Stochastic Safety in Space Conjunctions

Gomez, Aitor Ramirez; Wisniewski, Rafal

Published in:
2022 IEEE 61st Conference on Decision and Control (CDC)

DOI (link to publication from Publisher):
10.1109/CDC51059.2022.9992598

Publication date:
2022

Document Version
Accepted author manuscript, peer reviewed version

Stochastic Safety in Space Conjunctions

Aitor R. Gómez[1] and Rafael Wisniewski[1]

***Abstract*— The stochastic reach-avoid problem termed p-safety is further examined in the context of space debris and short-term orbital encounters. We define the collision probability problem, and reformulate it as a *strong* p-safety problem, which offers a computable solution. Enabling computation comes at the cost of a more restrictive formulation which requires several relaxation schemes. To this end, Bernstein forms are employed as polynomial approximation of the nonlinear dynamics, and sum-of-squares as bases to attain certificates of positivity. Finally, a stochastic version of the unperturbed planetary equations is used to model the dynamics.**

I. INTRODUCTION

Akin to past methodologies, this work is motivated by the never ceasing increase of human-made objects around the most populated regions of the Earth's orbits, particularly the Low-Earth Orbit (LEO). The increase is now driven by the revolution of small and medium-size satellites, which provide a cheaper and more versatile solution, specially for distributed problems such as networking and surveillance. With the extremely ambitious –and redundant– macro-constellation initiatives of *Starlink* (SpaceX), *Kuiper* (Amazon) and *OneWeb* (Airbus), just to name a few, the LEO is facing its most dense and packed age in the upcoming years. In 2019, the European Space Agency (ESA) already reported an avoidance maneuver performed against one of Starlink's satellites. In 2020, the National Aeronautics and Space Administration (NASA) showed a monotonous increase in the number of debris crossing the International Space Station (ISS) orbit, which doubled in the last decade [1]. And in 2021, ISS was hit by debris once more. On top of this, an anti-satellite weapon test has been carried out yet again in 2021. In order to ensure the operability and safety of Earth's orbits, this situation calls for autonomous safety systems that aim at preventing the addition of new space debris that threatens the catastrophic cascading collision phenomenon theorised by Donald J. Kessler [2] to become a reality.

That being the case, simplistic approaches to the probability of collision usually lead to very conservative results, thus reaching inconvenient conclusions when deciding weather a controlled object should maneuver away from that possibility. Bearing in mind that an in-orbit maneuver is expensive, the need of maneuvering can be severally reduced by finding more precise or advanced methods to determine the probability of collision, $\mathcal{P}_c$. Some of the most relevant methodologies as of today are [3]-[7], which are based on different integration schemes of a Gaussian probability density function, which describes the combined positional uncertainty of the two objects inside a volume of interest. The three dimensional integral is reduced to the following when projecting the problem into a plane orthogonal to the relative velocity vector, known as encounter plane, at time of closest approach, or TCA.

$$\mathcal{P}_c = \frac{1}{2\pi\sigma_x\sigma_y}\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} e^{\left[-\left(\frac{x-\mu_x}{\sqrt{2}\sigma_x}\right)^2-\left(\frac{y-\mu_y}{\sqrt{2}\sigma_y}\right)^2\right]} dydx, \tag{1}$$

where R is the combined radii of the objects, μ_x and μ_y are the components of the miss distance projected onto the encounter plane, and σ_x and σ_y are the respective standard deviations.

A differing variety of assumptions are considered by each of the previous authors in order to solve (1). Although, they all separate short- and long-term encounters as two significantly different problems that do not meet in the same collection of assumptions and, hence, need to be approached differently. A more detailed comparison between these methods can be found in [8].

We open the door to a novel –and more sophisticated– method to determine probabilities of collision between close approaching objects using the concept of p-safety [9]. The main idea is to determine an upper bound p to the probability of collision given an stochastic model of the objects and their set of initial conditions. Characterizing safety of space objects through p-safety represents the first necessary step towards a framework capable of encompassing a more complete analysis. For instance, enabling short- and long-term conjunction problems within the same formulation, as well as a framework capable of computing maneuvers to reduce the probability of collision to a desired p level. The latter being an idea that shares similarities to [10], [11] and the deterministic case [12].

II. METHODOLOGY

The calculation of the upper bound p assumes the propagated position estimation and the error covariances of the objects near TCA using high fidelity models. Subsequently, we switch to a more convenient set of equations that describe the stochastic motion of the objects. There exist several choices to model the drift of the states. For the sake of simplicity, the unperturbed planetary equations have been selected in this work. The diffusion terms are characterized by constant covariance matrices. Altogether, a stochastic differential equation (SDE) is defined, which describes the

*This work is supported by Innovation Fund Denmark in the project MARIOT 222230

[1]Section of Automation & Control, Aalborg University, 9220 Aalborg East, Denmark arg@es.aau.dk (A.R. Gómez), raf@es.aau.dk (R. Wisniewski)

evolution of the uncertain location of the objects in their nominal orbits as time unfolds.

Solving the p-safety problem to find the probability p for the given SDE is equivalent to solving a semi-definite programming problem to determine a polynomial stochastic barrier function h subject to certain constrains. The bound p, hence, represents a specific level set of h.

In the p-safety framework, the problem of finding h is restricted entirely to polynomials, as it will be shown. Consequently, the SDE must be governed, or approximated, by polynomials. In this paper, this approximation is carried out using Bernstein forms.

Lastly, three conceptually different sets, i.e. *S*, *U* and *A*, are defined in correlation with the stochastic process. The set *S* is a bounded subset of $\mathbb{R}^n$ describing the state-space of the process, namely containing all possible reachable states during the encounter. Sets *U* and *A* are closed non-intersecting subsets of *S*, describing the *unsafe* states and the initial states respectively. One may refer to *U* as the unsafe set and to *A* as the initial set.

It must be noted, that the complexity of the overall solution can easily grow, therefore; each step has been designed thoroughly not to fall into dimensionality problems.

III. CONCEPT OF P-SAFETY

For a detailed explanation of p-safety the reader is referred to [9]. Nonetheless, we introduce the concept briefly for completeness.

Let us begin by quickly reviewing the definition of the main variant: *strong* p-safety. To that end, we consider a probability space $(\Omega, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)$, and a process $(\mathbf{X}_t)$ in a measurable space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ governed by the following SDE

$$d\mathbf{X}_t = f(\mathbf{X}_t)dt + \sigma(\mathbf{X}_t)d\mathbf{W}_t, \tag{2}$$

with maps $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^l$ Lipschizt continuous and $(\mathbf{W}_t)$ denoting a Brownian motion. Then we define two measurable compact sets of $\mathbb{R}^n$, S and U, such that $U \subset S$. We call S the state-space of $(\mathbf{X}_t)$ and U the set of unsafe states.

The notion of safety in the p-safety framework is established by characterizing initial states $\mathbf{x} \in S$ of a process $(\mathbf{X}_t)$ according to the probability of $(\mathbf{X}_t)$ reaching the unsafe set U before leaving the state-space S. This can be formalized by means of stopping times, and extend the characterization to sets rather than just singletons in S. The following definition is then introduced for a random hitting time of the unsafe set, τ_U, and a random exit time of the state-space, ζ_S, for the process $(\mathbf{X}_t)$.

Definition 3.1: *(strong p-safety on a set) For an infinite time horizon, we call a measurable compact subset A a p-safe set iff*

$$P(A; U, S) := \sup_{\mathbf{x} \in A} \{P^{\mathbf{x}}\{\omega \in \Omega : \tau_U(\omega) < \zeta_S(\omega)\}\} \leq p \tag{3}$$

Remark 3.1: *There exists a variant termed weak p-safety that allows us to work with a probability measure μ_0 with $supp(\mu_0) \subset S$, hence providing a more realistic representation of the problem at hand. Determining a numerical solution for it, though, also demands more advanced theory and higher computational efforts. For this reason, we accommodate the upcoming assumptions and restrict our study to strong p-safety.*

Solving for $P(A; U, S)$ is shown to boil down to find a function $h : S \to \mathbb{R}_+$ in the space of measurable functions with extended generator of the process (2) defined as

$$\mathcal{L}h = \nabla h^T f + \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^T \nabla^2 h\right), \tag{4}$$

and domain $\mathcal{DL}$. A function $h \in \mathcal{DL}$ such that $\mathcal{L}h \leq 0$ implies that the process (h_t) defined as $h_t = h(\mathbf{X}_t)$ is a local super-martingale. Then, $P(A; U, S) \leq p$ with $p \in [0, 1]$ if

$$\begin{aligned} h(\mathbf{x}) &\leq p, \quad \forall \mathbf{x} \in A \\ h(\mathbf{x}) &\geq 1, \quad \forall \mathbf{x} \in U \end{aligned} \tag{5}$$

If (5) is satisfied, we call h a *super-martingale barrier function*, and employ dynamic programming to search among the set $\mathcal{H}$ of possible barrier functions to find the one that minimizes p. As of the time of writing this paper, methods to find h are limited, thus we advance here that finding a solution can only be attempted by restricting ourselves to polynomial forms. Henceforth, we define h as a non-negative polynomial on S and search on $\mathcal{K} = \mathcal{H} \cap \mathbb{R}_{d_h}[\mathbf{X}]$, where $\mathbb{R}_{d_h}[\mathbf{X}]$ is the space of polynomials of maximum degree d_h, using semi-definite programming to determine a solution of the following optimization problem.

$$\begin{aligned} P(A;\, U, S) = \inf_{h \in \mathcal{K}} p& \\ \text{subject to}& \\ (p, h) \in [0, 1] \times \mathcal{DL}& \\ -\mathcal{L}h(\mathbf{x}) \geq 0, \quad \forall \mathbf{x} \in S& \\ h(\mathbf{x}) \geq 0, \quad \forall \mathbf{x} \in S& \\ p - h(\mathbf{x}) \geq 0, \quad \forall \mathbf{x} \in A& \\ h(\mathbf{x}) - 1 \geq 0, \quad \forall \mathbf{x} \in U.& \end{aligned} \tag{6}$$

Certificates of positivity on a set are determined using sum-of-squares (SOS) polynomial and Putinar positivstellensatz [13], to the detriment of restricting S, U and A to be compact semi-algebraic sets.

IV. PROBABILITY OF COLLISION

Let us define two unperturbed nominal orbits, $\mathcal{O}_1$ and $\mathcal{O}_2$, based on their mean Keplerian elements, that is $\mathcal{O}_1(a_1, e_1, i_1, \omega_1, \Omega_1)$ and $\mathcal{O}_2(a_2, e_2, i_2, \omega_2, \Omega_2)$. Then, consider an SDE as in (2) governing the evolution of the eccentric angles X_t^1 and X_t^2 of two space objects $\mathcal{X}_1$ and $\mathcal{X}_2$ along their nominal orbits.

For this purpose, the drift becomes $f : \mathbb{R}^2 \to \mathbb{R}^2$ and is adopted from deterministic models that describe the mean motion of an orbiting object. Two models are generally used for this purpose, whether one employs Newton's law of gravity in the Cartesian description or the planetary equations in the Keplerian elements, or classical orbital elements (COE), description. To avoid the curse of dimensionality, we select the planetary equations and consider that the mean Keplerian elements do not osculate during the encounter. This implies that all time derivatives die out with the exception of the mean anomaly $M_t \in \mathbb{R}$. We now change the focus to the eccentric anomaly $X_t \in [0, 2\pi)$, which time evolution equation can be derived from the *Kepler equation*, $M_t = X_t - e \sin X_t$, as

$$\frac{dX_t}{dt} = \frac{m + \frac{de}{dt}}{1 - e \cos X_t} = \frac{m}{1 - e \cos X_t}, \tag{7}$$

where $m \in \mathbb{R}$ is the mean motion and $e \in [0, 1)$ is the eccentricity of the orbit. The state vector studying the eccentric angle of two space objects now becomes $\mathbf{X}_t(\boldsymbol{\omega}) = [X_t^1(\omega), X_t^2(\omega)]^T$ with mean initial condition $\mathbf{X}_0 = [X_0^1, X_0^2]^T$, constant covariance $\sigma(\mathbf{X}_t) = \Sigma \in \mathbb{R}^{2\times 2}$ and Brownian motion $\mathbf{W}_t(\boldsymbol{\omega}) = [W_t^1(\omega), W_t^2(\omega)]^T$. The two last items representing the diffusion. In summary, let us consider the following SDE.

$$\begin{bmatrix} dX_t^1 \\ dX_t^2 \end{bmatrix} = \begin{bmatrix} \frac{m_1}{1-e_1 \cos X_t^1} \\ \frac{m_2}{1-e_2 \cos X_t^2} \end{bmatrix} dt + \Sigma \begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix}. \tag{8}$$

A usual assumption for short-term conjunctions is to consider the error covariance in Cartesian coordinates constant during the encounter. Accordingly, we define $\mathbf{w}_1(\omega) \sim \mathcal{N}(\mathbf{0}, \Sigma_{w_1})$ and $\mathbf{w}_2(\omega) \sim \mathcal{N}(\mathbf{0}, \Sigma_{w_2})$ as random vectors in $\mathbb{R}^3$, with constant covariance matrices

$$\Sigma_{w_1} = \begin{bmatrix} \sigma_a^1 & 0 & 0 \\ 0 & \sigma_c^1 & 0 \\ 0 & 0 & \sigma_r^1 \end{bmatrix}^2, \quad \Sigma_{w_2} = \begin{bmatrix} \sigma_a^2 & 0 & 0 \\ 0 & \sigma_c^2 & 0 \\ 0 & 0 & \sigma_r^2 \end{bmatrix}^2, \tag{9}$$

obtained from the state propagation schemes. The elements in the diagonal represent the *along-track*, *cross-track* and *radial* standard deviations.

Then $(\mathbf{P}_t^1), (\mathbf{P}_t^2) \in \mathbb{R}^3$, are random vector processes derived using $(\mathbf{X}_t)$, $\mathbf{w}_1(\omega)$ and $\mathbf{w}_2(\omega)$, which describe the absolute position of the objects.

$$\begin{aligned} \mathbf{P}_t^1 &= p_1(X_t^1) + \mathbf{w}_1, \\ \mathbf{P}_t^2 &= p_2(X_t^2) + \mathbf{w}_2, \end{aligned} \tag{10}$$

where the maps $p_1, p_2 : \mathbb{R} \to \mathbb{R}^3$ transform the eccentric angles X_t^1 and X_t^2 to positional vectors pointing on $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively, and $\mathbf{w}_1$ and $\mathbf{w}_2$ randomize the position of the objects around $\mathcal{O}_1$ and $\mathcal{O}_2$. The geometrical interpretation of (10) can be visualized in Fig. 1.

Led by common practice, we disregard the attitude information of the space objects and model $\mathcal{X}_1$ and $\mathcal{X}_2$ as spheres. Each one with their respective largest dimension as radius, and centers at $(\mathbf{P}_t^1)$ and $(\mathbf{P}_t^2)$. We define $(\tilde{\mathbf{P}}_t)$ as the relative position process.

$$\tilde{\mathbf{P}}_t = \mathbf{P}_t^1 - \mathbf{P}_t^2. \tag{11}$$

Based on (11), we strive to compute the probability $\mathcal{P}_c$ of a collision event between $\mathcal{X}_1$ and $\mathcal{X}_2$ occurring. If we understand the collision event as the two spheres intersecting, then the probability can be formulated by studying the stopping-time of the process $(\tilde{\mathbf{P}}_t)$ hitting the set $B_R =: \{\mathbf{z} \in \mathbb{R}^3 : \|\mathbf{z}\|_2 \leq R\}$, with R being the combined radii. That is,

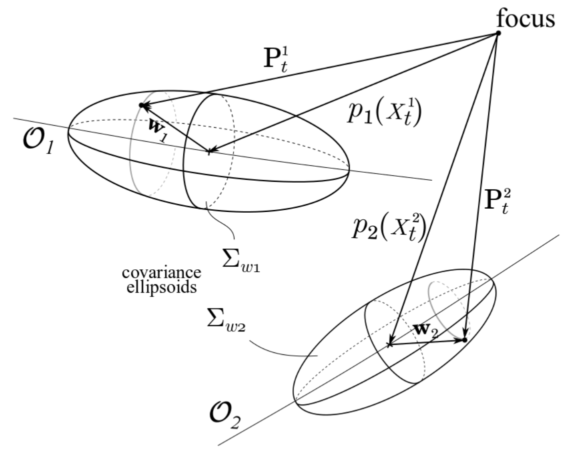


Fig. 1: Geometrical interpretation of a particular realization of the processes $(\mathbf{P}_t^1)$ and $(\mathbf{P}_t^2)$ in (10) for an arbitrary point of time. Vectors $\mathbf{w}_1(\omega)$ and $\mathbf{w}_2(\omega)$ are expected constant in the local reference frame of the objects.

$$\mathcal{P}_c = P^{\mathbf{x}_0}\{\omega \in \Omega : \tau_{B_R}^{\tilde{\mathbf{P}}_t}(\omega) < \infty\}, \tag{12}$$

where $\tau_{B_R}^{\tilde{\mathbf{P}}_t}(\omega)$ is the random hitting-time of the set B_R and $\mathbf{x}_0 \in \mathbb{R}^3$ is a singleton representing the mean initial state of $(\mathbf{X}_t)$. The challenge tackled in the next section is to reformulate $\mathcal{P}_c$ as a p-safety problem.

V. FORMULATION OF P-SAFE CONJUNCTIONS

Reformulating (12) means to find the upper bound p, such that $\mathcal{P}_c \leq p$, as a solution of a p-safety problem. The process of reformulating is two-fold: 1) we find polynomial versions of the non-polynomial forms and 2) we determine a triple (A, U, S) as compact semi-algebraic sets.

We show that the latter is very restrictive in the current formulation of p-safety, and we will just attempt to find an approximation based on a series of assumptions. In the next section, we employ Bernstein forms to perform the polynomial approximation of the drift f given their suitable approximating properties over any continuous region, e.g. S.

A. *POLYNOMIAL FORMS*

Bernstein forms are well known tools to determine certificates of positivity of a function on a simplex. Withal, they also present effective properties to uniformly approximate continuous functions over regions of space. Let us modify the basic definition of Bernstein approximation over a simplex [14] to approximate a function f on a segment. Also note that the functions composing f are completely uncoupled, thus each function f_i, $i \in \{1, 2\}$ can be approximated separately.

Definition 5.1: *(Bernstein approximation) [14] Consider the segment $D = [a, b]$ with $a, b \in \mathbb{R}$, and let $d \in \mathbb{N}$. Then, $\hat{f} \in \mathbb{R}_d[X]$ is a Bernstein polynomial of degree $\leq d$ with*

$X \in D$ approximating a function f on a segment D.

$$\hat{f}(X) = \sum_{k=0}^{d} \beta_k(f, d, D) \binom{k}{d} \lambda_0^{d-k} \lambda_1^k, \tag{13}$$

where $\lambda_1 = 1 - \frac{X-a}{b-a}$ and $\lambda_0 = \frac{X-a}{b-a}$ are the new barycentric coordinates and $\beta_k(f, d, D) \in \mathbb{R}$ is the k^{th} Bernstein coefficient of degree d on D.

The coefficients β_k are direct evaluations of f on D. A function $g : \mathbb{N} \to \mathbb{R}$ such that $g(k) = a + \frac{k}{d}(b-a)$ can be composed with f for this purpose. Altogether,

$$\beta_k(f, d, D) = (f \circ g)(k), \; k \in \{0, 1..., d\}. \tag{14}$$

Convergence of $\hat{f}$ to f can be achieved by applying either *degree elevation* of the Bernstein polynomial or *subdivision* of the region defined by D [15]. Focusing the study only in the vicinity of TCA makes f very easy to approximate by one single Bernstein polynomial with a suitable enough low degree. Hence, we elevate the degree of $\hat{f}$ up to a certain degree $d' > d$ to reach errors of the order 10^{-9}, and reformulate (8) as follows.

$$\begin{bmatrix} dX_t^1 \\ dX_t^2 \end{bmatrix} = \begin{bmatrix} \hat{f}_1(X_t^1) \\ \hat{f}_2(X_t^2) \end{bmatrix} dt + \Sigma \begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix}. \tag{15}$$

We denote $\hat{f}(\mathbf{X}_t) = [\hat{f}_1(X_t^1), \hat{f}_2(X_t^2)]^T$, or simply $\hat{f}$, the Bernstein approximation of f.

B. COMPACT SEMI-ALGEBRAIC SETS (A, U, S)

In this section, we address the procedure of determining the triple (A, U, S). Sets A and U describe regions of the state-space S with different significance in the reachability problem. Note that the location of the objects in space is described by stochastic processes. Hence, not only the initial conditions are represented by a probability distribution, but also the unsafe states. The p-safety formulation, however, disregards this fact by considering A and U as bounded semi-algebraic sets rather than distributions on S.

Bearing this in mind, we strive to perform some simplifications to fit as closely as possible to the original problem described in Sec. IV, while maintaining the p-safety formulation. To that end, we use the error covariances defined in (9) to determine A and U.

The set A encloses possible initial states of the process $(\mathbf{X}_t)$. To identify them, we use the standard deviation of the *along-track* direction in Cartesian coordinates, i.e. σ_a^1 and σ_a^2. We relate the along-track standard deviation to the standard deviation of the initial states, denoted as $\sigma_{X_0}^1$ and $\sigma_{X_0}^2$, by means of a linear (small angle) approximation.

$$\sigma_{X_0}^i = \frac{\sigma_a^i}{a_i(1 - e_i \cos X_0^i)}, \quad i \in \{1, 2\}, \tag{16}$$

the denominator being the instantaneous radius of the object in the mean initial state, and the nominator the *arc* of uncertainty described by the standard deviation in the direction of the objects' velocity vector. Then, A becomes an ellipse centered around $\mathbf{X}_0$, and we can expand the ellipse to include more possible initial conditions by taking $n \in \mathbb{N}$ standard deviations. Formally, for a given mean initial state $\mathbf{z}_0$

$$A := \left\{ \mathbf{z} \in \mathbb{R}^2 : (\mathbf{z} - \mathbf{z}_0)^T \Phi (\mathbf{z} - \mathbf{z}_0) \leq 1 \right\}, \tag{17}$$

where $\Phi \in \mathbb{R}^{2\times2}$ is the matrix expanding the ellipse semi-major/minor axes, defined as

$$\Phi = \begin{bmatrix} (n\sigma_{X_0}^1)^{-2} & 0 \\ 0 & (n\sigma_{X_0}^2)^{-2} \end{bmatrix}. \tag{18}$$

Approximating U is a more intricate job. Alike the initial states, the unsafe states are better represented by probability distributions rather than a finite collection of states of equal risk. Nonetheless, we determine U as the latter.

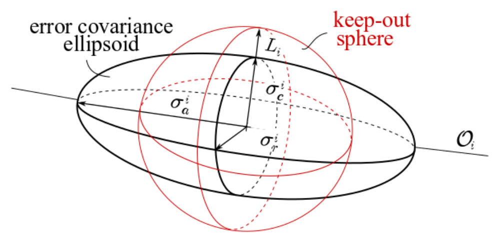


Fig. 2: A $1-\sigma$ ellipsoid and the keep-out sphere considering $\sigma_a^i > \sigma_c^i > \sigma_r^i$, $i \in \{1, 2\}$. The distance L_i is exaggerated for the sake of understanding.

Let us begin by considering the maps $p_1(X_t^1)$ and $p_2(X_t^2)$ defined in Sec. IV which transform the eccentric angles into vectors in $\mathbb{R}^3$ describing the position of the space objects in their nominal orbits. Then, we introduce the idea of *keep-out* spheres, as two spheres of fixed radii R_1 and R_2 with center at $p_1(X_t^1)$ and $p_2(X_t^2)$, respectively. These spheres define regions in Cartesian coordinates that the objects are potentially occupying. Thus, we are interested on both keep-out spheres being away from each other as they move along the nominal orbits, i.e. as $(\mathbf{X}_t)$ evolve. The radius of the spheres are calculated as (see Fig. 2)

$$R_i = n\sigma_i + L_i, \quad i = \{1, 2\}, \tag{19}$$

where $\sigma_i = \max\{\sigma_c^i, \sigma_r^i\}$ and L_i is the largest dimension of the object $\mathcal{X}_i$ w.r.t its center of mass.

Let s be the nominal *miss distance* defined as the minimum orthogonal distance between the sets $\mathcal{O}_1$ and $\mathcal{O}_2$. Herein, we obtain a necessary condition for the nominal miss distance s to enable the p-safety computation, that is

$$s \leq R_1 + R_2. \tag{20}$$

Then, the set U is identified as the following set

$$U := \left\{ (z_1, z_2) \in \mathbb{R}^2 : \|p_1(z_1) - p_2(z_2)\|_2 \leq R_1 + R_2 \right\}. \tag{21}$$

Clearly, (21) is not semi-algebraic. Determining U as a semi-algebraic set could be done in various ways, yet we seek for a computationally cheap approach. Thus, we start by defining the set in terms of the true anomaly $\theta_t \in [0, 2\pi)$, which is directly related to the eccentric anomaly by

$$\tan\frac{\theta_t}{2} = \sqrt{\frac{1+e_i}{1-e_i}} \tan\frac{X_t}{2}. \tag{22}$$

Consider in Cartesian coordinates the position of $\mathcal{X}_1$ and $\mathcal{X}_2$ in their nominal orbits as vectors in terms of θ_t^1 and θ_t^2, i.e. $q_1, q_2 : \mathbb{R} \to \mathbb{R}^3$ such that

$$q_i(\theta_i) = T_{O_i}^I \begin{bmatrix} r(\theta_i)\cos\theta_i \\ r(\theta_i)\sin\theta_i \\ 0 \end{bmatrix}, \quad i \in \{1, 2\} \tag{23}$$

where $T_{O_1}^I, T_{O_2}^I \in \mathcal{SO}(3)$ are transformation matrices from the orbital to the inertial frame of reference using the mean COEs, and $r(\theta_i) = \frac{a_i(1-e_i^2)}{1-e_i\cos\theta_i}$ is the distance from the focal point of the orbit to the center of each object. Then, the true anomalies characterized as unsafe (see Fig. 3) belong to the following set[1]

$$\left\{(z_1, z_2) \in \mathbb{R}^2 : \|q_1(z_1) - q_2(z_2)\|_2 \le R_1 + R_2\right\}. \tag{24}$$

For short-term encounters and keep-out spheres with standard values of n, between 3 and 8, U becomes a small set. If a change of variable is performed such that the origin is translated to the TCA point, then U contains the origin and its neighbourhood. In other words, the potential conjunction happens in the vicinity of the origin of the new variable. For such a change of variable a linear approximation can be employed.

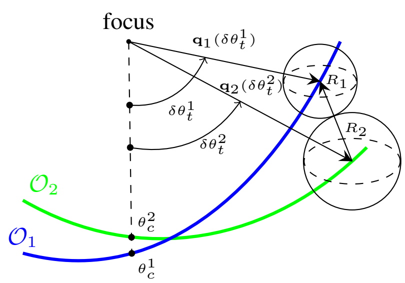


Fig. 3: Instance of a $\mathbf{\Theta}_t$ configuration satisfying (24).

We denote by $\theta_c^1, \theta_c^2 \in [0, 2\pi)$ the mean true anomalies and $X_c^1, X_c^2 \in [0, 2\pi)$ the mean eccentric anomalies, both at TCA. Then, (23) can be modified to accommodate the change of variable $\theta_t = \theta_c + \delta\theta_t$, combined with the following linear approximations

$$\begin{aligned} \cos(\theta_c + \delta\theta_t) &= \cos\theta_c - \delta\theta_t \sin\theta_c + O(\delta\theta_t^2) \\ \sin(\theta_c + \delta\theta_t) &= \sin\theta_c + \delta\theta_t \cos\theta_c + O(\delta\theta_t^2). \end{aligned} \tag{25}$$

Neglecting terms of order higher than two, one can work out the tedious calculations to approximate the inequality in (24) as a quadratic inequality of the form $\mathbf{\Theta}_t^T A_q \mathbf{\Theta}_t + B_q \mathbf{\Theta}_t + c \le 0$, in terms of $\mathbf{\Theta}_t = [\delta\theta_t^1, \delta\theta_t^2]^T$, which can be reformulated to take the standard form of an ellipse centered at the origin. By taking [16]

$$\Upsilon_\theta = A_q \left(\frac{1}{4} B_q^T A_q^{-1} B_q - c\right)^{-1}, \tag{26}$$

the previous quadratic expression becomes $\mathbf{\Theta}_t \Upsilon_\theta \mathbf{\Theta}_t \le 1$.

[1]Note that (24) resembles the set in (21). However, it is algebraically easier to work with the maps q_1 and q_2.

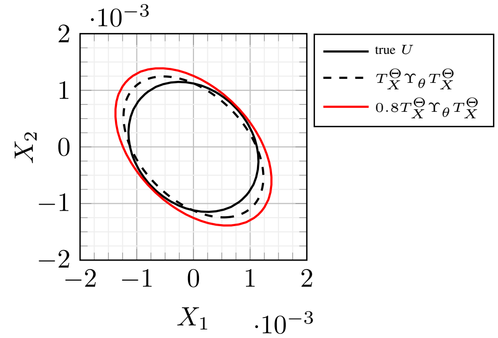


Fig. 4: True unsafe set and its compact semi-algebraic version of example in Sec. VI with $\eta = 0.8$.

Lastly, we use (22) and the following linear approximation

$$\tan(\theta_c + \delta\theta_t) = \frac{\tan\theta_c + \delta\theta_t}{1 - \delta\theta_t \tan\theta_c} + O(\delta\theta_t^2), \tag{27}$$

to determine a transformation matrix $T_X^\Theta : \mathbf{X}_t \mapsto \mathbf{\Theta}_t$ that maps from eccentric angles to true anomaly angles.

$$T_X^\Theta = \begin{bmatrix} \frac{1+\varrho_1\xi_1}{\varrho_1+\xi_1} & 0 \\ 0 & \frac{1+\varrho_2\xi_2}{\varrho_2+\xi_2} \end{bmatrix}, \tag{28}$$

with constants $\varrho_i = \sqrt{\frac{1-e_i}{1+e_i}}$ and $\xi_i = \tan\frac{X_c^i}{2}\tan\frac{\theta_c^i}{2}$, for $i \in \{1, 2\}$. Finally,

$$\Upsilon = \eta T_X^\Theta \Upsilon_\theta T_X^\Theta, \tag{29}$$

and the parameter $\eta \in \mathbb{R}_{>0}$ is meant to compensate for the approximation errors by uniformly enlarging the ellipse, so that it includes the original set.

If the geometric problem in (24) is well posed and a region that satisfy the inequality exist, meaning both keep-out spheres intersect somewhere along the mean orbits and condition (20) is satisfied, then Υ is a positive definite matrix. Negative definite otherwise.

Thus, U becomes (see example in Fig. 4)

$$U := \left\{\mathbf{z} \in \mathbb{R}^2 : \mathbf{z}^T \Upsilon \mathbf{z} \le 1\right\}. \tag{30}$$

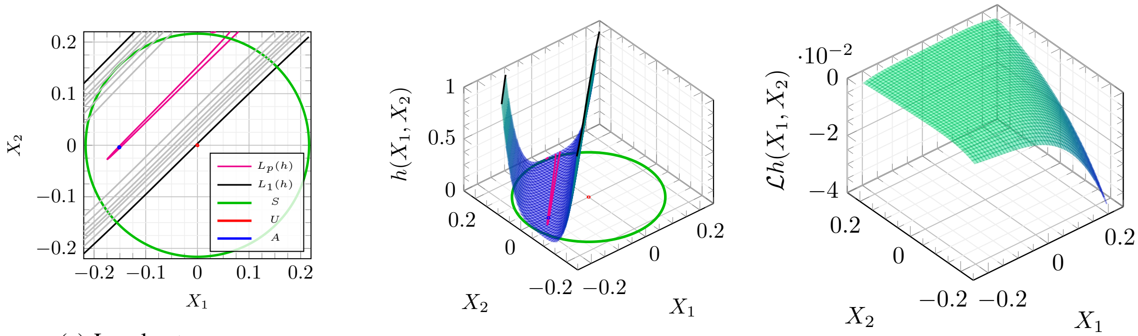
Determining the set S is more direct, given that any polynomial level set including A and U without boundary intersections can suffice to compute p-safety. Nevertheless, S has a direct impact on the stochastic barrier function h. To keep the framework simple, we propose a circle with center at the origin and radius

$$R_S = \alpha\left(\|\mathbf{X}_0\|_2 + \frac{1}{\underline{\lambda}_A^2}\right), \tag{31}$$

which ensures the enclosure of A and U if $\underline{\lambda}_A$ is the smallest eigen value of Φ. We use $\alpha \in \mathbb{R}$ to force non-intersecting boundaries between A and S by imposing $\alpha > 1$. Then,

$$S := \left\{\mathbf{z} \in \mathbb{R}^2 : \mathbf{z}^T \mathbf{z} \le R_S^2\right\}. \tag{32}$$

In the next section we illustrate some results via the numerical example of Kosmos 2251 – Iridium 33.



(a) Level sets (b) h (c) $\mathcal{L}h$

Fig. 5: Top *(a)* and 3d *(b)* view of h, (A, U, S), and the level sets $L_1(h)$ and $L_p(h)$, and *(c)* the generator $\mathcal{L}h$.

VI. EXAMPLE: KOSMOS 2251 – IRIDIUM 33

The formulation derived in this paper has been implemented[2] on MATLAB employing YALMIP's SOS module. The initial conditions of the collision are summarized in the following parameters[3], considering K-2251 as $\mathcal{X}_1$: $e_1 = 0.0016015$, $i_1 = 74.0357$ [deg], $\omega_1 = 95.9865$ [deg], $\Omega_1 = 95.9865$ [deg] $m_1 = 14.31135$ [rpm], $X_0^1 = 0.177528$ [rad] and $L_1 = 17$ [m]; and I-33 as $\mathcal{X}_2$: $e_2 = 0.0002253$, $i_2 = 86.3989$ [deg], $\omega_2 = 89.6115$ [deg], $\Omega_2 = 121.2960$ [deg] $m_2 = 14.34220263$ [rpm], $X_0^2 = 0.289902$ [rad] and $L_2 = 25$ [m]. For both objects it is assumed that $\sigma_a = 1.5$ [km], and $\sigma_r = \sigma_c = 0.5$ [km].

We choose $n = 8$ for illustrative purposes, a Bernstein degree $d = 5$ and assume the diffusion matrix to be $\Sigma = \text{diag}(10^{-4}, 10^{-4})$. For $\eta = 0.8$, $\alpha = \sqrt{2}$ and fixing the degree of the searching space to $d_h = 11$, the barrier function h found provides a bound $p = 1.1462964 \cdot 10^{-4}$. The level sets $L_p(h)$ and $L_1(h)$, where $L_\rho(h) := \{\mathbf{z} \in \mathbb{R}^2 : h(\mathbf{z}) = \rho, \rho \in \mathbb{R}\}$, are represented in Fig. 5 *a)* together with the triple (A, U, S). Fig. 5 *b)* and *c)* show that both, h and $\mathcal{L}h$, are non-negative and non-positive on S, respectively. Close-ups for A and U are also provided in Fig. 6 showing the fulfillment of the constraints.

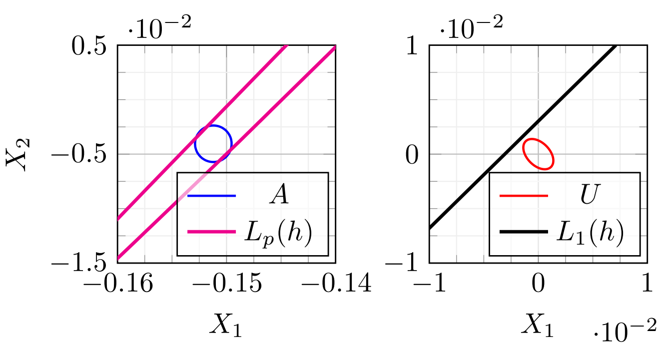


Fig. 6: Close-up of U and A, and level sets.

[2]Code available online at https://github.com/aitor-rg/p-safety

[3]The parameters do not represent the actual geometry at TCA or collision.

VII. CONCLUSIONS

We have successfully reformulated the collision probability as a *strong* p-safety problem. However, for several values of d_h suboptimal solutions are found due to bad or slow progress in the iteration process. We also show that some of the stochastic attributes of the problem are lost given the restrictive formulation in terms of semi-algebraic sets, hence a series of relaxation schemes were required. Moreover, one might find uncomfortable bearing with various new design parameters that impact the outcome, and which require further analysis and understanding. Finally, determining U and A became rather tedious given the states considered for the SDE model. Studying alternative SDE models in p-safety for short-term encounters is left as future work.

REFERENCES

[1] National Aeronautics and Space Administration, Orbital Debris, Quarterly News, Volume 24, Issue 3, August 2020.

[2] Kessler, Donald & Johnson, Nicholas & Liou, J.-C & Matney, Mark, The Kessler Syndrome: Implications to Future Space operations, Advances in the Astronautical Sciences. 137, 2010.

[3] Russell P. Patera, General Method for Calculating Satellite Collision Probability, Journal of Guidance, Control and Dynamics, AIAA, 2001.

[4] Salvatore Alfano, A Numerical Implementation of Spherical Object Collision Probability, Journal of the Astronautical Sciences, 2005.

[5] Akella and Alfriend, Probability of Collision Between Space Objects, Journal of Guidance, Control and Dynamics, AIAA, 2000.

[6] K. Chan, Improved Analytical Expressions for Computing Spacecraft Collision Probabilities, AAS/AIAA Space Flight Mechanics Meeting, Ponce, Puerto Rico, 2003.

[7] J. Foster, A Parametric Analysis of Orbital Debris Collision Probability and Maneuver Rate for Space Vehicles, NASA JSC-25898, ESA, 1992.

[8] S. Alfano, Review of Conjunction Probability Methods for Short-term Encounters, Advances in the Astronautical Sciences, 2007.

[9] R. Wisniewski and M. L. Bujorianu, Safety of Stochastic Systems: An Analytic and Computational Approach, Automatica, 2021.

[10] P. M. Esfahani, D. Chatterjee, J. Lygeros, The stochastic reach-avoid problem and set characterization for diffusions, Automatica, 2016.

[11] S. Prajna A. Jadbabaie G. Pappas. Stochastic Safety Verification Using Barrier Certificates.Conference on Decision and Control, 2004.

[12] M. Zakiyullah and B. Jayawardhana, Stabilization with guaranteed safety using Control Lyapunov Barrier Function, Automatica, 2016.

[13] M. Marshall, Positive polynomials and sums of squares, Mathematical Surveys and Monographs, American Mathematical Society, 2008.

[14] Richard Leroy, Certificates of positivity in the simplicial bernstein basis, hal-00589945, hal.archives-ouvertes.fr, 2009.

[15] T. Leth, Polynomials in the Bernstein Basis and their use instability theory, PhD thesis, Aalborg University, 2017.

[16] Philip J. Schneider and David H. Eberly, Geometric Tools for Computer Graphics, Chapter 5, Morgan Kaufmann, San Francisco, 2003.