On the one-dimensional steady and unsteady porous flow equations

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Abstract

Porous flow in coarse granular media is discussed theoretically with special concern given to the variation of the flow resistance with the porosity. For steady state flow, the Navier-Stokes equation is applied as a basis for the derivations. A turbulent flow equation is suggested. Alternative derivations based on dimensional analysis and a pipe analogy, respectively, are discussed. For non-steady state flow, the derivations are based on a cylinder/sphere analogy leading to a virtual mass coefficient. For the fully turbulent flow regime, existing experimental data values of the quadratic flow resistance coefficients are presented. Moreover, a simple formula for estimation of the turbulent flow coefficient is given. Virtual mass coefficients based on existing data are presented, however, no definite conclusions can be given due to the scarce data available.

1. Introduction

The physics of porous flow in coarse granular materials plays an important role in evaluation of scale effects in scale models and in formulation of numerical models for wave-rubble mound structure interactions.

With the purpose of providing insight into the nature of porous flow, different theoretical expressions are derived and discussed. For the sake of practical applications in connection with e.g. rubble mound structures, special emphasis is put on the variation of the flow resistance with the porosity, the gradation, the grain shape and the surface roughness.

Traditionally, the hydraulic radius concept, i.e. the ratio of the pore volume to the total surface area of the grains within a unit volume, has been applied in the description of steady porous flow. In the present paper, this line is followed in combination with alternative basic approaches: First, the Navier-Stokes equation is applied, secondly a dimensional analysis is carried out, and finally a cylinder/sphere analogy is established. A pipe analogy leading to a friction factor formulation is mentioned but not discussed further.
Before discussing any coefficients for the flow resistance, it is important to notice the various characters of the flow, i.e. creeping flow, laminar flow with non-linear convective inertia forces and fully turbulent flow. These flow regimes are referred to as the Darcy, the Forchheimer and the fully turbulent flow regimes, respectively. Darcy flow is not relevant for the case of coarser rubble mounds. Because no sharp limits exist, the separations between the flow regimes are quantified by Reynolds number ranges.

In order to describe the local acceleration and the associated virtual mass for the case of non-steady porous flow, it is necessary to distinguish between the volume of water in the porous matrix and the displaced volume of water, i.e. the volume of solids. In this paper, the virtual mass coefficient is related to the volume of the solids corresponding to the usual approach for calculation of flow forces on single bodies.

2. Historical background

2.1. Stationary flow

In 1856 D’Arcy described empirically the relation between the hydraulic gradient, \( I \), and the discharge velocity, \( V \), through porous sands and sandstones as

\[ V = K I \]  

which may also be written as

\[ I = K^{-1} V \]  

where \( K \) is the permeability coefficient (m/s).

These expressions, which have later been referred to as the Darcy law, apply to the laminar flow case without convective inertia forces, i.e. creeping flow. As a heuristic correlation Forchheimer in 1901 extended this expression with a quadratic term,

\[ I = aV + b|V|^2 \]  

where \( a \) and \( b \) are coefficients.

The Forchheimer expression applies to laminar flow regimes in which convective inertia forces and to some extent also a minor proportion of turbulence are present.

In order to describe the coefficients \( a \) and \( b \), various analogies have been applied. The Darcy law compares to the Hagen-Poiseuille equation, see e.g. Scheidegger (1974), which is an exact solution to the Navier-Stokes equation for laminar flow in a number of parallel circular capillaries with diameter \( d_c \), i.e. without convective accelerations. This yields a parabolic velocity distribution, and the relation between the hydraulic gradient and the average velocity reads

\[ I = \frac{32}{g d_c^2} V \]  

where \( v \) is the kinematic viscosity.

The Darcy law is often written in the form
where $K_s$ is the specific or intrinsic permeability (m²).

Kozeny in 1927 applied a pipe analogy based on the pore velocity, $V/n$, $n$ being the porosity defined as the fluid volume divided by the total volume, and a hydraulic radius, $R$ was defined as the ratio between the pore volume and the wet surface, i.e. the surface of the grains. Carman (1937) expressed the surface of the grains as a function of the pore geometry assuming spherical obstacles with diameter $d$. These assumptions together lead to

$$I = \frac{\nu}{gK_s} V$$

Comparing to experiments Carman found $\kappa = 5$. Eq. (6) has later been referred to as the Kozeny–Carman expression.

Also in the Forchheimer flow regime both terms can be expressed as function of the pore geometry. Ergun (1952) kept the Kozeny–Carman expression for the linear flow resistance, although with a new constant applicable in the Forchheimer regime. For the quadratic term he applied a pipe analogy with hydraulic radius, $R$, valid for Reynolds numbers ranging between approximately 1 and 1300 and arrived at the expression

$$I = 36\kappa \left(\frac{1-n}{n^3}\right)^2 \frac{\nu}{gd^2} V$$

Ergun verified by experiments the expressions involving $n$, cf. Fig. 3 and the related discussion presented later in the paper.

Engelund (1953) also applied a pipe analogy with hydraulic radius, $R$, for the quadratic term, but for the linear term he fitted an expression of the type

$$I = 150 \left(\frac{1-n}{n^3}\right)^2 \frac{\nu}{gd^2} V + 1.75 \left(\frac{1-n}{n^3}\right) \frac{1}{gd} |V|V$$

Engelund (1953) also applied a pipe analogy with hydraulic radius, $R$, for the quadratic term, but for the linear term he fitted an expression of the type

$$I = \alpha_E \left(\frac{1-n}{n^2}\right)^3 \frac{\nu}{gd^2} V$$

to experiments with porous flow in sand.

Dybbbs and Edwards (1984) presented the results of physical model tests with flow in porous structures. The porous structures consisted of plexiglass spheres in a hexagonal packing and glass and plexiglass rods arranged in a complex, fixed three-dimensional geometry. Both water and oil were used for the testing and laser anemometry and flow visualisation were applied as basis for flow regime characterisation.

Four flow regimes were identified, the description of which is quoted directly from the above reference:

1) The Darcy or creeping flow regime where the flow is dominated by viscous forces and the exact nature of the velocity distribution is determined by local geometry. This type of flow occurs at $Re < 1$. At $Re = 1$, boundary layers begin to develop near the solid boundaries of the pores.

2) The inertial flow regime. This initiates at $Re$ between 1 and 10 where the boundary layers become more pronounced and an ‘inertial core’ appears. The developing of these ‘core’ flows outside the boundary layers is the reason for the non-linear relationship between
pressure drop and flow rate. As the Re increases, the 'core' flows enlarge in size and their influence becomes more and more significant on the overall flow picture. This steady non-linear laminar flow regime persists to a $Re \sim 150$.

(3) An unsteady laminar flow regime in the Reynolds number range of 150 to 300. At a $Re \sim 150$, the first evidence of unsteady flow is observed in the form of laminar wake oscillations in the pores. These oscillations take the form of travelling waves characterised by distinct periods, amplitudes and growth rates. In this flow regime, these oscillations exhibit preferred frequencies that seem to correspond to specific growth rates. Vortices form at $Re \sim 250$ and persist to $Re \sim 300$.

(4) A highly unsteady and chaotic flow regime for $Re > 300$, qualitatively resembling turbulent flow."

In the following, the flow regimes will be denoted as follows:

1. The Darcy flow regime
2. The Forchheimer flow regime
3. The fully turbulent (rough turbulent) flow regime

Flow regime (3) is a transitional flow regime between the Forchheimer and the fully turbulent flow regimes.

Fand et al. (1987) examined experimentally the variation of the $a$ and $b$ coefficients in Eq. (3) with the Reynolds number. From the test results, it appears that three different flow regimes can be identified: the Darcy, the Forchheimer and the fully turbulent flow regimes. For the variation of the $a$ and $b$ terms with the pore geometry, they applied the expressions given by Ergun (1952) with new constants specific for each flow regime.

Some authors, e.g. Ward (1964), have established expressions for both terms in the Forchheimer flow regime with no direct dependency on neither the porosity nor the grain diameter. From dimensional analysis and fitting to experiments he found

$$ I = \frac{\nu}{gK_s} V + \frac{c}{g\sqrt{K_s}} |V|V $$

(9)

where $K_s$ is the specific permeability (m$^2$) and $c$ is a dimensionless constant which has the same value 0.550 for all porous media. The first term compares to the Darcy law, Eq. (5). Some authors apply Eq. (9) also in the turbulent flow regime with a lower value of $c$ than given by Ward. As also the linear term is different from the Forchheimer to the turbulent flow regime, a factor should in fact be applied to the constant "1" as well.

Several authors have proposed expressions for the variation of $a$ and $b$ with the porosity. Some of these are shown in Table 1, in which the Kozeny–Carman expression refers to the Darcy flow regime, and for the other expressions the constants refer to the Forchheimer flow regime.

The steady flow resistance can be derived from a pipe analogy, cf. Burchart and Christiansen (1991):

$$ I_{\text{steady}} = 3f \left( \frac{1-n}{n^3} \frac{1}{gd} \right) |V|V $$

(10)

From basic hydrodynamics, it can be deduced that the friction factor $f$ varies with the Reynolds number, the gradation and grain shape as well as the relative surface roughness.
Table 1  
Expressions for the a and b coefficients in Eq (3)  

<table>
<thead>
<tr>
<th>Authors</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kozeny (1927) (see Carman, 1937)</td>
<td>[36\kappa \frac{(1-n)^2 \nu}{n \gamma d^2}, \kappa = 5]</td>
<td></td>
</tr>
<tr>
<td>Ergun (1952)</td>
<td>[150 \frac{(1-n)^2 \nu}{n \gamma d^2}, \kappa = 1.75 \frac{1-n}{n \gamma d} ]</td>
<td></td>
</tr>
<tr>
<td>Engelund (1953)</td>
<td>[\alpha_0 \frac{(1-n)^3 \nu}{n^2 \gamma d}, \beta ]</td>
<td></td>
</tr>
<tr>
<td>Ward (1964)</td>
<td>[\frac{\nu}{gK_s}, \frac{c}{g \sqrt{K_s}}, c = 0.550]</td>
<td></td>
</tr>
</tbody>
</table>

of the grains in the rough turbulent flow region. The friction factor approach has been dealt with by several researchers, see e.g. Ergun (1952), but is not discussed further in this paper. Reference is made to Hannoura and Barends (1981).

2.2. Non-stationary flow

For the case of non-stationary flow in coarse granular media, the Forchheimer expression can be extended with an inertia term

\[ I = aV + b|V| \frac{\partial V}{\partial t} + c \frac{\partial V}{\partial t} \] (11)

This was originally suggested by Polubarinova-Kochina in 1952. Like the stationary terms, the inertia term varies with the porosity. In case of non-uniform flow, macroscopic convective accelerations are present in addition to the local acceleration, \( \partial V/\partial t \). For a breakwater, the boundary conditions for the porous medium are the finite extension of the waves, the free surface and the impervious bottom. Due to the non-linear interaction between the grains in the porous medium in combination with the boundary conditions, the inertia coefficient, \( c \), defined from Eq. (11) cannot in general be applied to the convective accelerations, i.e. the local derivative, \( \partial V/\partial t \), in Eq. (11) cannot in general be substituted by the total derivative, \( dV/dt \). This is in contradiction to single obstacles exposed to a non-steady and non-uniform flow, where it is sometimes possible to merge the local and convective accelerations into one inertia term, i.e. the coefficients are identical, see Sarpkaya and Isaacson (1981). Considering porous flow, it is thus necessary to treat the local and the macroscopic convective accelerations separately. As the local accelerations are usually dominating over the convective accelerations and as the local accelerations are much easier to deal with than the total accelerations in theory, \( \partial V/\partial t \) is applied in the definition of \( c \), cf. Eq. (11). Hence, in case of non-uniform flow, the macroscopic convective accelerations must be added as a separate term in supplement to the inertia term in Eq. (11). Usually, a quadratic term of the type \( (1/g)(V/n)(\partial /\partial x(V/n)) \) is applied. The procedure described above is applied in most literature. It should be noticed that for a typical breakwater, the quadratic term in Eq. (11) is dominating over the inertia term in Eq. (11), which is dominating over the gradient associated with the macroscopic convective accelerations.
Sollitt and Cross (1972) considered a condition for a fluid element together with the pore velocity as characteristic velocity. In addition to the acceleration of the fluid element, they added a virtual mass term related to the volume of solids

$$C_m (1 - n) \frac{\partial}{\partial t} \left( \frac{V}{n} \right)$$  \hspace{1cm} (12)

which was then, according to the authors, distributed over the volume of water and considered as an extra force acting on the fluid element. Sollitt and Cross explain it as follows: "The resistance force due to the virtual mass is equal to the product of the displaced fluid mass, the virtual mass coefficient, and the acceleration in the approach velocity. The resulting force is distributed over the fluid mass within the pore so that the force per unit mass of fluid is simply

$$C_m \frac{1 - n}{n} \frac{\partial}{\partial t} \left( \frac{V}{n} \right),$$  \hspace{1cm} (13)

The entire force balance then reads

$$- \frac{1}{\rho g} \frac{\partial p}{\partial x} = \frac{1}{g} \frac{\partial}{\partial t} \left( \frac{V}{n} \right) + \frac{1 - n}{g} C_m \frac{1 - n}{n} \frac{\partial}{\partial t} \left( \frac{V}{n} \right) = \frac{1 + C_m^s}{ng} \frac{1 - n}{n} \frac{\partial V}{\partial t}$$ \hspace{1cm} (14)

However, the applied procedure seems unclear and the result not applicable as the pressure gradient does not only act on the fluid element but on the entire sample of water and solids.

Madsen (1974) and Hannoura and McCorquodale (1978) considered a unit volume of the sample and related the inertia term both to the volume of water and the volume of solids. As characteristic velocity they applied the pore velocity and obtained

$$- \frac{\partial p}{\partial x} = \rho n \frac{\partial}{\partial t} \left( \frac{V}{n} \right) + \rho C_m^s (1 - n) \frac{\partial}{\partial t} \left( \frac{V}{n} \right)$$ \hspace{1cm} (15)

leading to

$$- \frac{1}{\rho g} \frac{\partial p}{\partial x} = \frac{1 + C_m^s}{g} \frac{1 - n}{n} \frac{\partial V}{\partial t}$$ \hspace{1cm} (16)

$C_m^s$ is the virtual mass coefficient. Eq. (16) is general as no assumptions are made about the geometry of the porous medium, cf. the cylinder analogy presented later in the paper which, based on the pore velocity, leads to the same inertia term as in Eq. (16).

Wang and Gu (1988) also related the inertia term to the volume of water and the volume of solids and used the pore velocity as characteristic velocity. For the pressure, however, they applied only the part acting on the relative area of the pores in a cross-section, $n_a$, i.e.

$$n_a \left( - \frac{\partial p}{\partial x} \right) = \rho n \frac{\partial}{\partial t} \left( \frac{V}{n} \right) + \rho C_m^w (1 - n) \frac{\partial}{\partial t} \left( \frac{V}{n} \right)$$ \hspace{1cm} (17)

$n_a$ was assumed (correctly) equal to $n$ (see e.g. Underwood, 1970), resulting in an expression equal to the expression of Sollitt and Cross (1972)
Van Gent (1992) derived an expression for the inertia term from a condition for a fluid element. In his derivation, as done by Sollitt and Cross (1972), the pressure forces related to the volume of solids are transformed to the volume of water. This leads to:

\[
-\frac{1}{\rho g} \frac{\partial p}{\partial x} = \frac{1 + C_m^w \frac{1-n}{n}}{ng} \frac{\partial V}{\partial t}
\]

(18)

As discussed later in the present paper, only the formulation by Madsen (1974) and Hannoura and McCorquodale (1978) is considered to be correct.

3. One-dimensional steady flow equation

3.1. Considerations based on the Navier-Stokes equation

The kinematic and dynamic conditions for a fluid element in laminar porous flow can in principle be described by the Navier-Stokes equation

\[
\frac{\partial v_i}{\partial t} + \frac{\partial v_j v_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\partial^2 v_i}{\partial x_i \partial x_j} + g_i + \nu \frac{\partial^2 v_i}{\partial x_i \partial x_j}
\]

(20)

with the appropriate boundary conditions along the grain surfaces and the boundary of the space in question. \( t \) and \( x \) are the independent time and space variables, respectively. \( v \) is the velocity, \( p \) is the pressure, \( \rho \) is the density, \( \nu \) is the viscosity and \( g \) is the acceleration of gravity.

Introducing the hydraulic pressure gradient

\[
I_i = -\frac{1}{\rho g} \frac{\partial p}{\partial x_i}
\]

(21)

and considering only closed conduit flow, we obtain

\[
I_i = -\nu \frac{\partial^2 v_i}{\partial x^2} + \frac{1}{g} \frac{\partial v_i}{\partial x_j} v_j + \frac{1}{g} \frac{\partial v_i}{\partial t}
\]

(22)

For the one-dimensional (1D) stationary case, Eq. (22) simplifies to

\[
I = -\nu \frac{\partial^2 v_1}{\partial x^2} + \frac{\nu}{g} \frac{\partial^2 v_1}{\partial x^2} + \frac{1}{g} \frac{\partial v_1}{\partial x_i}
\]

(23)

Introducing \( U \) and \( D \) as any characteristic velocity and length, respectively, Eq. (23) can be written in the dimensionally correct form


\[ I = \frac{\nu}{g} \frac{U}{D^2} + \frac{1}{g} \frac{U^2}{D} \tag{24} \]

or

\[ I = AU + BU^2 \tag{25} \]

Eq. (25) is identical to Eq. (3), the Forchheimer equation. The coefficients \( A \) and \( B \) (or \( \alpha \) and \( \beta \)) are often taken as constants for a given fluid viscosity and a given geometry of the porous structure. This, however, is not a correct assumption because the coefficients depend on the kinematics of the flow including curvature of the flow paths, cf. the discussion related to the historical information.

The various flow domains are usually characterised by a Reynolds number, \( \text{Re} \). In case of "creeping flow", in which the velocities are very small, the convective inertia term can be neglected, and we obtain the solution

\[ I = \frac{\nu}{gD^2} U = A''U \tag{26} \]

which is well-known as the "Darcy" equation, cf. Eq. (4). Creeping flow is as mentioned earlier not relevant for the case of coarser rubble mounds.

If the velocities are larger, but the flow still stationary and laminar, then curvatures (perturbations) of the flow paths introduce additional pressure drop which is described by the non-linear convective inertia term. For such conditions, the flow can be described by Eqs. (24) and (25).

For large velocities, turbulence will occur. Also turbulent porous flow can in principle be described by Eq. (20) with appropriate boundary conditions. The inertia terms will for fully turbulent (rough turbulent) flow completely dominate over the viscous term, and we obtain an equation of the form

\[ I = \beta \frac{1}{g} \frac{U^2}{D} = B'U^2 \tag{27} \]

If for fully turbulent flow an equation of the form (24) or (25) is used, it is important to notice that the linear term is only a fitting term which has no physical meaning if we assume viscous forces to be negligible.

The Navier-Stokes equation (20) is never used for solving turbulent flow problems because the complexity of the flow makes it impossible. Instead, Eq. (20) is reformulated by introducing velocity mean values and velocity fluctuations. The effect of the latter is the so-called Reynolds stresses signified by an extra term in Eq. (20), arising from the convective acceleration term:

\[
\frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j} v_j = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + g_i + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{1}{\rho} \left( -u_i u_j \right) \tag{28}
\]

where \( p \) and \( v_i \) now represent time averages, and \( u_i \) and \( u_j \) are the velocity fluctuations.

This re-formulated equation is known as the Reynolds equation. Written in the form of Eq. (23) for the one-dimensional stationary closed conduit case, Eq. (28) yields:
Assuming the velocity fluctuations to vary proportionally to the velocity time average, represented by the characteristic value $U$, and taking $D$ as a characteristic length scale, very naturally the Reynolds stress term takes the same form as the convective term, and hence they can be merged together into one term, cf. Eq. (27). It should be noticed that it is not necessary to apply the Reynolds equation, as the Navier-Stokes equation leads to (24) and (25) and by neglecting the viscous term, also to (27). The above considerations on the Reynolds equation are included in order to demonstrate the relation to turbulent flow problems which are solved from the Reynolds equation.

The coefficients $a''$, $\alpha$, $\beta$ and $\beta'$ (and $A''$, $A$, $B$ and $B'$) depend on the flow regime characterised by $Re$. In principle, Eqs. (26) and (27) represent two asymptotic expansions for very small and very large $Re$, respectively.

For smooth spheres with uniform diameter, the transition zones between the different flow regimes are expected to be narrow and easy to identify. For irregular and graded materials, the transition zones are likely to be blurred and difficult to identify. Burcharth and Christensen (1991) use as a practical engineering approach a separation into a Forchheimer flow regime, given by Eq. (24) and a turbulent flow regime, given by Eq. (27), within each of which the coefficients can be taken as constants with good accuracy, cf. Fig. 1 and the following section. Experimental evidence for Fig. 1 is given in Fand et al. (1987).

If in Eq. (24) $U$ is substituted by $V/n$, where $V$ is the discharge velocity, $n$ is the porosity, and $D$ is substituted by a hydraulic radius, $R$, defined as the ratio of pore volume over pore surface area, i.e. $R = n/1 - n \cdot d/6$ for spheres with diameter $d$, we obtain
\[ I = \alpha \left( \frac{1-n}{n} \right)^2 \frac{V}{\text{gd}^2 n} + \beta \frac{1-n}{n} \frac{1}{\text{gd}^2 n} \left( \frac{V}{n} \right)^2 \]  

(30)

where \( \alpha \) depends on \( Re \), the gradation and the grain shape, and \( \beta \) depends on the same parameters plus the relative surface roughness of the grains.

Irmay (1958) considered the derivation of the Darcy and Forchheimer equations from Navier-Stokes’ equations. The detailed microscopic flow pattern including separation was discussed. After averaging over the sample, this lead to the same expression as Eq. (30). In Irmay’s derivation, however, the expression for the hydraulic gradient also included the gradient in \( \rho V^2/2 \). In the one-dimensional case, the gradient in the velocity squared vanishes, and the difference in the definition of the hydraulic gradient has no influence. Further, Irmay regarded the local acceleration term negligible.

3.2. Turbulent flow equation

It follows from the previous considerations that for fully turbulent flow, it is not correct to use a series expression consisting of both a linear term and a quadratic term in the sense that the linear term is insignificant, although this is generally the conventional approach, cf. Fig. 1. A more correct approach is depicted in Fig. 2.

\( Re_c \) is in principle the critical Reynolds number signifying a lower value for the turbulent flow regime, and \( V_c \) is the corresponding discharge velocity. According to Fand et al. (1987), the Reynolds number range for the transition between the Forchheimer flow and the turbulent flow is rather narrow, \( 80 \leq Re \leq 120 \) for randomly packed spherical particles. For this case, it can be assumed as a close approximation that \( Re_c = 100 \) separates the Forchheimer flow range and the turbulent range. For stone samples, the corresponding Reynolds number ranges are wider and a larger value of \( Re_c \) must be chosen. A re-analysis of available data

![Fig. 2. Representation of the turbulent flow equation. From Burcharth and Christensen (1991).](image)
Table 2
Engelund (1953) coefficients transformed to fit Eq (30). From Burcharth and Christensen (1991)

<table>
<thead>
<tr>
<th>Grain Type</th>
<th>α</th>
<th>β</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform, spherical particles</td>
<td>~190</td>
<td>~1.8</td>
</tr>
<tr>
<td>Uniform, rounded sand grains</td>
<td>~240</td>
<td>~2.8</td>
</tr>
<tr>
<td>Irregular, angular grains</td>
<td>up to 360 or larger</td>
<td>up to 3.6 or larger</td>
</tr>
</tbody>
</table>

on porous flow in coarse granular media has shown that \( Re_c = 300 \) is a characteristic value, see Burcharth and Christensen (1991).

The turbulent flow equation (Burcharth and Christensen, 1991) is given by

\[
I = I_c + b(V - V_c)^2
\]

where \( b = \beta' \left[ (1 - n)/n^3 \right] (1/gd) \), cf. Eqs. (27) and (30). \( I_c \) can be calculated from the Forchheimer flow equation with \( V = V_c \) equal to

\[
V_c = \frac{Re_c \nu}{d}
\]

Inserting Eq. (32) into Eq. (30), we obtain

\[
I_c = Re_c \alpha \left( \frac{1 - n}{n^3} \right) \frac{\nu^2}{gd^3} + Re_c \beta \frac{1 - n}{n^3} \frac{\nu^2}{gd^3}
\]

or

\[
I_c = \frac{\nu^2}{gd^3} \frac{1 - n}{n^3} \left[ \alpha(1 - n)Re_c + \beta Re_c^2 \right]
\]

where \( \alpha \) and \( \beta \) correspond to the Forchheimer flow range and to the related definition of the characteristic diameter, \( d \), cf. Table 2.

In order to evaluate Eq. (32), Table 3 shows typical values of \( I_c \) and the related \( V_c \) calculated for various characteristic grain diameters using \( \nu = 1.14 \times 10^{-6} \text{ m}^2/\text{s}, Re_c = 300 \), and \( \alpha = 500, \beta = 5.0 \) and \( n = 0.45 \), which may be taken as values for irregular, angular grains, cf. Table 2.

It is seen from Table 3 that for all breakwaters with core material of quarry run (\( d > 0.03 \text{ m} \)) or coarser material, \( I_c \) will be smaller than approximately \( 10^{-2} \) and the corresponding

<table>
<thead>
<tr>
<th>Characteristic diameter, ( d ) (m)</th>
<th>( I_c )</th>
<th>( V_c ) (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>430</td>
<td>0.34</td>
</tr>
<tr>
<td>0.01</td>
<td>( 43 \times 10^{-2} )</td>
<td>0.034</td>
</tr>
<tr>
<td>0.03</td>
<td>( 1.6 \times 10^{-2} )</td>
<td>0.011</td>
</tr>
<tr>
<td>0.06</td>
<td>( 2.0 \times 10^{-3} )</td>
<td>0.006</td>
</tr>
<tr>
<td>0.20</td>
<td>( 5.3 \times 10^{-5} )</td>
<td>0.002</td>
</tr>
</tbody>
</table>
critical bulk velocity, $V_c$, smaller than approximately $10^{-2}$ m/s. In this case, $I_c$ and $V_c \approx 0$ and Eq. (31) reduces to

$$I = \beta' \frac{1 - n}{n^3} \frac{V^2}{gd}$$

(35)

where $\beta'$ depends on the relative surface roughness of the grains and the grading.

For the quasi-steady flow in breakwater sand cores, the viscous effects will be present and consequently the Forchheimer equation (30) with the $\alpha$ and the $\beta$ values given in Table 2 might be used. The very large $I_c$ value of 430 given in Table 3 for sand with $d = 0.001$ m indicates that fully turbulent flow in sand will never occur in a breakwater situation. Even related to permeameter tests, such a large hydraulic gradient is extreme.

The ratio between the linear and the quadratic terms in Eq. (33) for the transition between Forchheimer flow and fully turbulent flow is, cf. also the Engelund Reynolds number equation, $\xi = \frac{bV}{a}$:

$$\xi = \frac{\beta Re_c}{\alpha (1-n)} \approx 5 \text{ for } Re_c = 300$$

(36)

A somewhat larger value of $\xi$ was expected. This points towards a larger value of $Re_c$, i.e. $Re_c \approx 600-1000$. However, if this is the case, then it can be concluded that the empirically determined $\alpha$ and $\beta$ values by Engelund and other researchers dealing with sand size grains and $Re < 150$ have not been fitted to results covering the whole regime from Darcy flow to fully turbulent flow. Consequently, it is doubtful if the reported small grain $\beta$-values by Engelund and others can be taken as the asymptotic values for turbulent flow regime. Instead, it is recommended that $\beta$-values determined from experiments with fully turbulent flow are used.

As to the $\alpha$-value of 190 for uniform spherical particles reported by Engelund, cf. Table 2, it represents truly the lower asymptotic value for the Forchheimer flow regime because it is quantitatively identical to the uniform diameter sphere coefficient, $36\kappa = 36 \cdot 5.34 = 192$, given in the Darcy flow equation

$$I = 36\kappa \frac{(1 - n)^2}{n^3} \frac{\nu}{gd^2} V$$

(37)

where $\kappa$ is the Kozeny–Carman constant, see Fand et al. (1987).

3.3. Dimensional analysis

Burcharth and Christensen (1991) applied a dimensional analysis in order to obtain expressions for $A$ and $B$ in Eq. (25). A very similar derivation is given here:

$$I = I \left( \frac{V}{g}, \nu, g, R, \text{ geometry} \right)$$

(38)

where the hydraulic radius $R$ is taken as

$$\frac{n \frac{d}{1 - n}}{6}$$
and the geometry is characterised partly by the surface roughness, \( k \), of the grains and partly by a shape-gradation parameter, \( G \).

Dimensional analysis yields the gradient \( I \) to be a function of a linear term \( I_1 \) dependent on the viscosity, and a quadratic term, \( I_2 \), i.e. \( I = I_1 + I_2 \) where

\[
I_1 = I_1 \left[ \left( \frac{1-n}{n} \right)^2 \nu \frac{V}{gd^2 n} G \right]
\]

\[
I_2 = I_2 \left[ \frac{1-n}{n} \frac{1}{gd} \left( \frac{V^2}{n} \right)^{\frac{1-n}{n}} \left( \frac{1-n}{n} k \right)^N \right]
\]

for all \( N \)

If the assumption is made that \( I \) equals the sum of \( I_1 \) and \( I_2 \), we obtain

\[
I = I_1 + I_2 = I_1 \left[ \left( \frac{1-n}{n} \right)^2 \nu \frac{V}{gd^2 n} G \right] + I_2 \left[ \frac{1-n}{n} \frac{1}{gd} \left( \frac{V^2}{n} \right)^{\frac{1-n}{n}} \left( \frac{1-n}{n} k \right)^N \right]
\]

for all \( N \)

For the case of non-linear laminar flow, as it is found in the lower end of the Forchheimer regime, the gradient is not dependent on \( k \), and hence, \( N = 0 \) yielding the same expression as (30)

\[
I = I_1 + I_2 = \alpha \left( \frac{1-n}{n} \right)^2 \nu \frac{V}{gd^2 n} + \beta \left( \frac{1-n}{n} k \right)^2 \left( \frac{V^2}{n} \right)
\]

where \( \alpha \) and \( \beta \) both depend on the gradation and grain shape.

In the upper end of the Forchheimer regime, there are three contributions to the flow resistance: viscous, inertia and turbulent forces. Consequently, the gradient also depends on \( k \), which can be retained if for instance in Eq. (41) \( N \) is set equal to 1

\[
I = I_1 + I_2 = \alpha \left( \frac{1-n}{n} \right)^2 \nu \frac{V}{gd^2 n} + \beta^* \left( \frac{1-n}{n} k \right)^2 \left( \frac{V^2}{n} \right)
\]

or

\[
I = \frac{1}{gd^2} \left( \frac{1-n}{n} \right)^2 \left[ \alpha \nu \left( \frac{V}{n} \right) + \beta^* \left( \frac{V}{n} \right)^2 \right]
\]

where \( \alpha \) and \( \beta^* \) both depend on the gradation and grain shape.

The formulation Eq. (43) makes it necessary to define and quantify the surface roughness parameter, \( k \). This is not easy, because when dealing with real samples of stones, it is impossible to vary the various geometrical parameters independently over significant ranges. For this reason, the formulation given by Eq. (42) is preferred for the whole Forchheimer regime, in which case the influence of \( k \) must be included in the \( \beta \) values.

The expressions (39) and (40) leading e.g. to Eqs. (42), (43) and (44) are not unique. Any dimensionless factor, like \( n \), can be applied to the parameters in (38) without violating the dimensional analysis. It will still give combined parameters which in a mathematical sense are correct, but not necessarily physically the most relevant or meaningful. It is difficult from theory to verify which is the best formulation of the factor \( B \) with respect to the
dependency on the porosity. The suggestions given in Eqs. (42) and (43) might be equally consistent within the relevant parameter ranges. One effect which is not implemented in the equations is a possible change of $\beta$ and $\beta^*$ with $n$. When $n$ is increased without changing the shape of the grains, the pores and the flow become less tortuous leading to a relative reduction in the non-linear term, i.e. a reduction in $\beta$ or $\beta^*$. However, this effect is probably less significant within the practical rather narrow range of $n$.

Finally, for the case of fully turbulent (rough turbulent) flow, the viscous term vanishes. For $N=0$, we obtain

$$I = I_2 = \beta' \left( 1 - \frac{n}{3} \right) \frac{1}{\frac{1}{g} d \left( \frac{1}{n} \right)}$$

which is identical to the non-linear term in Eq. (30). If, for instance, $N = 1$, we get:

$$I = I_2 = \beta'^* \left( \frac{1}{n} \right) \frac{1}{\frac{1}{g} d \left( \frac{1}{n} \right)}$$

Both $\beta'$ and $\beta'^*$ depend on the gradation and grain shape. Additionally, $\beta'$ will depend on the relative roughness $k/d$.

As in the case for formulations Eq. (42) and Eq. (43) belonging to the Forchheimer regime, it is difficult from theory to verify which of the formulations Eq. (45) or Eq. (46) is the most correct with respect to the influence of $n$.

### 3.4. Experimental verification of the dependence of the Forchheimer coefficients on the porosity

Ergun's equation (7) follows the general Forchheimer equation form, Eq. (3), in which the coefficients are denoted $a$ and $b$. In order to verify the variation of $a$ and $b$ with the porosity, Ergun performed experiments with porous gas flow in the Forchheimer regime. Crushed porous material was packed with different porosities, ranging between 0.44 and 0.8.

![Graphs showing the dependence of $a$ and $b$ on porosity ($n$).](image)

Fig. 3. Dependence of $a$ and $b$ on porosity ($n$). From Ergun (1952).
0.53. It appeared that the variation with the porosity conforms to Eqs. (7) and (42), cf. Fig. 3.

3.5. *Experimental values of α′ and β′ coefficients for fully turbulent flow*

The data in Table 4, showing characteristic values of β′ to be applied in Eq. (45), have been corrected for wall effects where it was necessary, cf. Burcharth and Christensen (1991). All data cover the fully turbulent flow regime and were fitted to the Forchheimer equation (30) which involve determination of both an α′ and a β′ value. The data for rock from Williams (1992) are related to the equivalent spherical diameter. The original data from Williams were related to the nominal diameter (equivalent cube length). It is uncertain which reference diameter has been applied for the data from Hannoura and McCorquodale (1978). The other data for rock are related to either the equivalent spherical diameter or the sieve diameter, which based on experience, are approximately identical. For the tests of

<table>
<thead>
<tr>
<th>Material</th>
<th>Packing</th>
<th>$d_{95}/d_{15}$</th>
<th>α′</th>
<th>β′</th>
<th>Re</th>
<th>Data source *</th>
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</thead>
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<tr>
<td>Spheres</td>
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<td>900–6000</td>
<td>1.0–1.3</td>
<td>630–14 000</td>
<td>Sm</td>
</tr>
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<td>640–900</td>
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<td>1.1</td>
<td>630–14 000</td>
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<td>410–1700</td>
<td>1.1–1.5</td>
<td>180–9000</td>
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<td>3700–7700</td>
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<td>~10 000</td>
<td>2.2</td>
<td>&lt;2100–8050</td>
<td>B</td>
</tr>
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<td></td>
<td></td>
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<td>1400–15 000</td>
<td>2.2–2.9</td>
<td>500–3600</td>
<td>D</td>
</tr>
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<td>1.7–2.2</td>
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<td></td>
<td></td>
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<td>?</td>
<td>2.4</td>
<td>750–7500</td>
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<td>Irregular rock</td>
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<td>1400–13 000</td>
<td>2.4–3.0</td>
<td>600–10 300</td>
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<td>300–5700</td>
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<td></td>
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<td>?</td>
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<td>W</td>
</tr>
<tr>
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<td>Sm</td>
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<td></td>
<td></td>
<td>1.2</td>
<td>?</td>
<td>3.7</td>
<td>750–7500</td>
<td>W</td>
</tr>
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</table>

Hannoura and McCorquodale, the direction between the flow and the underlayer during construction of the sample is not known. In the tests of Smith, the flow was parallel to the underlayer during construction of the sample, and in all other tests the flow was perpendicular to the underlayer during construction.

Solvik and Svee (1976) found the following values of $\beta'$ related to a reference diameter equal to $1.7d_{10}$:

- crushed stones, a little rounded: $\beta' = 3.1$
- crushed stones, sharp edged: $\beta' = 3.6$

The direction between the flow and the underlayer during construction is not known.

Andersen et al. (1993) carried out experiments with cylinders, spheres and rock in steady and oscillatory flow, cf. the following section on unsteady flow. As the $\beta'$ values are lower than found in most literature and as experimental problems occurred with the sphere and rock samples, these $\beta'$ values are not included in Table 4.

In order to evaluate the relative importance, the ratio between the quadratic and the linear flow resistance terms in the fully turbulent flow regime is calculated according to Eq. (36) with $\beta'$ and $\alpha'$ inserted instead of $\beta$ and $\alpha$. Taking $\beta'$ equal to 3.0 and $\alpha'$ equal to 1000, we find $\xi = 10$ for $Re = 2000$ and $\xi = 25$ for $Re = 5000$.

### 3.6. Estimate of $\beta'$ based on the natural angle of repose

As the gross shape of the stones and to some extent also the surface texture (roughness) are governing for the natural angle of repose, $\phi$, as well as for the flow resistance, it is attempted to establish a simple relationship between the two latter. For the stone material tested in stationary flow by Burcharth and Christensen (1991) and Williams (1992), $\phi$ has been found from a simple test. A cone was formed on a circular tray by adding stones successively from a small drop height and $\phi$ was found. Fig. 4 shows a plot of $\beta'$ vs. $\tan \phi$. In addition, a set of spheres was tested in the same manner. The $\beta'$ value of 1.4 for the

![Fig. 4. $\beta'$ vs. $\tan \phi$ for selected tests with narrow graded materials. Flow perpendicular to the underlayer during construction of the sample. (■) Aalborg University stone material used in tests by Burcharth et al. (1991). (□) Hydraulic Research stone material used in tests by Williams (1992).](image-url)
spheres is taken as an average of the data for randomly packed spheres from Fand et al. (1987) and Dudgeon (1966), cf. Table 4.

The closest fit is shown in Fig. 4 with a dotted line. A simple linear equation is given by

$$\beta' = 7.1\tan \phi - 3.0$$ (47)

This simple equation gives a first engineering estimate on $\beta'$ for rather narrow graded materials with the shapes: round — irregular — equant. $\phi$ can easily be measured from a cone shaped pile of the stones. The 90% confidence bands of Eq. (47) are approximately $\beta' \pm 20\% \beta'$. Note that the data to which Eq. (47) is fitted corresponds to conditions where the flow is perpendicular to the underlayer when placing the stones. Most likely, this orientation will correspond to the maximum flow resistance for the sample.

4. One-dimensional unsteady flow equations

The one-dimensional unsteady porous flow equation is, cf. the discussion related to Eq. (11), often taken as

$$I = aV + b(V|V| + c\frac{\partial V}{\partial t}$$ (48)

where $I$ is the hydraulic gradient and $V$ is a characteristic velocity. Eq. (11) compares to the Morison equation if the linear viscous term is either neglected or included in the quadratic term through the variation of the coefficient, $b$. The coefficients $b$ and $c$ depend on the geometry (inclusive surface texture) of the stones, on $Re$, on $\partial V/\partial t$ and the flow history. Thus, the coefficients are not constants and should in principle be treated as instantaneous values, even for oscillatory flow conditions. However, in engineering practice, for the sake of simplicity, the coefficients are taken as constants dependent on characteristic Reynolds and Keulegan-Carpenter ($KC$) numbers signifying the oscillatory flow, cf. Eq. (49). The Keulegan-Carpenter number is defined as

$$KC = \frac{V_m T}{d}$$ (49)

where $V_m$ is the maximum velocity and $T$ the period of the oscillatory flow. Instantaneous values of $b$ and $c$ are too complicated to deal with in practice, for which reason the following discussion is based on time invariant coefficients within a cycle. This on the other hand involves fitting of the coefficients over a complete cycle and the values of the coefficients will then specifically relate to cyclic flow. The question still remains to which extent such values can be used for non-cyclic flow for which the standard $KC$ number is not defined, and consequently, some other definition, e.g. based on some zero-crossing splitting of the velocity time series, must be applied.

4.1. Cylinder analogy

In order to describe the local acceleration and the associated virtual mass for the case of non-steady porous flow, it is necessary to distinguish between the volume of water in the
porous matrix and the displaced volume of water, i.e. the volume of solids. In this section, the virtual mass coefficient is related to the volume of the solids corresponding to the usual approach for calculation of flow forces on single bodies.

Initially, the flow through an array of fixed pipes with “large porosity” is considered, Fig. 5. Within the entire sample space, the volume of the pipes equals $(1 - n) \, dx \, dy \, dz$, and the number of pipes equals $4/ \pi d^2 (1 - n) \, dx \, dz$, where $d$ is the pipe diameter. In this case, it is obvious to use the far field velocity, $V$, as the characteristic velocity. The pressure acting on the entire sample of grains and water is considered. The force balance in the direction of the flow reads ($\partial p/\partial x$ is negative in the direction of the flow):

$$
- \frac{\partial p}{\partial x} \, dx \, dy \, dz - F_{\text{drag}}^{\text{pipes}} - F_{\text{inert}}^{\text{pipes}} - F_{\text{fluid}} = 0 \tag{50}
$$

where

$$
F_{\text{drag}}^{\text{pipes}} = C_a \frac{1}{2} \rho |V| Vd \, dy \, \frac{4}{\pi d^2} (1 - n) \, dx \, dz \tag{51}
$$

$$
F_{\text{inert}}^{\text{pipes}} = C_m \rho \, \frac{\pi d^2}{4} \, dy \, \frac{\partial V}{\partial t} \, \frac{4}{\pi d^2} (1 - n) \, dx \, dz \tag{52}
$$

$$
F_{\text{fluid}} = \rho n \, dx \, dy \, dz \, \frac{\partial V}{\partial t} \tag{53}
$$

The above equations yield:

$$
I = - C_a \frac{2}{\pi d} \frac{1 - n}{g} |V| V + \frac{n + C_m (1 - n)}{g} \frac{\partial V}{\partial t} \tag{54}
$$
In general for a fixed body exposed to an ambient flow, \( C_m = 1 + C_a \), where \( C_m \) is the virtual mass coefficient, 1 relates to the Froude-Krylov force and \( C_a \) is the added mass coefficient. For a single smooth cylinder \( C_a = 1 \) and \( C_m = 2 \). \( C_d \) and \( C_m \) correspond to the conventional definition of the Morison Equation and depend on \( Re, KC \), and the relative surface roughness, \( k/d \). However, for an array of cylinders, the drag coefficient, \( C_d \), depends also on \( n \). This is because for separated flow the pressure distribution around the cylinder depends on the velocity distribution which again depends on \( n \). Thus, a cylinder analogy, where the force on each cylinder is considered, seems more complicated than the hydraulic radius analogy where the forces are merely averaged over the sample of water and solid. However, as already discussed in relation to hydraulic radius theory, the drag coefficients \( \beta \) and \( \beta^* \) in Eqs. (42) and (43) must depend on \( n \) because the tortuosity depends on \( n \).

In case of a sphere analogy, the same structure of the formula appears:

\[
I = - \frac{\partial \left( \frac{P}{\rho g} \right)}{\partial x} = C_d \frac{3}{4d} \frac{1-n}{g} |V|V + \frac{n + C_m(1-n)}{g} \frac{\partial V}{\partial t}
\]  

(55)

For a single smooth sphere, \( C_a = 0.5 \) and \( C_m = 1.5 \).

A more realistic model of a porous medium is a dense sample of cylinders/spheres. The ambient flow is now taken as the pore velocity, \( V/n \). The force balance (50) is still valid, but now:

\[
F_{\text{drag}}^{\text{pipe}} = C_d \frac{1}{n} \frac{V}{n} \frac{V}{n} \frac{4}{\pi d^2} (1-n) \, dx \, dz
\]

(56)

\[
F_{\text{iner}}^{\text{pipe}} = C_m \frac{4}{\pi d^2} \frac{\partial (V/n)}{\partial t} \frac{4}{\pi d^2} (1-n) \, dx \, dz
\]

(57)

\[
F_{\text{iner}}^{\text{fluid}} = \rho n \, dx \, dy \, dz \, \frac{\partial (V/n)}{\partial t}
\]

(58)

This yields:

\[
I = - \frac{\partial \left( \frac{P}{\rho g} \right)}{\partial x} = C_d^* \frac{2}{\pi d} \frac{1-n}{n} \frac{1}{g} |V|V + \frac{1 + C_m^*}{g} \frac{1-n}{n} \frac{\partial V}{\partial t}
\]

(59)

Also for this case of densely spaced cylinders, a dependency of \( C_d^* \) and \( C_m^* \) on \( Re, KC \), \( k/d \) and \( n \) is expected, for the same reasons as explained for the widely spaced cylinders.

It is evident from Eqs. (54) and (59) that neither \( C_m \) nor \( C_m^* \) can be independent of \( n \). Note that the \( n \)-relationship in the drag term in Eq. (59) is different from the non-linear term in Eqs. (42) and (7). However, there is no contradiction because the conventional formulation of the drag term in Eq. (59) contains also the effect of the linear term in Eq.
Fig. 6. Cross-section cut through pores and grain contact points.

(42) $C^*$ and $\beta'$ are different, also with respect to the variation with $Re$. The drag term in Eq. (59) can be replaced by the right-hand side of Eq. (42).

Some authors, e.g. Wang and Gu (1988), use for the driving force

$$\frac{\partial p}{\partial x} dx_n\, dy\, dz$$

where $n_a = n$ is the relative area of the pores in a cross-section.

This yields (still using $V/n$ as characteristic velocity):

$$I = -\frac{\partial}{\partial x} \left( \frac{p}{\rho g} \right) = C_d^{w} \frac{2}{\pi d} \frac{1-n}{n^2} \frac{1}{g} |V|V^+ \frac{1}{ng} \frac{n}{\partial V}{\partial t}$$

However, this equation involving the area factor, $n_a = n$, should not be applied because integration of the $x$-axis component of the pressure forces acting on a cross-section, cutting through pores and grain contact points only, yields $p\, dy\, dz$, where $p$ is taken as the average pressure along the cut section, cf. Fig. 6. This reasoning has for many years been applied within highdam engineering related to calculation of cross-section stresses in concrete exposed to large pore pressures. Another way of arriving at the same conclusion is to consider the pressure drop over the sample length recorded by transducers placed just outside each end of the sample (boundary effects can be disregarded for long sample lengths), in which case it is clear that the driving force is $\Delta p\, dy\, dz$, cf. Fig. 6.

4.2. Dependence of the $\beta'$ coefficients on $KC$

Andersen et al. (1993) carried out experiments with cylinders, spheres and rock in steady and oscillatory flow. For the tests with spheres and rock, a certain flow of water under the porous sample took place. In the data analysis, the velocities have been corrected accordingly, however, some uncertainty is introduced in the results. Hence, in Fig. 7, based on data from Andersen et al. (1993), is shown the ratio between the oscillatory and stationary coefficients, i.e. $\beta' / \beta'_{stat}$ vs. $KC$, whereas the absolute $\beta'$ values are omitted. For some tests, there is a tendency towards increasing flow resistance for decreasing $KC$ values. Some uncertainty is associated with Fig. 7 as the $\beta' / \beta'_{stat}$ does not approach unity for large $KC$ values for all tests.
4.3. Inertia coefficients determined from experiments

Hannoura and McCorquodale (1978) carried out experiments with non-stationary flow through coarse granular media, applying a free fall U-tube technique. Large accelerations only appeared in time intervals of 0.15 to 0.25 s. Four types of material were tested. Only one of the test series showed some consistency, resulting in an average value of $C_m^*$ equal to 2.41 and with a standard deviation of 2.48, cf. Table 5. Values of $C_m^* < 1$ were found in a number of tests. This, however, implies negative added mass coefficients which from a physical point of view makes no sense. The occurrence of the negative $C_a^*$ values is most likely due to either experimental uncertainties and/or the averaging method related to the determination of the per definition time-invariant coefficients $C_d^*$ (or $a$ and $b$) and $C_m^*$. The latter problem is well-known from fitting of the Morison equation with time invariant coefficients to flow forces in oscillatory flow. The values of $c$ and $C_m$ in Table 5 have been calculated for the present purpose.

Burcharth and Christensen (1991) also applied a free fall U-tube technique. Eight rock samples with different grading and shape class were tested. Like with the tests of Hannoura and McCorquodale large accelerations only appeared in short time intervals, typically in the order of 0.3 s. From the proceeding deceleration phase values of $C_m^*$ between 12 and 35 were found. However, the authors do not regard these results as reliable due to the limitations of the experimental method.

Smith (1991) carried out experiments in oscillatory flow, but with relatively small accelerations. The results are shown in Table 6. The values of $c$ are average values based

<table>
<thead>
<tr>
<th>Material</th>
<th>$d$ (m)</th>
<th>$n$</th>
<th>$c$ (s$^2$/m)</th>
<th>$C_m$</th>
<th>$C_m^*$</th>
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<tr>
<td>Crushed rock</td>
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<td>0.441</td>
<td>0.413</td>
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<td>2.41</td>
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</table>

Table 5
Experiments of Hannoura and McCorquodale (1978). Average values
Table 6
Experiments of Smith (1991). Average values

<table>
<thead>
<tr>
<th>Matr. No.</th>
<th>n</th>
<th>c (s²/m)</th>
<th>C_m</th>
<th>C_m̂</th>
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<tr>
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<td>0.37</td>
<td>4.55</td>
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<tr>
<td>C75</td>
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<td>0.23</td>
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<td>3.82</td>
<td>1.47</td>
</tr>
<tr>
<td>S</td>
<td>0.47</td>
<td>0.32</td>
<td>5.04</td>
<td>1.90</td>
</tr>
</tbody>
</table>

* C: spheres, cubic packing; R: spheres, rhombohedral packing; S: Tabular rock.

Table 7
Inertia coefficients from Andersen et al. (1993)

<table>
<thead>
<tr>
<th>Matr. No.</th>
<th>c (s²/m)</th>
<th>V_m (m/s)</th>
<th>Re</th>
<th>T (s)</th>
<th>KC</th>
<th>a_m (m/s²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cl</td>
<td>0.14-0.26</td>
<td>0.17-0.73</td>
<td>7700</td>
<td>33 000</td>
<td>2-4</td>
<td>10-53</td>
</tr>
<tr>
<td>C2</td>
<td>0.29-0.46</td>
<td>0.22-0.68</td>
<td>9900</td>
<td>31 000</td>
<td>2-4</td>
<td>8.7-40</td>
</tr>
<tr>
<td>C3</td>
<td>0.60-1.26</td>
<td>0.15-0.28</td>
<td>6800</td>
<td>13 000</td>
<td>2-4</td>
<td>7.9-20</td>
</tr>
<tr>
<td>St</td>
<td>0.41-0.58</td>
<td>0.24-0.52</td>
<td>9700</td>
<td>21 000</td>
<td>2-4</td>
<td>12-38</td>
</tr>
<tr>
<td>R1</td>
<td>0.31-0.63</td>
<td>0.16-0.51</td>
<td>11000</td>
<td>34 000</td>
<td>2-4</td>
<td>4.2-27</td>
</tr>
<tr>
<td>R2</td>
<td>0.27-0.74</td>
<td>0.14-0.47</td>
<td>7500</td>
<td>25 000</td>
<td>2-4</td>
<td>4.6-28</td>
</tr>
<tr>
<td>R4</td>
<td>0.41-0.62</td>
<td>0.16-0.51</td>
<td>8500</td>
<td>27 000</td>
<td>2-4</td>
<td>5.2-34</td>
</tr>
<tr>
<td>R5</td>
<td>0.23-0.78</td>
<td>0.062-0.28</td>
<td>1400</td>
<td>6100</td>
<td>2-4</td>
<td>4.9-45</td>
</tr>
<tr>
<td>R8</td>
<td>0.50-0.63</td>
<td>0.13-0.36</td>
<td>4400</td>
<td>12 000</td>
<td>2-4</td>
<td>6.5-37</td>
</tr>
</tbody>
</table>

* C: cylinders, qua. packing; S: spheres, cubic packing; R1, R5, R8: irreg. rock; R3: semi round rock; R4: round rock.

Table 8
Virtual mass coefficients from Andersen et al. (1993)

<table>
<thead>
<tr>
<th>Matr. No.</th>
<th>C_m</th>
<th>C_m̂</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cl</td>
<td>2.8-8.5</td>
<td>1.4-5.9</td>
</tr>
<tr>
<td>C2</td>
<td>5.5-9.5</td>
<td>2.6-5.0</td>
</tr>
<tr>
<td>C3</td>
<td>8.2-17.8</td>
<td>2.3-5.4</td>
</tr>
<tr>
<td>S1</td>
<td>6.8-10.0</td>
<td>2.7-4.3</td>
</tr>
<tr>
<td>R1</td>
<td>4.7-10.3</td>
<td>1.6-4.1</td>
</tr>
<tr>
<td>R3</td>
<td>4.0-12.5</td>
<td>1.4-5.2</td>
</tr>
<tr>
<td>R4</td>
<td>6.0-9.4</td>
<td>2.0-3.3</td>
</tr>
<tr>
<td>R5</td>
<td>3.3-13.1</td>
<td>1.0-5.4</td>
</tr>
<tr>
<td>R8</td>
<td>7.4-9.5</td>
<td>2.5-3.3</td>
</tr>
</tbody>
</table>

* C: cylinders, qua. packing; S: spheres, cubic packing; R1, R5, R8: irreg. rock; R3: semi round rock; R4: round rock.

on eight tests. The porosities are the values obtained during testing.

The experiments by Andersen et al. (1993) lead to the inertia and virtual mass coefficients shown in Tables 7 and 8. As mentioned previously, a certain flow of water under the sample took place, and corrections were introduced in the data analysis. Hence, the coefficients are associated with some uncertainty.
4.4. Significance of the inertia term

In order to compare the relative importance of the quadratic flow resistance term and the inertia term for sinusoidal motion, the ratio between the maximum values of these has been derived from Eqs. (35) and (59) showing the significance of the \( KC \) number.

\[
\frac{I_{\text{qua}}}{I_{\text{iner}}} = \frac{\beta' \frac{1-n}{n^3} \frac{V_m^2}{gd}}{1 + C_m^* \frac{1-n}{n} \left( \frac{\partial V}{\partial t} \right)_m} = \frac{\beta' \frac{1-n}{n^3} \frac{KC}{2\pi}}{1 + C_m^* \frac{1-n}{n}}
\]

(62)

It should be noticed that the two components appear with a phase shift of 90°. For a typical breakwater, \( KC \) is in the order of 10 in the surface layers. With \( n = 0.41, \beta' = 3.0 \) and for instance \( C_m^* = 3.0 \), the above ratio equals 8.

With respect to the performance of physical model tests, the above ratio indicates that it is difficult to accurately extract the inertia term from the entire signal in order to determine the virtual mass coefficient. For the oscillatory tests of Smith (1991), the above ratio varied between 1.6 and 20, considering due to measurement errors only the tests where the maximum velocity exceeded 0.1 m/s and the maximum acceleration exceeded 0.1 m/s². For the rock material, the tests with velocity above 0.1 m/s, acceleration above 0.1 m/s² and \( I_{\text{qua}}/I_{\text{iner}} < 5 \), \( C_m^* \) ranged between 1.38 and 1.81. In the tests by Andersen et al. (1993) for the cylinder samples, \( I_{\text{qua}}/I_{\text{iner}} \) ranged between 0.075 and 5.3 and \( C_m^* \) was found at 3.9 on average. For the sphere and rock samples, \( I_{\text{qua}}/I_{\text{iner}} \) ranged between 0.75 and 11 and \( C_m^* \) was found at 3.1 on average. The available experimental results are too scattered to support definitive conclusions about the values of \( C_m^* \).

5. Conclusions

The one-dimensional steady porous flow equation, including the variation of the coefficients with the porosity, is derived from the Navier-Stokes equation in combination with the hydraulic radius concept. Alternatively, the same equation can be found from a dimensional analysis. A turbulent flow equation is suggested. The presented \( \beta' \) values for fully turbulent flow refer to Reynolds numbers typically up to 10000, which are smaller than those in a real breakwater case, which in the surface layers are in the order of \( 10^5 \). It has still not been proved that \( \beta' \) is constant over this large range of Reynolds numbers, although it is most likely from a theoretical point of view.

The one-dimensional unsteady porous flow equation and the variation with the porosity is derived from a cylinder/sphere analogy, and virtual mass coefficients selected from various tests are presented. However, it is concluded that the results are too scarce to form definite conclusions about the values of the virtual mass coefficient.
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References


