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Publication date:
2006

Document Version
Publisher's PDF, also known as Version of record

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ISSN 1901-726X
DCE Technical Report No. 10

AALBORG UNIVERSITY
Department of Civil Engineering

Aalborg University
Department of Civil Engineering
Division of Water and Soil

DCE Technical Report No. 10

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by

Lars Bo Ibsen
Morten Liingaard

December 2006

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Published 2006 by
Aalborg University
Department of Civil Engineering
Sohngaardsholmsvej 57,
DK-9000 Aalborg, Denmark

Printed in Denmark at Aalborg University

ISSN 1901-726X
DCE Technical Report No. 10

Preface

The technical report "Experimental modal analysis" is divided into four numbered sections, and a list of references is situated after the last section. Tables, equations and figures are indicated with consecutive numbers. Cited references are marked as e.g. Bendat and Piersol (1986), with author specification and year of publication in the text.

The work within this report has only been possible with the financial support from the Energy Research Programme (ERP)[1] administered by the Danish Energy Authority. The project is associated with the ERP programme "Soil–Structure interaction of Foundations for Offshore Wind Turbines". The funding is sincerely acknowledged.

Aalborg, December 6, 2006 Lars Bo Ibsen & Morten Liingaard

[1] In danish: "Energiforskningsprogrammet (EFP)"

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Chapter 1
Experimental modal analysis

This technical report concerns the basic theory and principles for experimental modal analysis. The sections within the report are: Output-only modal analysis software (section 1.1), general digital analysis (section 1.2), basics of structural dynamics and modal analysis (section 1.3) and system identification (section 1.4).

1.1 Output-only Modal Analysis Software

The experimental modal analysis of the wind turbine prototype is performed by means of the software package ARTeMIS (Ambient Response Testing and Modal Identification Software). The ARTeMIS software is fully compatible with the hardware of the monitoring system. The software package consists of two tools, the ARTeMIS Testor and the ARTeMIS Extractor (SVS 2006).

1.1.1 Output-only Modal Identification

The experimental modal analysis of the wind turbine makes use of "Output-only modal identification" which is utilized when the modal properties are identified from measured responses only. "Output-only modal identification" is also known by the terms "ambient identification" or "ambient response analysis" within the field of civil engineering. The following description is based on SVS (2006).

Modal Identification

The basic principle in Modal identification is the determination of modal parameters from experimental data. The usual modal parameters are natural frequencies (the resonance frequencies), damping ratios (the degree to which the structure itself is able of damping out vibrations) and mode shapes (the way the structure moves at a certain resonance frequency). The common way is to use input-output modal identification where the modal parameters are found by fitting a model to a Frequency Response Function, a function relating excitation and response. The traditional techniques in input-output modal identification is described frequently in the literature, see for instance (Ewins 1995; Maia and Silva 1997).

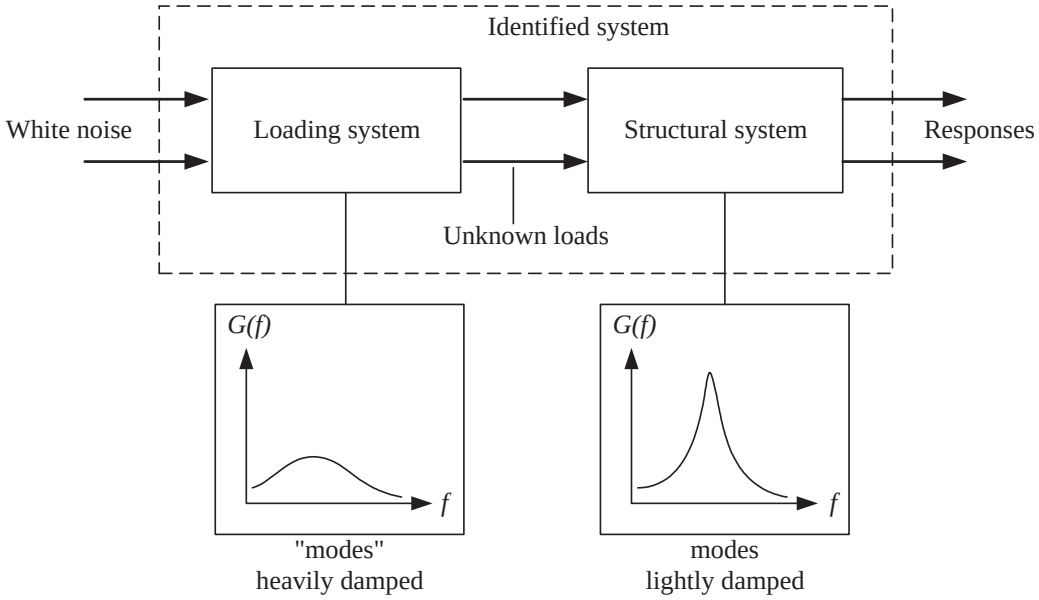

Figure 1.1: Principles of output-only modal analysis.

Output-only modal Identification

When modal identification is based on the measured response (output) only, things become more complicated for several reasons, the excitation (input) is unknown and the measured response (output) is often noisy.

Output-only modal identification is used for analyzing large civil engineering structures, operating machinery or other structures that are not easily excited artificially. Large civil engineering structures are often excited by natural loads that cannot easily be controlled, for instance wave loads (offshore structures), wind loads (Buildings) or traffic loads (bridges). For operating machinery the problems are the same. They are also excited by natural sources like noise from bearings or vibrations from the environment around the structure. In these cases, it is an advantage to use output-only modal identification. Instead of exciting the structure artificially and dealing with the natural excitation as an unwanted noise source, the natural excitation is used as the excitation source. The idea of output-only modal identification is illustrated in Figure 1.1.

The unknown loading conditions of the structure are assumed to be produced by a virtual system loaded by white noise. The white noise is assumed to drive both the real structural system and the virtual loading system as a total system and not only the structural system.

For that reason the user is identifying two types of modes, one type of modes that belongs to the real structural system and another type of "modes" that belong to the virtual loading system. The real structural modes are characterized by light damping,

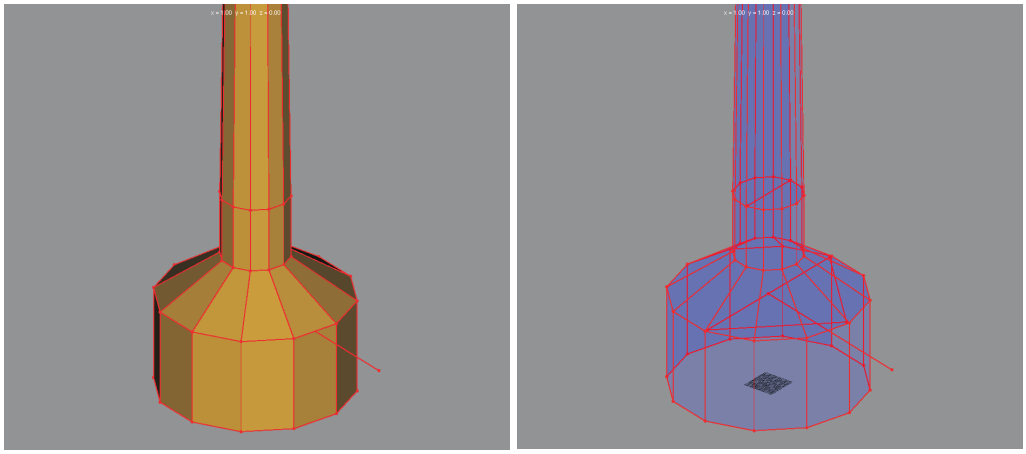

Figure 1.2: Geometry of wind turbine tower and foundation. Left, with opaque surfaces and right, without surfaces.

whereas the "modes" of the virtual loading system usually are heavily damped, see Figure 1.1. Furthermore, the user might also identify computational modes that appear because the signals are contaminated with noise. This means, that it is of outmost importance that the real structural modes are separated from noise modes and excitation modes during the modal identification process.

1.1.2 ARTeMIS Testor

The ARTeMIS Testor is a test planning tool where the geometry of the structure and the sensor settings and locations are defined. There are three main tasks to be carried out: Geometry generation, hardware definition and test planning. The tasks are briefly described in the following.

Geometry generation

The geometry of the system consists of two subsets. The first subset of the geometrical model is the active master system defined by the coordinates of the actual sensors. In this case it is the xyz-coordinates of the 15 accelerometers (the positions are given in the main paper). The second subset is the slave system of nodes. The slave system represents the physical appearance of the structure. The displacements of the nodes of the slave system are coupled to the master system by means of slave equations. The slave equations are influence relations that states how much a slave node moves if the corresponding master node is displaced by 1 unit. Lines and opaque surfaces can be added into the geometry in order to make a realistic and uncomplicated representation of the structure in the subsequent analyses, see Figure 1.2.

Hardware Definition

The hardware is defined by one or more a virtual data acquisition units (a front-end) that each represents a measurement session. Each front-end unit holds as many transducer objects as there are measurement channels in the session. The front-end unit contains information about the number of data points, sampling frequency and the Nyquist frequency of the particular session. The Transducer object is a virtual measurement channel. This object contains the actual measurements of a single channel as well as the parameters necessary to describe them.

Test Planning

The Test Planning task is used to assign each of the transducer objects to the geometry. Each transducer object must be linked to one of the master nodes and the orientation of the transducer must be set as well, i.e. the degree of freedom (DOF). The location and orientation of the transducer objects are shown in Figure 1.3

1.1.3 ARTeMIS Extractor

The ARTeMIS Extractor is the key application of the ARTeMIS software package. The software allows the user to perform accurate modal identification under operational conditions and in situations where the structure is impossible or difficult to excite by externally applied forces. The typical outputs of the analyses are modal information about the natural frequencies, mode shapes and damping ratios.

Analysis assumptions

The modal analysis within this software is based on the assumptions that the underlying physical system of the structure is linear and time-invariant. The linearity imply that the physical system comply with the rules of linear superposition. The time-invariance implies that the underlying mechanical or structural system does not change in time. Within this frame the program is based on two different estimation techniques, one in time domain and one in frequency domain, see Figure 1.4.

Stochastic Subspace Identification

The time domain estimation is based on Stochastic Subspace Identification (SSI) technique. In the SSI techniques a parametric model is fitted directly to the raw time series data obtained from the accelerometers. The parametric models are characterized by the assumption of a mathematical model constructed from a set of parameters, where the mathematical model is a linear and time-invariant system of differential equations. The task of the SSI technique is to adjust the parameters in order to change the way the model fits to the data. In general the objective is to estimate a set of parameters that will minimize the deviation between the predicted system response (predicted transducer signal) of the model and measured system response (transducer signal). The parametric models and Stochastic Subspace Identification are described in Section 1.4. For references, see (Andersen 1997; Brincker and Andersen 1999).

Figure 1.3: Test Planning. Each of the transducers (green arrows) is placed at the master nodes of the geometry. The arrows at the transducers nodes indicate the orientation.

Frequency Domain Decomposition

The frequency domain estimation is a non-parametric model (also known as spectral models) based on a Frequency Domain Decomposition (FDD) method. The FDD method is an extension of the well-known frequency domain approach that is based on mode estimations directly from the Power Spectral Density (PSD) matrix, i.e. well separated modes can be identified at the peaks of the PSD matrix.

The basic principle of the Frequency Domain Decomposition (FDD) technique is to perform an approximate decomposition of the system response into a set of independent single degree of freedom (SDOF) systems; each corresponding to an individual mode. In the FDD the Spectral Density matrix is decomposed by means of the Singular Value Decomposition (SVD) into a set of auto spectral density functions, each corresponding to a single degree of freedom system. The steps of the FDD technique are illustrated in

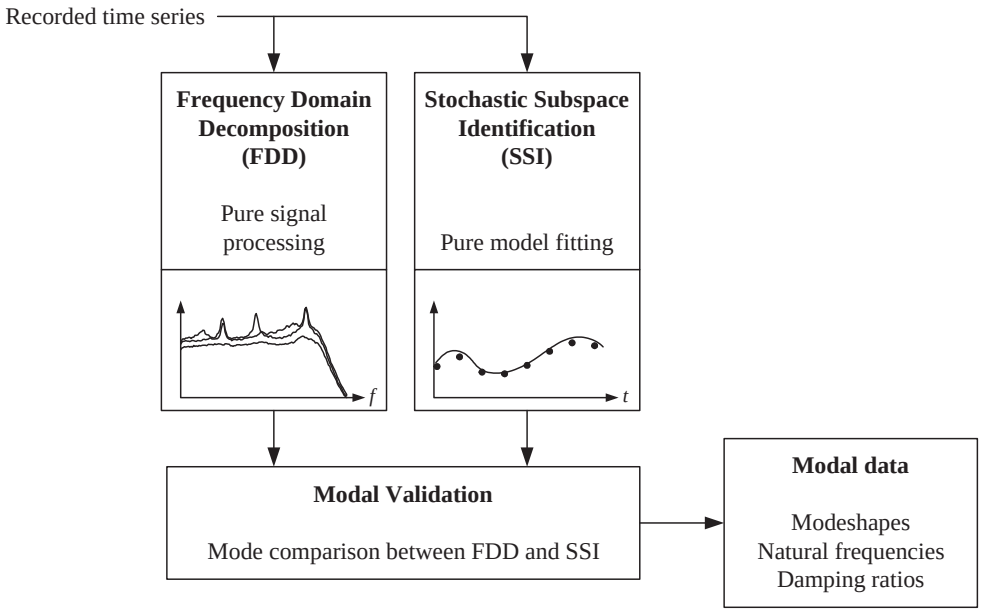

Figure 1.4: Modal analysis with ARTeMIS Extractor.

Figure 1.5.

The key feature is that the singular values are estimates of the Auto Spectral density of the SDOF systems, and the singular vectors are estimates of the mode shapes. The basic theory concerning identification by FDD is given in Section 1.4. For references, see (Brincker, Andersen, and Zhang 2000; Brincker, Zhang, and Andersen 2000).

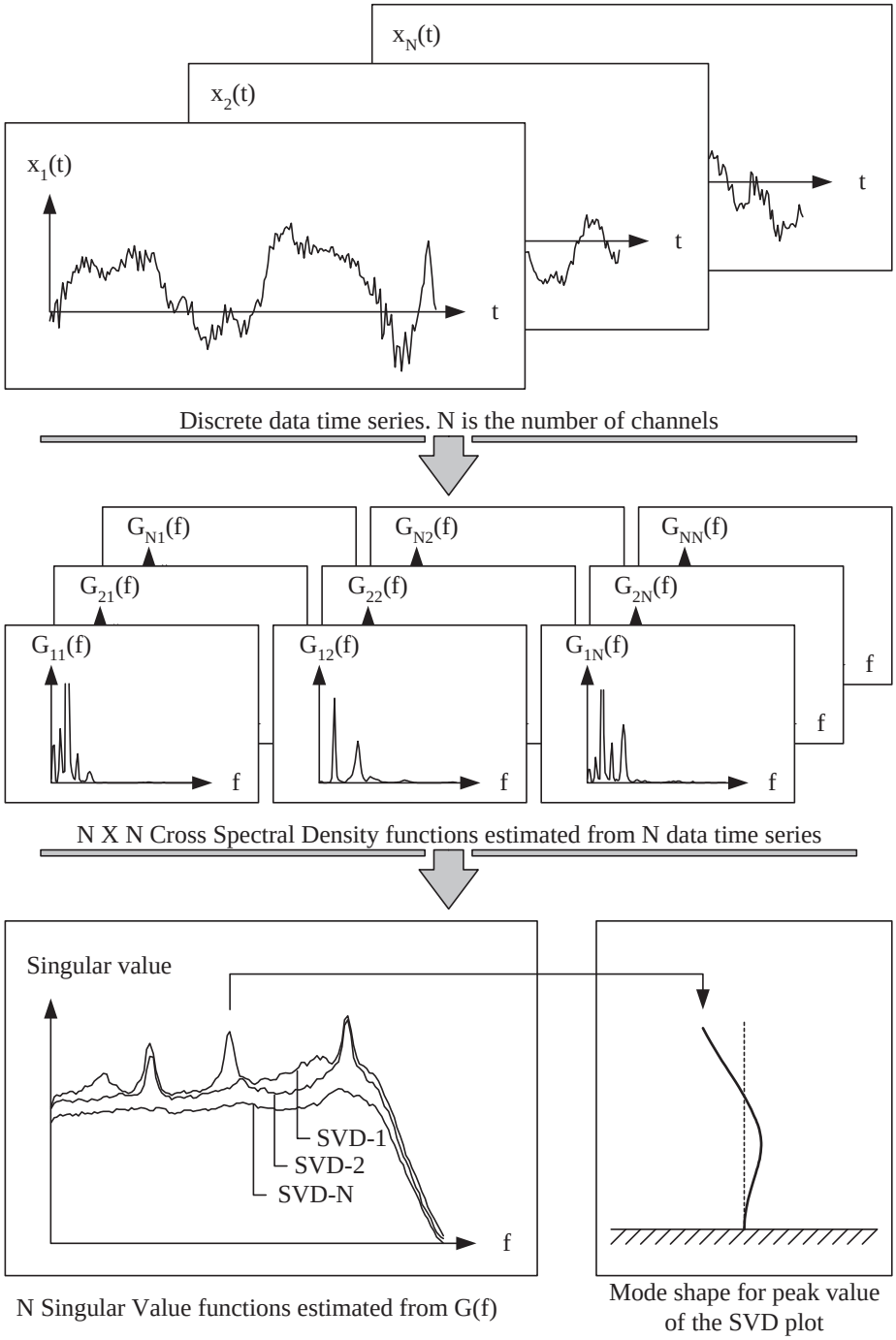

Figure 1.5: Main steps of the Frequency Domain Decomposition (FDD) technique.

Example—Peak picking by FDD

The following example shows the main output of peak picking by FDD. The FDD peak pricking is based on measured data recorded February 15, 2005. The data set consists of a 1 hour measurement in 15 channels. The sampling frequency was 200 Hz and the data was decimated by an order of 20. The main screen image for the FDD modal identification technique is shown in Figure 1.6. Note that four modes are identified. The second screen image shows the first mode shape of the wind turbine.

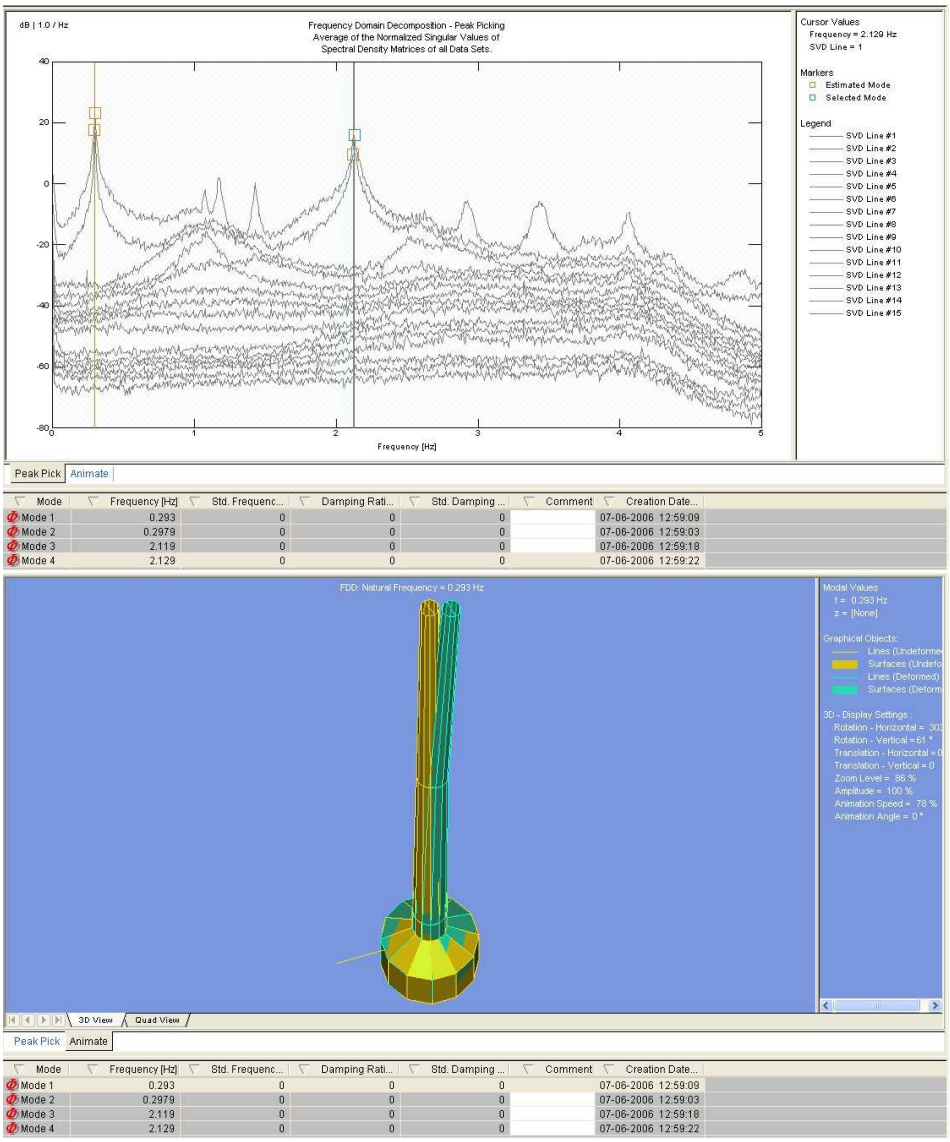

Figure 1.6: Top: main screen image for the FDD modal identification technique. Bottom: screen image of the first mode shape of the wind turbine.

1.2 General digital data analysis

This section explains the basic digital operations that are required prior to the main estimation of the modal parameters by the two identification procedures. Before the measured data is useful as input data to the modal identification estimation routines several pre-processing procedures are required. The typical pre-processing steps are described in the following.

1.2.1 Data Sampling and aliasing

The analog input signals from the transducers are continuous and processed by means of a analog filter and a analog-to-digital conversion in order to manage the information on a digital computer. The process prior to the digital signal processing is shown in Figure 1.7.

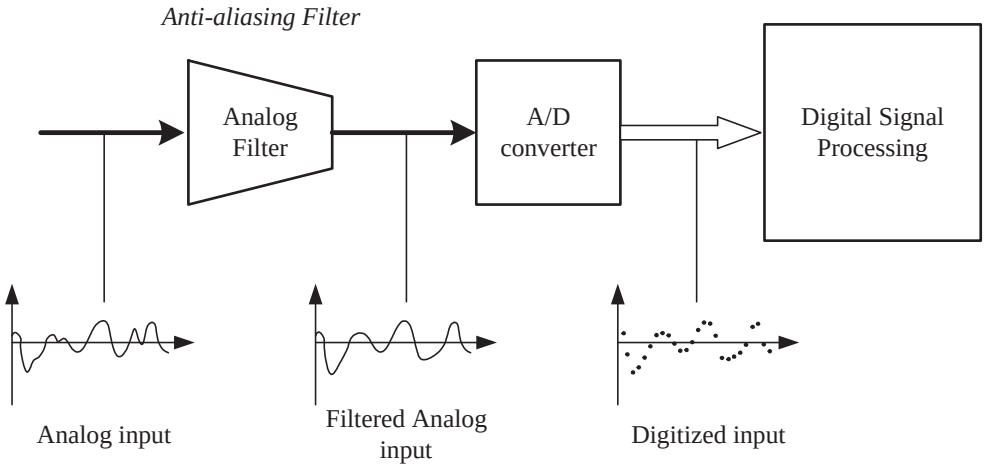

Figure 1.7: Filtering and A/D conversion of analog input signal prior to digital signal processing.

1.2.2 Structure of measured data

The measured data from the accelerometers are considered as sample records of a random process, i.e. the data are physical realizations of a random process. It is assumed that the random process is stationary, which means that the loading and structural system is assumed to be time invariant.

The measured data are digital representations of a continuous signal from the transducers. Two data time series x, y are illustrated in Figure 1.8. The time series are sampled with a fixed sample frequency f_s. The equally spaced time interval between the data points is denoted the sampling interval Δt. Δt is equal to $1/f_s$. It is assumed that

the recorded time series can be separated into data segments x_n and y_n of the length T containing N numbers of data points. By this segmentation of the time series the data are assumed to be periodic with a return period of T.

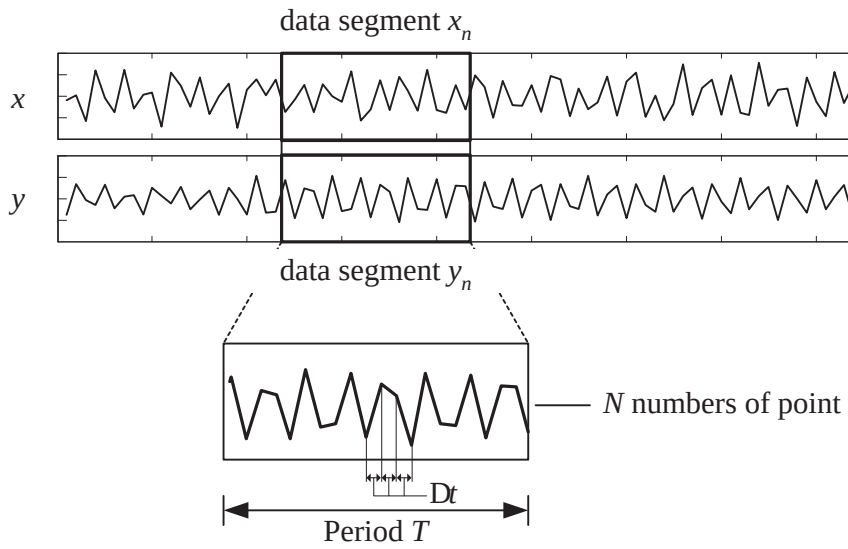

Figure 1.8: Digital representation of a continuous signal. The time series x and y are digitized signals with equally spaced time intervals Δt.

1.2.3 Nyquist frequency

To present the frequency content of the data the Fourier Transform $X(f)$ of $x(t)$ is imposed. Each frequency component (or cycle) of the original data requires at least two samples, which means that the highest frequency that can be defined by a sampling rate of $f_s = 1/\Delta t$ is $f_s/2$. This particular band-limiting frequency is denoted the Nyquist frequency (or folding frequency):

$$f_{nyq} = \frac{f_s}{2} = \frac{1}{2\Delta t} \tag{1.1}$$

1.2.4 Aliasing

Frequencies or vibration cycles above f_{nyq} in the original data will appear below f_{nyq} in the frequency domain and could be misinterpreted as low frequency content, see Figure 1.9. This phenomenon is known as aliasing. To avoid aliasing the frequency content of the original data above f_{nyq} should be removed prior to the subsequent signal processing procedures. The high frequency information can be removed by "anti-aliasing filters" by applying a low pass filter that cuts off frequency content higher than f_{nyq}. Real filters does not have an infinitely sharp cut-off shape, so the anti-aliasing filter cut-off

frequency is set to approx. 80 % of f_{nyq} to assure that any data at frequencies above f_{nyq} are strongly suppressed (Bendat and Piersol 1986).

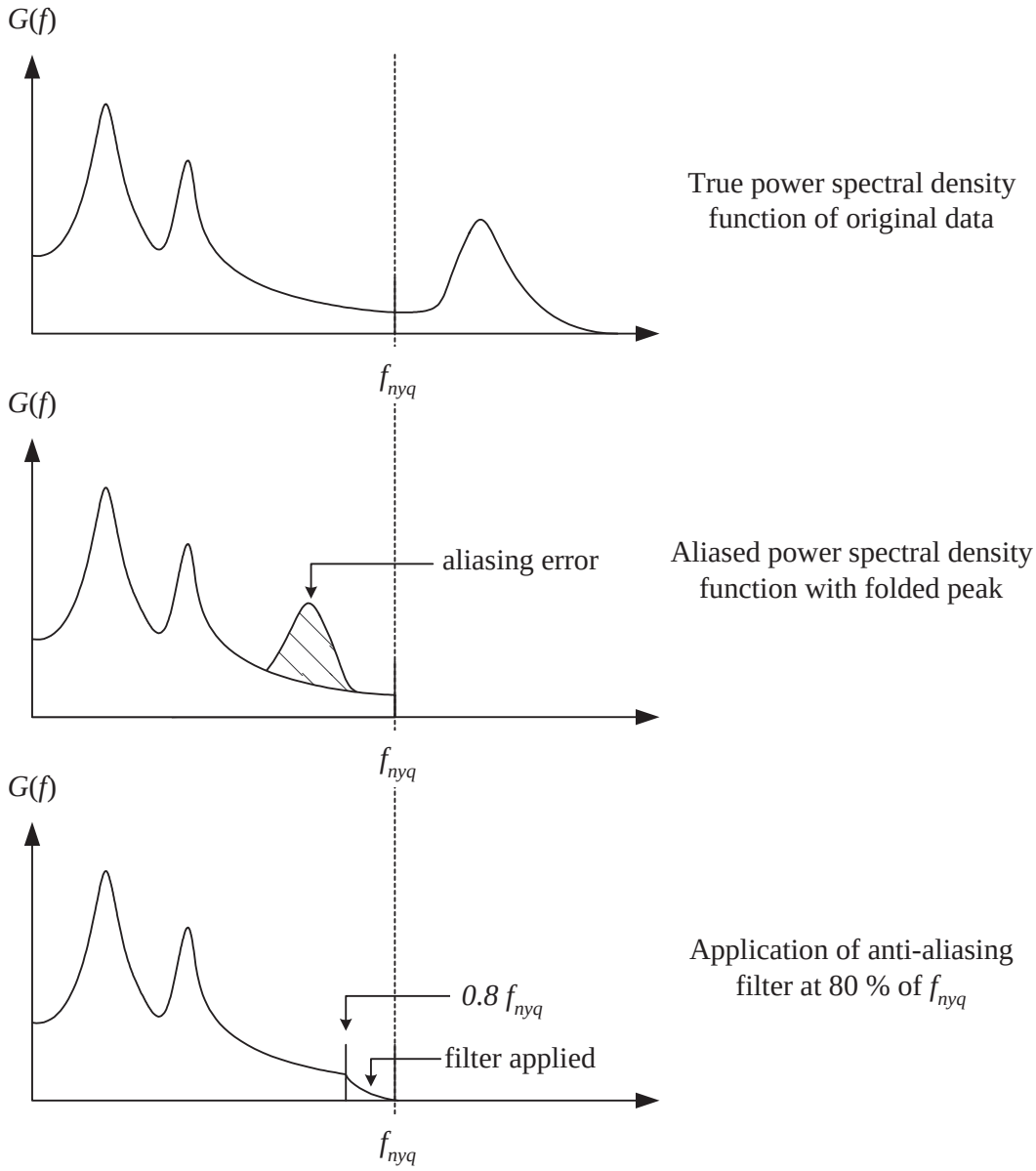

Figure 1.9: Aliased power spectrum due to folding.

1.2.5 Signal processing (digital data analysis)

The traditional non-parametric models for system identification are primarily based of spectral analysis that makes use of Fourier Transform techniques. The spectral analysis is employed for analysing stochastically excited systems and in this case the excitation and the system response can be characterized by spectral densities in frequency domain. The basic signal processing steps in the non-parametric methods are described in the following.

Spectral Analysis—example

When operating with spectral analysis techniques the shape of the time domain waveform of the vibrating structure is not dealt with; the key information is the frequency, phase and amplitude of the component sinusoids. The Discrete Fourier Transform (DFT) technique is used to extract this information. This general concept is shown by an example, based on a description from Smith (1997).

The measuring device is a transducer (here an accelerometer) where the data is sampled by a rate of 200 Hz and thereby a Nyquist frequency of 100 Hz. An analog low-pass filter (anti-aliasing filter) is used to remove all frequencies above 100 Hz, and the cut-off frequency is set to 80 % of the Nyquist frequency. A sample of 1024 data point of a measured signal is shown in Figure 1.10(a). This corresponds to a data segment of a time series, as shown in Figure 1.8. The DFT technique makes use of the Fast Fourier transform (FFT) algorithm. When the FFT is applied for transforming a sample of 1024 data points, this result in a 513 point frequency spectrum in the frequency domain, i.e. the frequency range from 0 to 100 Hz is divided into 513 frequency points. By using the FFT algorithm it is assumed that the signal to be transformed is periodic within the transformation window (here corresponding to the 1024 samples). Many types of signals, such as random signals are non-periodic in the transformation window, which may lead to distortion of the frequency spectrum. This distortion is referred to as "spectral leakage" and results in inaccurate spectral information of the measured signal. To suppress the spectral leakage the measured signal is tapered before applying the FFT, so the discontinuities at the edges of the transformation window are reduced. This time history tapering is done by multiplying the measured signal in Figure 1.10(a) by a suitable time window as shown in Figure 1.10(b). This specific window is denoted a Hamming Window but other time windows are available, see e.g. Bendat and Piersol (1986). The resulting signal is shown in Figure 1.10(c), where the samples near the ends have been reduced in amplitude. The windowed signal in Figure 1.10(c) is transformed by means of DFT into a 513 point frequency spectrum (here a Power Spectral Density spectrum) as shown in Figure 1.10(d). This plot is filled with noise because there is not enough information in the original 1024 points to obtain a well defined spectrum. The noise is not reduced by refining the FFT to 2048 points (=1025 point frequency spectrum), because using a longer FFT provides better frequency resolution, but the same noise level.

In order to reduce the noise more data is needed. This is carried out by separating the data into multiple 1024 data point segments. Each segment is multiplied by the Hamming Window, processed by the 1024 FFT algorithm and converted into a frequency spectrum. The resulting spectrum is constructed by averaging all the frequency spectra,

as shown in Figure 1.10(e). Here the spectrum is an average of 200 spectra. The noise level has been reduced and the relevant features of the signal can be investigated. It should be noted that the number of segments should be sufficiently large, e.g. 100 or more.

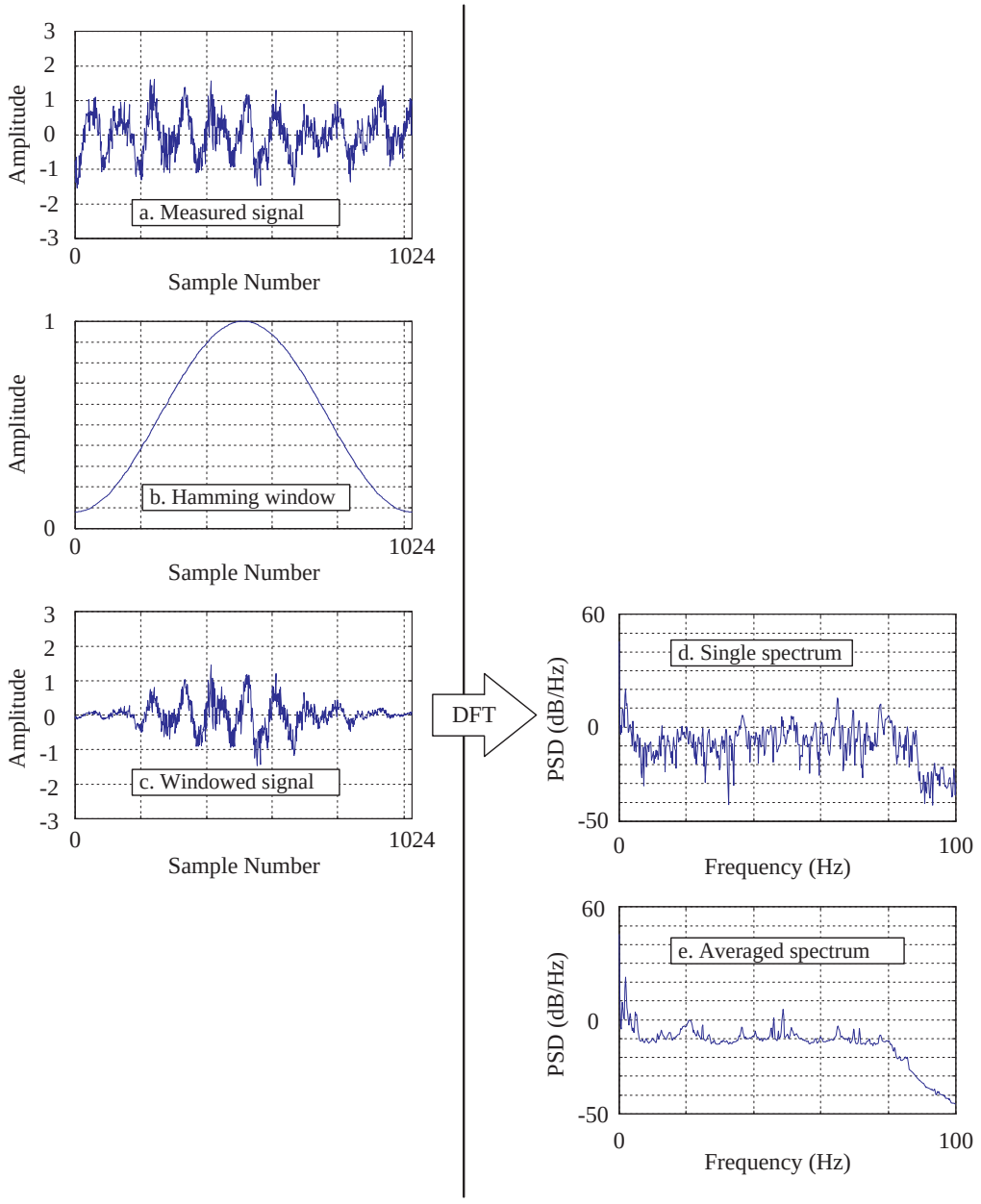

Figure 1.10: Example of spectral analysis of a signal. (a) shows 1024 samples taken from a transducer with a sample frequency of 200 Hz. The signal is multiplied by a Hamming window (b), resulting in the windowed signal in (c). The Power Spectral Density (PSD) of the windowed signal is calculated by means of the Discrete Fourier Transform (DFT) and followed by multiplication in frequency domain, as displayed in (d). When averaging e.g. 200 of these spectra the random noise is reduced, resulting in the averaged spectrum shown in (e).

Spectral analysis—General concepts

The spectral analysis is used for identifying the frequency composition of random signals in the frequency domain. In the following some basic descriptive properties used for describing random signals are presented. These are:

- Autocorrelation functions
- Cross-correlation functions
- Spectral density functions

Autocorrelation Function
The definition of an autocorrelation function is: The expected value of the product of a random variable or signal realization with a time-shifted version of itself. The autocorrelation function contains information about how quickly random signals or processes changes with respect to time, and whether the process has a periodic component and what the expected frequency might be.

A pair of random variables from the same process $x(t)$ is considered, that is $x_1 = x(t_1)$ and $x_2 = x(t_2)$. Then the autocorrelation $\Pi_{xx}(t_1, t_2)$ of x_1 and x_2 can be written as:

$$\Pi_{xx}(t_1, t_2) = E[x_1 x_2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 p(x_1, x_2) dx_1 dx_2 \tag{1.2}$$

where $p(x_1, x_2)$ is the joint probability density function of $x_1 = x(t_1)$ and $x_2 = x(t_2)$. The above equation is valid for stationary and non-stationary random processes. For a stationary process the expression can be generalized, and it can be proven that the expected values of the random process will be constant and independent of time. Therefore, the autocorrelation function will depend only on the time difference and not the absolute time. The time difference is introduced as $\tau = t_1 - t_2$ and the autocorrelation $\Pi_{xx}(\tau)$ can be expressed as:

$$\Pi_{xx}(t, t+\tau) = \Pi_{xx}(\tau) = E[x(t)x(t+\tau)] \tag{1.3}$$

Usually the whole random process is not available or described properly. In these cases, the autocorrelation can be estimated for a given interval, 0 to T seconds, of the sample function of the random process. The estimation of the autocorrelation $\hat{\Pi}_{xx}(\tau)$ is given as:

$$\hat{\Pi}_{xx}(t, t+\tau) = \frac{1}{T-\tau} \int_0^{T-\tau} x(t)x(t+\tau) dt \tag{1.4}$$

This is given for the continuous case. It is usually not possible to describe the complete continuous-time function for the random signals, so the equation is modified in order to treat the information in a discrete-time formula. The discrete-time formulation for estimating the autocorrelation is as follows:

$$\hat{\Pi}_{xx}[m] = \frac{1}{N-m} \sum_{n=0}^{N-m-1} x[n]x[n+m] \tag{1.5}$$

where N is the number of data points in the sample, m is the data point corresponding to τ and n the data point corresponding to t.

Cross-correlation Function
When dealing with multiple random processes, it is important to describe the relationship between the processes. This may for example occur if more than one random signal is applied to a system. The cross correlation function is defined as the expected value of the product of a random variable from one random process with a time-shifted, random variable from a different random process.

For a stationary process the expression for the cross-correlation can be written in terms of $\tau = t_1 - t_2$ since the expected values of the random process will be constant and independent of time, as described for the autocorrelation. Consider two random processes $x(t)$ and $y(t)$. Then the cross correlation function is defined as:

$$\Pi_{xy}(t, t+\tau) = \Pi_{xy}(\tau) = E[x(t)y(t+\tau)] \tag{1.6}$$

The cross-correlation can be estimated for a given interval, 0 to T seconds, of the sample functions of the random processes. The estimation of the cross-correlation $\hat{\Pi}_{xy}(\tau)$ is given as:

$$\hat{\Pi}_{xy}(\tau) = \frac{1}{T-\tau}\int_0^{T-\tau} x(t)y(t+\tau)dt \tag{1.7}$$

The discrete-time formulation for estimating the cross-correlation is as follows:

$$\hat{\Pi}_{xy}[m] = \frac{1}{N-m}\sum_{n=0}^{N-m-1} x[n]y[n+m] \tag{1.8}$$

where N is the number of data points in the sample, m is the data point corresponding to τ and n the data point corresponding to t.

Spectral density Function
The spectral density functions can be defined in several ways. These are:

- Definition via correlation functions
- Definition via finite Fourier transforms
- Definition via filtering-squaring-averaging operations

Only the first two definitions will be mentioned here, for further details, see Bendat and Piersol (1986). The most common way to define the spectral density function is by means of the correlation function described in the previous section. The spectral density function is defined by taking a single Fourier Transform of the correlation function. The auto-spectral density function $S_{xx}(f)$ is thus defined as:

$$S_{xx}(f) = \int_{-\infty}^{\infty} \Pi_{xx}(\tau)\mathrm{e}^{-\mathrm{i}2\pi f\tau}d\tau \tag{1.9}$$

Where i ($= \sqrt{-1}$) is the imaginary unit and f is the frequency. This approach gives a two-sided spectral density function $S_{xx}(f)$, which is defined for $f \in [-\infty, \infty]$. It

should be noted that the auto-spectral density function $S_{xx}(f)$ also is denoted the "power spectral density function". As well as the auto-spectral density function is defined for a autocorrelation function there exists a cross-spectral density function $S_{xy}(f)$, defined as:

$$S_{xy}(f) = \int_{-\infty}^{\infty} \Pi_{xy}(\tau) \mathrm{e}^{-\mathrm{i}2\pi f\tau} d\tau \tag{1.10}$$

The second way of defining the spectral density function is based on finite Fourier Transforms of the original data series. Two random processes $x(t)$ and $y(t)$ are considered. For a finite time interval $0 \leq t \leq T$ the spectral density function can be defined as:

$$S_{xx}(f,T) = \frac{1}{T} X^*(f,T) X(f,T) \tag{1.11a}$$

$$S_{xy}(f,T) = \frac{1}{T} X^*(f,T) Y(f,T) \tag{1.11b}$$

where

$$X(f,T) = \int_0^T x(t) \mathrm{e}^{-\mathrm{i}2\pi ft} dt \tag{1.12a}$$

$$Y(f,T) = \int_0^T y(t) \mathrm{e}^{-\mathrm{i}2\pi ft} dt \tag{1.12b}$$

$X(f,T)$ and $Y(f,T)$ represent finite Fourier transforms of $x(t)$ and $y(t)$, respectively, and $X^*(f,T)$ is the complex conjugate of $X(f,T)$. The estimate of $S_{xx}(f)$ or $S_{xy}(f)$ when T tends toward infinity is given by:

$$S_{xx}(f) = \lim_{T\to\infty} E[S_{xx}(f,T)] \tag{1.13a}$$

$$S_{xy}(f) = \lim_{T\to\infty} E[S_{xy}(f,T)] \tag{1.13b}$$

It can be shown that (1.13) is equal to (1.9) and (1.10) (Bendat and Piersol 1986).

It is not convenient to describe the frequency composition in the frequency range from $-\infty$ to ∞. Hence, the spectral density function $S(f)$ is converted into a one-sided spectral density function $G(f)$ where $f \in [0, \infty]$. The one-sided auto-spectral and cross-spectral density functions $G_{xx}(f)$ and $G_{xy}(f)$ are defined as:

$$G_{xx}(f) = 2S_{xx}(f) \qquad 0 \leq f \leq \infty \qquad \text{otherwise zero} \tag{1.14a}$$

$$G_{xy}(f) = 2S_{xy}(f) \qquad 0 \leq f \leq \infty \qquad \text{otherwise zero} \tag{1.14b}$$

1.3 Basics of structural dynamics and modal analysis

In this section the basic equations for structural dynamics and modal analysis are presented. The section is based on Bendat and Piersol (1986) and Andersen (1997).

1.3.1 Dynamic model of second-order structural system

Dynamic structural systems subjected to external loading are often modelled as a lumped mass-spring-dashpot parameter model given by:

$$\mathbf{M}\ddot{\mathbf{z}}(t) + \mathbf{C}\dot{\mathbf{z}}(t) + \mathbf{K}\mathbf{z}(t) = \mathbf{f}(t) \tag{1.15}$$

$\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices with the dimensions $p \times p$. $\mathbf{z}(t)$ and $\mathbf{f}(t)$ are $p \times 1$ displacement and force vectors at the mass points, respectively. Equation (1.15) is a second-order differential equation that represents the force equilibrium of the structural system. The inertial forces $\mathbf{M}\ddot{\mathbf{z}}$ are balanced by a set of linear-elastic restoring forces $\mathbf{K}\mathbf{z}$, viscous damping forces $\mathbf{C}\dot{\mathbf{z}}$ and the external forces $\mathbf{f}(t)$.

The general solution of the linear constant-parameter can be described by a weighting function $\mathbf{h}(\tau)$, also known as the unit impulse response function, which is defined as the output of the system at any time to a unit impulse input applied a time before (Bendat and Piersol 1986). $\mathbf{h}(\tau)$ has the dimension $p \times p$. If it is assumed that the initial conditions are zero, i.e. $\mathbf{z}(0) = \mathbf{0}$ and $\dot{\mathbf{z}}(0) = \mathbf{0}$, then the solution can be written in terms of the following convolution integral:

$$\mathbf{z}(t) = \int_0^t \mathbf{h}(\tau)\mathbf{f}(t-\tau)d\tau, \qquad \begin{cases} \mathbf{z}(0) = \mathbf{0} \\ \dot{\mathbf{z}}(0) = \mathbf{0} \end{cases} \tag{1.16}$$

The expression in (1.16) states that the output $\mathbf{z}(t)$ is given as a weighted linear sum over the entire history of the input $\mathbf{f}(t)$.

The unit impulse response function $\mathbf{h}(\tau)$ describes the system in time domain. The system can also be described in the frequency domain by means of a frequency response function $\mathbf{H}(f)$. If the system parameters are constant and the system is linear then $\mathbf{H}(f)$ is defined as the Fourier Transform of $\mathbf{h}(\tau)$:

$$\mathbf{H}(f) = \int_0^\infty \mathbf{h}(\tau)\mathrm{e}^{-\mathrm{i}2\pi f\tau}d\tau, \qquad \text{or} \qquad \mathbf{H}(\omega) = \int_0^\infty \mathbf{h}(\tau)\mathrm{e}^{-\mathrm{i}\omega\tau}d\tau \tag{1.17}$$

where f is frequency, ω is angular frequency and i is the imaginary unit.

1.3.2 Modal Analysis

Within the field of system identification is assumed that the estimated models can serve as a basis for a subsequent modal analysis of the structure. In the following it is shown how the modal information can be extracted from the second-order structural system in (1.15). For a particular mode, the jth mode, can be represented by various modal parameters. These are (Andersen 1997):

- Modal frequency:

- Eigenvalue λ_j
- Angular eigenfrequency ω_j
- Natural eigenfrequency f_j

♦ Modal damping:

- Damping ratio ζ_j

♦ Modal vector:

- eigenvector $\mathbf{\Psi}_j$
- Mode shape $\mathbf{\Phi}_j$

♦ Modal scaling:

- Modal mass m_j
- Residues $\mathbf{R}_j$

The different parameters will be introduced and explained as they appear in the description.

State space representation of the dynamic system

The vibrations of the system in (1.15) are assumed to be viscously damped, and for that reason it is necessary to evaluate the eigenvalue problem of the system as complex in order to determine the modal parameters. The solution of the complex eigenvalue problem requires the construction of a $2p \times 2p$ system matrix and a $2p$ response vector. The response vector is denoted as the state vector of the system in (1.15). The state vector is defined in term of displacements and velocities of the system:

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix} \tag{1.18}$$

By means of the state vector in (1.18) the second-order system in (1.15) can be reduced to a first-order differential equation system:

$$\begin{aligned} &\mathbf{A}\dot{\mathbf{z}}(t) + \mathbf{B}\mathbf{z}(t) = \mathbf{u}(t) \\ &\mathbf{A} = \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} \mathbf{f}(t) \\ \mathbf{0} \end{bmatrix} \end{aligned} \tag{1.19}$$

The differential equation in (1.19) is denoted as the "state space representation" of the dynamic system.

Free vibrations of the dynamic system

The free vibration of the system in (1.19) is given by:

$$\mathbf{A}\dot{\mathbf{z}}(t) + \mathbf{B}\mathbf{z}(t) = \mathbf{0} \tag{1.20}$$

The solution for (1.20) is assumed to be on the following form:

$$\mathbf{x}(t) = \mathbf{\Psi} \mathrm{e}^{\lambda t} \tag{1.21}$$

Where $\mathbf{\Psi}$ is a complex vector with dimensions $2p \times 1$ and λ is a complex constant. When (1.21) is inserted into (1.20) it shows that (1.21) is a solution if and only if $\mathbf{\Psi}$ is a solution to the first-order eigenvalue problem given as:

$$(\lambda \mathbf{A} + \mathbf{B})\, \mathbf{\Psi} = \mathbf{0} \tag{1.22}$$

This leads to the characteristic polynomial of the eigenvalue problem:

$$\det(\lambda \mathbf{A} + \mathbf{B}) = \mathbf{0} \tag{1.23}$$

The order of the polynomial is $2p$ and has $2p$ roots λ_j, $j = 1, \ldots, 2p$. For each of the roots λ_j a non-trivial solution $\mathbf{\Psi}_j$ to (1.23) exists. The solution vector $\mathbf{\Psi}_j$ is denoted an eigenvector. The system is assumed to be underdamped (typical for a broad range of civil engineering structures) and this means that the eigenvalues λ_j can be represented by complex conjugated pairs, given by:

$$\begin{aligned} \lambda_j, \lambda_{j+1} &= -2\pi f_j \zeta_j \pm \mathrm{i}2\pi f_j \sqrt{1-\zeta_j^2} = -\omega_j \zeta_j \pm \mathrm{i}\omega_j \sqrt{1-\zeta_j^2} \\ \zeta_j &< 1, \quad j = 1, 3, \ldots, 2p-1 \end{aligned} \tag{1.24}$$

Where f_j is the natural eigenfrequency, ω_j is the angular eigenfrequency and ζ_j the damping ratio of the jth underdamped mode. Note that both λ_j and λ_{j+1} is given in (1.24). From (1.18) and (1.19) it follows that the eigenvector has the form:

$$\mathbf{\Psi}_j = \begin{bmatrix} \mathbf{\Phi}_j \\ \lambda_j \mathbf{\Phi}_j \end{bmatrix}, \quad j = 1, 2, \ldots, 2p \tag{1.25}$$

The standard eigenvalue problem of the second-order system can be formulated if $\mathbf{A}$ and $\mathbf{B}$ and (1.25) is inserted into (1.22). This gives the following equation:

$$\left(\lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K}\right) \mathbf{\Phi}_j = \mathbf{0}, \quad j = 1, 2, \ldots, 2p \tag{1.26}$$

The vector $\mathbf{\Phi}_j$ is the non-trivial solution for the standard eigenvalue problem of the second-order system, and is denoted as the "mode shape". The eigenvectors $\mathbf{\Psi}_j$ for all the modes from $j = 1$ to $2p$ can be assembled in one matrix $\mathbf{\Psi}$ which defines the complex modal matrix for the system. $\mathbf{\Psi}$ is given as:

$$\mathbf{\Psi} = \begin{bmatrix} \mathbf{\Phi}_1 & \mathbf{\Phi}_2 & \cdots & \mathbf{\Phi}_{2p} \\ \lambda_1 \mathbf{\Phi}_1 & \lambda_2 \mathbf{\Phi}_2 & \cdots & \lambda_{2p} \mathbf{\Phi}_{2p} \end{bmatrix} \tag{1.27}$$

The matrix in (1.28) has important orthogonal properties with respect to the matrix $\mathbf{A}$. The special properties are as follows:

$$\boldsymbol{\Psi}^{\mathrm{T}}\mathbf{A}\boldsymbol{\Psi} = \mathbf{M}_d, \quad \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{B}\boldsymbol{\Psi} = \boldsymbol{\Lambda}\mathbf{M}_d, \quad \begin{cases} \boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_j\} \\ \mathbf{M}_d = \mathrm{diag}\{m_j\} \end{cases} \tag{1.28}$$

where $\boldsymbol{\Lambda}$ and $\mathbf{M}_d$ are diagonal matrices containing $2p$ eigenvalues λ_j and damped modal masses m_j, respectively.

The solution to the system in (1.15) is given by the convolution integral in (1.16). This function can conveniently be expressed in terms of the modal decomposed system as:

$$\mathbf{h}(\tau) = \sum_{j=1}^{2p} \frac{\boldsymbol{\Phi}_j \boldsymbol{\Phi}_j^{\mathrm{T}}}{m_j} \mathrm{e}^{\lambda_j \tau} = \sum_{j=1}^{2p} \mathbf{R}_j \mathrm{e}^{\lambda_j \tau} \tag{1.29}$$

m_j is the jth diagonal element of $\mathbf{M}_d$ and $\mathbf{R}_j$ is the residue matrix that corresponds to the jth eigenvalue.

The mode shapes are called "shapes" because they are unique in shape but not in value. That is, the mode shape vector for $\boldsymbol{\Phi}_j$ each mode j does not have unique values. The mode shape vector can be arbitrary scaled to any set of values, but relationship of one shape component to another is unique. In the system masses are known it is possible to scale the mode shapes so that the modal masses are unity. However, when the modal data is obtained from experimental spectral analyses (from experimental frequency transfer function measurements), no mass matrix is available for scaling. Without the mass matrix the experimental mode shapes can still be scaled to unit modal masses by using the relationship between residues and mode shapes:

$$\mathbf{R}_j = \alpha_j \boldsymbol{\Phi}_j \boldsymbol{\Phi}_j^{\mathrm{T}}, \qquad \alpha_j = \frac{1}{m_j \omega_j} \tag{1.30}$$

where α_j is a scaling constant for the jth mode. The relation between α_j and m_j is also shown in (1.30).

1.3.3 Spectral analysis of dynamic excited system

Stochastically excited system can be analysed in the frequency domain, if certain characteristics are satisfied. It is assumed that the system is linear and the applied excitation $\mathbf{f}(t)$ is a stationary zero mean Gaussian distributed stochastic process. In this case the response $\mathbf{z}(t)$ of the system is also a Gaussian distributed stochastic process. Since $\mathbf{f}(t)$ is assumed zero mean, it can be fully described by its correlation function $\boldsymbol{\Pi}_{ff}(\tau)$. The system is assumed linear so the response $\mathbf{z}(t)$ of the system is also described by its correlation function $\boldsymbol{\Pi}_{zz}(\tau)$.

By using (1.9) the auto spectral density functions $\mathbf{S}_{ff}(\omega)$ and $\mathbf{S}_{zz}(\omega)$ for $\mathbf{f}(t)$ and $\mathbf{z}(t)$ can be established. Note that ω is an arbitrary angular frequency. By introducing the frequency response function $\mathbf{H}(\omega)$ from (1.17) it is possible to describe $\mathbf{S}_{zz}(\omega)$ by means of $\mathbf{S}_{ff}(\omega)$ in the following way:

$$\mathbf{S}_{zz}(\omega) = \mathbf{H}(\omega)\mathbf{S}_{ff}(\omega)\mathbf{H}^{\mathrm{H}}(\omega) \tag{1.31}$$

where the superscript $^{\mathrm{H}}$ is the Hermitian conjugate (equal to complex conjugate and transpose). In (1.17) it is shown that the frequency response function $\mathbf{H}(\omega)$ is the Fourier transform of the unit impulse response function $\mathbf{h}(\tau)$. Using this relation means that (1.29) can be transformed into frequency domain and the frequency response function $\mathbf{H}(\omega)$ can be given as a partial fraction expansion:

$$\mathbf{H}(\omega) = \sum_{j=1}^{2p} \frac{\mathbf{R}_j}{\mathrm{i}\omega - \lambda_j} \tag{1.32}$$

where λ_j and $\mathbf{R}_j$ are the poles and residues of the partial fraction expansion, respectively. Suppose that the input $\mathbf{f}(t)$ is Gaussian white noise, then the auto spectral density function $\mathbf{S}_{ff}(\omega)$ is constant intensity matrix denoted by $\mathbf{F}$. The spectral density function $\mathbf{S}_{zz}(\omega)$ of the response $\mathbf{z}(t)$ of a Gaussian white noise excited second-order system is then given by:

$$\mathbf{S}_{zz}(\omega) = \mathbf{H}(\omega)\mathbf{F}\mathbf{H}^{\mathrm{H}}(\omega) = \sum_{j=1}^{2p}\sum_{k=1}^{2p} \frac{\mathbf{R}_j\mathbf{F}\mathbf{R}_j^{\mathrm{H}}}{(\mathrm{i}\omega - \lambda_j)(\mathrm{i}\omega - \lambda_k)} \tag{1.33}$$

1.4 System Identification

The system identification by ARTeMIS (SVS 2006) operates with two different identification techniques, one in time domain and one in frequency domain. The two models are:

- Frequency Domain Decomposition (FDD)
- Stochastic Subspace Iteration (SSI)

The models are described briefly in the following.

1.4.1 ID by Frequency Domain Decomposition (FDD)

The frequency domain estimation is a non-parametric model (also known as spectral models) based on a Frequency Domain Decomposition (FDD) method. The FDD method is an extension of the well-known frequency domain approach that is based on mode estimations directly from the Power Spectral Density (PSD) matrix, i.e. well separated modes can be identified at the peaks of the PSD matrix.

The basics of the identification algorithm are as follows. The estimate of power spectral density matrix $\hat{\mathbf{G}}_{yy}(f)$ is determined by means of signal processing of the measured accelerations. $\hat{\mathbf{G}}_{yy}(f)$ is a $N \times N$ matrix where N is the number of channels, known at discrete frequencies $f = f_i$. The estimate of power spectral density matrix $\hat{\mathbf{G}}_{yy}(f_i)$ is then decomposed by means of a Singular Value Decomposition (SVD) into a matrix of the form:

$$\hat{\mathbf{G}}_{yy}(f_i) = \mathbf{U}_i \mathbf{\Sigma}_i \mathbf{U}_i^{\mathrm{H}} \tag{1.34}$$

where $\mathbf{U}_i = [\mathbf{u}_{i1}, \mathbf{u}_{i2}, \ldots, \mathbf{u}_{iN}]$ is unitary matrix containing N singular vectors $\mathbf{u}_{ij}$. $\mathbf{\Sigma}_i$ is a diagonal matrix containing N scalar singular values σ_{ij}^2:

$$\mathbf{\Sigma}_i = \begin{bmatrix} \sigma_{i1}^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_{iN}^2 \end{bmatrix} \tag{1.35}$$

According to the theory of Frequency Domain Decomposition the first singular vector $\mathbf{u}_{i1}$ is an estimate of the mode shape $\hat{\mathbf{\Phi}}$ (Brincker et al. 2000; Brincker et al. 2000):

$$\hat{\mathbf{\Phi}} = \mathbf{u}_{i1} \tag{1.36}$$

The corresponding singular value σ_{iN}^2 is then part of a power spectral density function of an equivalent single degree of freedom (SDOF) system. The relation in (1.36) may not seem obvious, but it becomes clear when (1.34) is compared to (1.33). Remember that $\hat{\mathbf{G}}_{yy}(f_i)$ is a one-sided spectrum equal to $2\hat{\mathbf{S}}_{yy}(f_i)$. In (1.33) the spectrum is given in terms of residues, and these residues are again given by mode shapes. This means that the modes shapes are related to the singular vectors in (1.34).

The power spectral density function of the SDOF system is identified around a peak (mode k in Figure 1.11) by comparing the mode shape estimate $\hat{\mathbf{\Phi}}$ with singular vectors

for the frequencies around the mode. The comparison is done by means of a Modal Assurance Criterion (MAC):

$$\mathrm{MAC}(\boldsymbol{\Phi}_r, \boldsymbol{\Phi}_s) = \frac{|\boldsymbol{\Phi}_r^{\mathrm{H}} \boldsymbol{\Phi}_s|^2}{|\boldsymbol{\Phi}_r^{\mathrm{H}} \boldsymbol{\Phi}_r||\boldsymbol{\Phi}_s^{\mathrm{H}} \boldsymbol{\Phi}_s|} \tag{1.37}$$

The MAC value is the square of correlation between two modal vectors $\boldsymbol{\Phi}_r$ and $\boldsymbol{\Phi}_s$. If the MAC value is unity the two vectors are identical within a modal scale factor. Further information about modal indicators is given in Zhang et al. (2001).

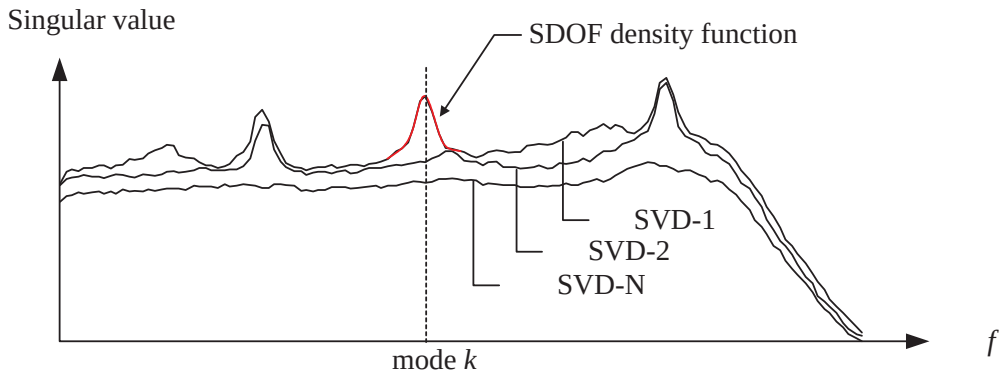

Figure 1.11: Plot of singular values of the power spectral density matrix as a function of frequency. The singular values around the kth mode of the system (structure) belong to the SDOF power spectral density function.

If the singular vectors for the frequencies around the peak have a high MAC value with respect to the mode shape estimate $\hat{\boldsymbol{\Phi}}$, the corresponding singular values belong to the SDOF density function. This is illustrated in Figure 1.11 where the red part of the power spectral density function is a SDOF density function.

When the SDOF power spectral density function has been estimated for a mode, the corresponding singular vectors are averaged together to obtain an improved estimate of the mode shape. The natural frequency and the damping ratio of the mode is estimated by transforming the SDOF Spectral Bell to time domain by inverse FFT. This results in a SDOF Correlation Function, and by simple regression analysis the estimates of both the natural frequency as well as the damping ratio can be obtained.

1.4.2 ID by Stochastic Subspace Iteration (SSI)

In the SSI techniques a parametric model is fitted directly to the raw time series data obtained from the accelerometers. The parametric models are characterized by the assumption of a mathematical model constructed from a set of parameters, where the mathematical model is a linear and time-invariant system of differential equations.

A dynamic structural model can be described by a set of linear second-order differential equations with constant coefficients as stated in the previous section. The model is reproduced here:

$$\mathbf{M}\ddot{\mathbf{z}}(t) + \mathbf{C}\dot{\mathbf{z}}(t) + \mathbf{K}\mathbf{z}(t) = \mathbf{f}(t) \tag{1.38}$$

$\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices, and $\mathbf{z}(t)$ and $\mathbf{f}(t)$ displacement and force vectors, respectively. (1.38) can be rewritten as a first-order system (rearrangement of (1.19)), given by:

$$\begin{aligned}
&\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x}(t) + \mathbf{B}_c\mathbf{u}(t) \qquad \mathbf{x}(t) = \begin{bmatrix}\mathbf{z}(t)\\ \dot{\mathbf{z}}(t)\end{bmatrix}\\
&\mathbf{A}_c = \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}\end{bmatrix}, \quad \mathbf{B}_c = \begin{bmatrix}\mathbf{0}\\ -\mathbf{M}^{-1}\mathbf{B}_2\end{bmatrix}, \quad \mathbf{f}(t) = \mathbf{B}_2\mathbf{u}(t)
\end{aligned} \tag{1.39}$$

Where $\mathbf{A}_c$ is the state matrix and $\mathbf{B}_c$ the system control influence coefficient matrix. Note that the excitation force $\mathbf{f}(t)$ is factorized into a matrix $\mathbf{B}_2$ describing the inputs in space and a vector $\mathbf{u}(t)$ describing the inputs in time.

In practice, not all the degrees of freedom are monitored. The measurements (accelerations, velocity or displacement) are evaluated at a subsystem of nodes (or locations). The observation equation for the measurements is given by:

$$\mathbf{y}(t) = \mathbf{C}_a\ddot{\mathbf{z}}(t) + \mathbf{C}_v\dot{\mathbf{z}}(t) + \mathbf{C}_d\mathbf{z}(t) \tag{1.40}$$

where $\mathbf{y}(t)$ corresponds to the output in the monitored subsystem. $\mathbf{C}_a$, $\mathbf{C}_v$ and $\mathbf{C}_d$ are the output matrices for acceleration, velocity and displacement, respectively. The output vector $\mathbf{y}(t)$ can be transformed into:

$$\begin{aligned}
&\mathbf{y}(t) = \mathbf{\Omega}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\\
&\mathbf{\Omega} = \begin{bmatrix}\mathbf{C}_d - \mathbf{C}_a\mathbf{M}^{-1}\mathbf{K} & \mathbf{C}_v - \mathbf{C}_a\mathbf{M}^{-1}\mathbf{C}\end{bmatrix}, \quad \mathbf{D} = \mathbf{C}_a\mathbf{M}^{-1}\mathbf{B}_2
\end{aligned} \tag{1.41}$$

where $\mathbf{\Omega}$ the output matrix and $\mathbf{D}$ is a direct transmission matrix. (1.39) and (1.41) constitute a continuous-time state-space model of a dynamic system. Since experimental data are discrete in nature the continuous system is reformulated into a discrete system. The measurements are then available at discrete time instances $k\Delta t$. The discrete state space model is then given by:

$$\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k\\
\mathbf{y}_k &= \mathbf{\Omega}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k
\end{aligned} \tag{1.42}$$

where $\mathbf{x}(k\Delta t)$ is the discrete time state vector, $\mathbf{A} = \exp(\mathbf{A}_c\Delta t)$ is the discrete state matrix and $\mathbf{B} = [\mathbf{A} - \mathbf{I}]\mathbf{A}_c^{-1}\mathbf{B}_c$ is the discrete input matrix. The equation in (1.42) forms the discrete-time state space model of a dynamic system. The model in (1.42) does not contain any uncertainties, such as process and measurement noise. There is always noise in practice, so the model in (1.42) is extended by including stochastic components. The noise is included by two components, $\mathbf{w}_k$ and $\mathbf{v}_k$, where $\mathbf{w}_k$ is process noise due to disturbances and modeling inaccuracies and $\mathbf{v}_k$ is measurement noise due to sensor

inaccuracy. When the stochastic components are included the following discrete-time state space model is obtained:

$$\begin{aligned}\mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{w}_k \\ \mathbf{y}_k &= \mathbf{\Omega}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k + \mathbf{v}_k\end{aligned} \tag{1.43}$$

In (1.43) it is assumed that the input $\mathbf{u}_k$ is known. This is not the case when the input is an unmeasurable stochastic process. In the case of such ambient vibrations it is impossible to distinguish the input term $\mathbf{u}_k$ from the noise terms $\mathbf{w}_k$ and $\mathbf{v}_k$. Modeling the input term $\mathbf{u}_k$ by the noise terms $\mathbf{w}_k$ and $\mathbf{v}_k$ results in a purely stochastic system:

$$\begin{aligned}\mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{w}_k \\ \mathbf{y}_k &= \mathbf{\Omega}\mathbf{x}_k + \mathbf{v}_k\end{aligned} \tag{1.44}$$

The equation in (1.44) constitutes the basis for the time-domain system identification technique, based on output only. The Stochastic Subspace Identification (SSI) technique is a class of techniques that are formulated and solved using the state space formulation in (1.44).

Principle of SSI solution
In order to solve (1.44), the system is reformulated. This includes three steps. First step is to determine $\mathbf{x}_k$. $\mathbf{x}_k$ is denoted as Kalman sequences that in SSI are found by means of a so-called orthogonal projection technique, see e.g. Van Overschee and De Moor (1996). Second step is to solve the regression problem for the matrices $\mathbf{A}$ and $\mathbf{\Omega}$ and for the residual noise components $\mathbf{w}_k$ and $\mathbf{v}_k$. The third step is to estimate a so-called Kalman gain matrix $\mathbf{K}_k$ so that the system can be written as a full covariance equivalent model:

$$\begin{aligned}\hat{\mathbf{x}}_{k+1} &= \mathbf{A}\hat{\mathbf{x}}_k + \mathbf{K}_k\mathbf{e}_k \\ \mathbf{y}_k &= \mathbf{\Omega}\hat{\mathbf{x}}_k + \mathbf{e}_k\end{aligned} \tag{1.45}$$

where $\mathbf{K}_k$ is the Kalman gain matrix, $\mathbf{e}_k$ is called the innovation and is a zero-mean Gaussian white noise process and $\hat{\mathbf{x}}_k$ is the predicted state vector. It can be shown that by performing a modal decomposition of the $\mathbf{A}$ matrix as $\mathbf{A} = \mathbf{V}[\mu_j]\mathbf{V}^{-1}$ and introducing a new state vector $\mathbf{z}_k$ the equation in (1.45) can also be written as:

$$\begin{aligned}\mathbf{z}_{k+1} &= [\mu_j]\mathbf{z}_k + \mathbf{\Psi}\mathbf{e}_k \\ \mathbf{y}_k &= \mathbf{\Phi}\mathbf{z}_k + \mathbf{e}_k\end{aligned} \tag{1.46}$$

where $[\mu_j]$ is a diagonal matrix containing the discrete eigenvalues. The natural frequencies f_j and damping ratios ζ_j are extracted using the following definition:

$$\mu_j = \exp\left(-2\pi f_j\left(\zeta_j \pm \sqrt{1-\zeta_j^2}\right)\Delta t\right) \tag{1.47}$$

where Δt is the sampling interval. The mode shape that are associated with the jth mode is given by the jth column of the matrix $\mathbf{\Phi}$. The last matrix $\mathbf{\Psi}$ that completes the modal decomposition contains a set of row vectors. The jth row vector corresponds to the jth mode. This vector distributes the white noise excitation $\mathbf{e}_k$ in modal domain to all the degrees of freedom.

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