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# SPARSELY-PACKETIZED PREDICTIVE CONTROL BY ORTHOGONAL MATCHING PURSUIT \*

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**Abstract.** We study packetized predictive control, known to be robust against packet dropouts in networked systems. To obtain sparse packets for rate-limited networks, we design control packets via an  $\ell^0$  optimization, which can be effectively solved by orthogonal matching pursuit. Our formulation ensures asymptotic stability of the control loop in the presence of bounded packet dropouts.

**1. Packetized Predictive Control.** Let us consider the following discrete-time, LTI plant model with a scalar input:

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + Bu(k), \quad k \in \mathbb{N}_0, \quad \mathbf{x}(0) = \mathbf{x}_0, \quad (1.1)$$

where  $\mathbf{x}(k) \in \mathbb{R}^n$  and  $u(k) \in \mathbb{R}$  for  $k \in \mathbb{N}_0$ . We assume that  $(A, B)$  is reachable. We are interested in a networked control architecture where the controller communicates with the plant actuator through an erasure channel, see Fig. 1.1. This channel introduces bounded packet-dropouts. In packetized predictive control (PPC), as described, e.g., in [4, 2] at each time instant  $k$ , the controller uses the state  $\mathbf{x}(k)$  of the plant (1.1) to calculate and send a control packet of the form

$$\mathbf{u}(\mathbf{x}(k)) \triangleq [u_0(\mathbf{x}(k)), u_1(\mathbf{x}(k)), \dots, u_{N-1}(\mathbf{x}(k))]^\top \in \mathbb{R}^N \quad (1.2)$$

to the plant input node.

To achieve robustness against packet dropouts, buffering is used. To be more specific, suppose that at time instant  $k$  the data packet  $\mathbf{u}(\mathbf{x}(k))$  defined in (1.2) is successfully received at the plant input side. Then, this packet is stored in a buffer, overwriting its previous contents. If the next packet,  $\mathbf{u}(\mathbf{x}(k+1))$ , is dropped, then the plant input  $u(k+1)$  is set to  $u_1(\mathbf{x}(k))$ , the second element of  $\mathbf{u}(\mathbf{x}(k))$ . The subsequent elements of  $\mathbf{u}(\mathbf{x}(k))$  are then successively used until some control packet  $\mathbf{u}(\mathbf{x}(k+\ell))$ ,  $\ell \geq 2$ , is successfully received, i.e., no dropout occurs.

**2. Sparse Control Packet Design for Asymptotic Stability.** In the PPC method discussed above, control packets  $\mathbf{u}(\mathbf{x}(k))$  are transmitted at each time  $k \in \mathbb{N}_0$  through an erasure channel. It is often the case that the bandwidth of the channel is limited, and hence one has to compress control packets to a smaller data size. To achieve an efficient compression of packets, we adopt a technique developed in *compressed sensing* [1] that considers the *sparsity* of a signal (or a vector). A sparse vector contains many 0-valued elements and can be highly compressed by only coding

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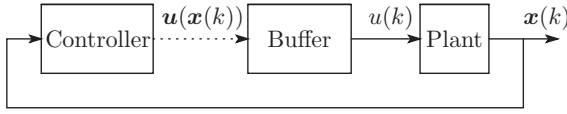


FIG. 1.1. Networked Control System with PPC

its few nonzero components. Based on this notion, in [2] we presented a sparsity-inducing  $\ell^1/\ell^2$  optimization for PPC which gives *practical* stability, i.e., existence of a bounded invariant set. In the present work, we embellish the approach of [2] by showing how to find *sparse* control packets  $\mathbf{u}(\mathbf{x}(k))$  ensuring that the networked system with bounded packet dropouts is *asymptotically stable*. For this, we propose to design the control packets via the following sparsity-promoting optimization:

$$\mathbf{u}(\mathbf{x}(k)) \triangleq \arg \min_{\mathbf{u} \in \mathbb{R}^N} \|\mathbf{u}\|_0 \text{ subject to } \|\mathbf{x}_{N|k}\|_P^2 + \sum_{i=0}^{N-1} \|\mathbf{x}_{i|k}\|_Q^2 \leq \mathbf{x}(k)^\top W \mathbf{x}(k), \quad (2.1)$$

where  $\|\mathbf{u}\|_0$  is the number of the nonzero elements in  $\mathbf{u} = [u_0, u_1, \dots, u_{N-1}]^\top$  and

$$\mathbf{x}_{0|k} = \mathbf{x}, \quad \mathbf{x}_{i+1|k} = A\mathbf{x}_{i|k} + Bu_i, \quad i = 0, 1, \dots, N-1. \quad (2.2)$$

In (2.2),  $N$  is the horizon length, taken here equal to the buffer size. The matrix  $Q > 0$  is a design parameter which allows one to shape the response of the plant state components;  $P > 0$  and  $W > 0$  are chosen such that the feedback system is asymptotically stable, as indicated in Theorem 2.1 below. To state our subsequent results, we introduce the matrices  $\Phi$ ,  $\Upsilon$  and  $\bar{Q}$  by

$$\Phi \triangleq \begin{bmatrix} B & 0 & \dots & 0 \\ AB & B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix}, \quad \Upsilon \triangleq \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad \bar{Q} \triangleq \text{blockdiag}\{\underbrace{Q, \dots, Q}_{N-1}, P\}.$$

**THEOREM 2.1 (Asymptotic Stability).** *Assume that the number of consecutive packet-dropouts is uniformly bounded by  $N$ . Suppose the matrices  $P$ ,  $Q$ , and  $W$  are chosen by the following procedure:*

- i. Choose  $Q > 0$  arbitrarily.
- ii. Solve the following Riccati equation to obtain  $P > 0$ :

$$P = A^\top P A - A^\top P B (B^\top P B)^{-1} B^\top P A + Q. \quad (2.3)$$

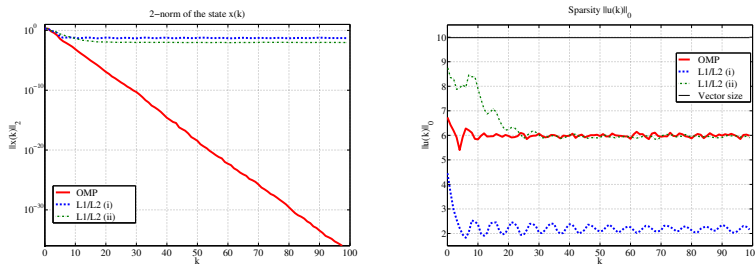
- iii. Compute the constants  $\rho \in [0, 1)$  and  $c > 0$  via

$$\rho \triangleq 1 - \lambda_{\min}(QP^{-1}), \quad c \triangleq \frac{1 - \rho^N}{1 - \rho} \max_i \lambda_{\max}\{\Phi_i^\top P \Phi_i (\Phi^\top \bar{Q} \Phi)^{-1}\},$$

where  $\Phi_i$  is the  $i$ -th column of matrix  $\Phi$ .

- iv. Choose a matrix  $\mathcal{E}$  such that  $0 < \mathcal{E} < (1 - \rho)P/c$ .
- v. Set  $W := P - Q + \mathcal{E}$ .

Then for any choice of the control vector from the feasible set of the optimization (2.1) (which includes the sparsest control packets  $\mathbf{u}(\mathbf{x}(k))$ ), the networked control system is asymptotically stable, that is,  $\lim_{k \rightarrow \infty} \mathbf{x}(k) = \mathbf{0}$ .



(a) Performance  $\|\mathbf{x}(k)\|_2$  (b) Sparsity  $\|\mathbf{u}(\mathbf{x}(k))\|_0$   
 FIG. 4.1. Results by OMP (solid) and  $\ell^1/\ell^2$  optimization (dash)

**3. Orthogonal Matching Pursuit.** The optimization (2.1) can be rewritten as follows:

$$\mathbf{u}(\mathbf{x}(k)) \triangleq \arg \min_{\mathbf{u} \in \mathbb{R}^N} \|\mathbf{u}\|_0 \quad \text{subject to} \quad \|G\mathbf{u} - H\mathbf{x}(k)\|_2^2 \leq \mathbf{x}(k)^\top W\mathbf{x}(k), \quad (3.1)$$

where  $G \triangleq \bar{Q}^{1/2}\Phi$  and  $H \triangleq -\bar{Q}^{1/2}\Upsilon$ . To solve this combinatorial optimization, we adopt an iterative greedy algorithm called *Orthogonal Matching Pursuit* (OMP) [3]. The algorithm is very simple and significantly faster than exhaustive search. Moreover, OMP always gives a vector in the feasible set of the optimization (2.1), and hence the networked control system will be asymptotically stable by Theorem 2.1.

**4. Example.** To show the effectiveness of the proposed method, we run 500 simulations with a fixed 4-th order unstable plant (the poles are  $-1.4396$ ,  $1.0808 \pm 0.6664i$ , and  $0.0220$ ). The packet size  $N$  is 10. We generate a packet dropout at each time  $k$  with dropout probability  $1/2$  (if there have been  $N - 1$  consecutive dropouts, we set the next dropout probability to be 0, i.e., the dropout process is Markovian). Figure 4.1 compares averaged results of the proposed method with that of [2]. Two plots (i) and (ii) are displayed for the method of [2], with different design parameters for sparsity. We can see that the OMP control exhibits asymptotic (even exponential) stability, whereas the  $\ell^1/\ell^2$  method of [2] is only practically stable. The  $\ell^1/\ell^2$  method produces much sparser vectors as in (i) than the OMP formulation, but the system does not asymptotically stable even if one relaxes the sparsity constraint in the  $\ell^1/\ell^2$  optimization as in (ii). In fact, it is proved in [2] that if the controlled plant is unstable, asymptotic stability is impossible by the  $\ell^1/\ell^2$  method. The reason is that the  $\ell^1/\ell^2$  method amounts to considering a fixed upper bound for the inequality constraint in (2.1) or (3.1) instead of time-varying  $\mathbf{x}(k)^\top W\mathbf{x}(k)$ . This leads to  $\mathbf{0}$ -valued control vector when  $\|\mathbf{x}(k)\|_2$  is sufficiently small (note that  $\mathbf{u} = \mathbf{0}$  is the sparsest vector among all vectors).

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