparrow IX = X \times \uparrow I$ is defined by

$$\eta(x,t) = (x,t).$$

The diagram (7) is the directed homotopy pushout from ∂_0 to ∂_1.

Consider the colimit (recall that $\mathrm{d}\mathbf{Top}$ is cocomplete) of the following diagram

$$\begin{tikzcd} Y \arrow[drrr, "u"'] & X \arrow[l, "f"'] \arrow[r, "\partial_0"] \arrow[drr, "h_0"] & \uparrow IX \arrow[dr, "h"] & X \arrow[l, "\partial_1"'] \arrow[r, "g"] \arrow[d, "h_1"] & Z \arrow[dl, "v"] \\ & & & A. & \end{tikzcd}$$

The map $h :\uparrow IX \to A$ defines homotopy $h \circ \eta : h_0 \to h_1 : X \to A$, where $h_\alpha = h \circ \partial_\alpha,\ \alpha = 0, 1$. From the diagram $u \circ f = h_0$ and $v \circ g = h_1$, and we conclude that the desired d-homotopy λ in the diagram 6 is the composition $\lambda = h \circ \eta$. The uniquenes up to isomorphisms now also follows.

Remark 2: The d-space A is the quotient of the sum

$$(\uparrow IX \sqcup Y \sqcup Z)/\sim,$$

where $\sim$ is the equivalence relation identifying $(x, 0)$ with $f(x)$ and $(x, 1)$ with $g(x)$. The construction is illustrated in Fig. 3. You may think of Y and Z as two "continuous" subsystems of a hybrid system. The d-subspace $f : X \subset Y$ is the guard and g is the reset function, c.f. [4].

B. Geometric Realization of Hybrid Systems

We follow the lines of [2] and define a geometric realization of a topological hybrid system $(G, H : \int G \to \mathrm{d}\mathbf{Top})$ as a directed homotopy colimit of H.

Proposition 3: Let $G \in \mathbf{Graph}$ and let $H : \int G \to \mathrm{d}\mathbf{Top}$ be a functor. The d-homotopy colimit of H exists and it is unique up to isomorphism.

Proof 2: It is enough to show that there is the d-homotopy colimit of the diagram

$$\begin{tikzcd} H(0,a) \arrow[drr] & H(1,p) \arrow[l, "H\delta^0"'] \arrow[r, "H\delta^1"] \arrow[r, phantom, shift right=6, "\overset{\lambda}{\Rightarrow}"] & H(0,b) \arrow[d] & H(1,q) \arrow[l, "H\delta^0"'] \arrow[r, "H\delta^1"] \arrow[l, phantom, shift right=6, "\overset{\theta}{\Rightarrow}"] & H(1,c) \arrow[dll] \\ & & C & & \end{tikzcd}$$

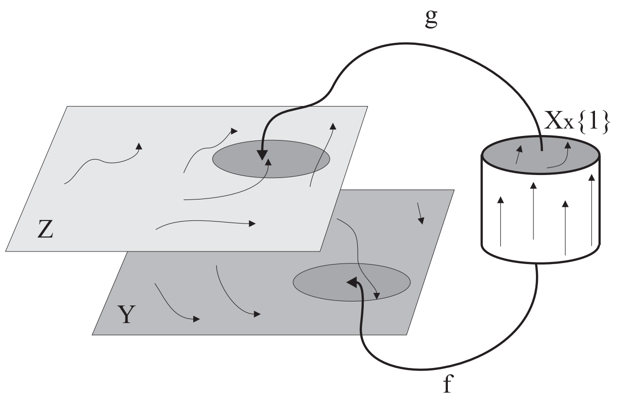

Fig. 3. Illustration of the d-homotopy pushout. You may think Y and Z are two "continuous" systems of a hybrid system. The d-subspace $f : X \subset Y$ is the guard and g is the reset function.

We see that the d-homotopy colimit follows from the diagram below

$$\begin{tikzcd}
H(0,a) \arrow[dr] & H(1,p) \arrow[l, "H\delta^0"'] \arrow[r, "H\delta^1"] \arrow[d, phantom, "\lambda'"] & H(0,b) \arrow[dl] \arrow[dd] \arrow[dr] & H(1,q) \arrow[l, "H\delta^0"'] \arrow[r, "H\delta^1"] \arrow[d, phantom, "\theta'"] & H(0,c) \arrow[dl] \\
& A \arrow[dr] & & B \arrow[dl] & \\
& & C, & &
\end{tikzcd}$$

where the left and the right squares are d-homotopy pushouts, whereas the central square is the d-pushout. □

Definition 15: Let $(G, H : \int G \to \mathrm{d}\mathbf{Top})$ be a topological hybrid system. A *geometric realization* of (G, H) is a d-homotopy colimit of H.

From Proposition 3 we conclude that the geometric realization of a topological hybrid system is a *single* d-space. Therefore the behavior of hybrid systems can be studied through its geometric realization.

Example 6: Consider Example 1, the category depicted in the diagram (3) gives rise to the hybrid system

$$(\uparrow IH(1,p) \sqcup H(0,a) \sqcup H(0,b) \sqcup \uparrow IH(1,q))/\sim,$$

where $\sim$ is the equivalence relation identifying $(x,0)$ with $H\delta_p^0(x)$, $(x,1)$ with $H\delta_p^1(x)$, $(y,0)$ with $H\delta_q^0(y)$ and $(y,1)$ with $H\delta_q^1(y)$, for any $(x,0),(x,1) \in \uparrow IH(1,p)$, $(y,0),(y,1) \in \uparrow IH(1,q)$, Fig. 4.

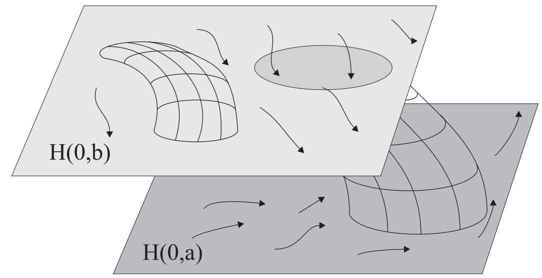

Fig. 4. Illustration of the d-homotopy colimit in Example 6.

VIII. Conclusions

The "discrete part" part of a hybrid system is a directed graph G, to which we associates the diagram category $\int G$. The "continuous parts" of a hybrid system are modelled as d-spaces, that is topological spaces equipped with the d-structure - a set of continuous paths closed under reparametrization, concatenation and which includes all constant paths. The d-spaces with maps preserving the d-structures form the category $\mathrm{d}\mathbf{Top}$. We define a topological hybrid system as a pair (G, H) consisting of a directed graph G and a functor $H : \int G \to \mathrm{d}\mathbf{Top}$. Geometric realization of the hybrid system (G, H) - d-homotopy colimit of H - is a single d-space. Now the behavior of a hybrid system can be studied in terms of the behavior od dynamical systems through its d-homotopy colimit.

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