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by

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# Total domination in partitioned graphs

Allan Frendrup\*, Preben Dahl Vestergaard† and Anders Yeo‡

## Abstract

We present results on total domination in a partitioned graph  $G = (V, E)$ . Let  $\gamma_t(G)$  denote the total dominating number of  $G$ . For a partition  $V_1, V_2, \dots, V_k$ ,  $k \geq 2$ , of  $V$ , let  $\gamma_t(G; V_i)$  be the cardinality of a smallest subset of  $V$  such that every vertex of  $V_i$  has a neighbour in it and define the following

$$f_t(G; V_1, V_2, \dots, V_k) = \gamma_t(G) + \gamma_t(G; V_1) + \gamma_t(G; V_2) + \dots + \gamma_t(G; V_k)$$

$$f_t(G) = \max\{f_t(G; V_1, V_2, \dots, V_k) \mid V_1, V_2, \dots, V_k \text{ is a partition of } V\}$$

$$g_t(G) = \max\{\sum_{i=1}^k \gamma_t(G; V_i) \mid V_1, V_2, \dots, V_k \text{ is a partition of } V\}$$

We summarize known bounds on  $\gamma_t(G)$  and for graphs with all degrees at least  $\delta$  we derive the following bounds for  $f_t(G)$  and  $g_t(G)$ .

- (i) For  $\delta \geq 2$  and  $k \geq 3$  we prove  $f_t(G) \leq 11|V|/7$  and this inequality is best possible.
- (ii) for  $\delta \geq 3$  and  $k = 2$  we prove that  $f_t(G) \leq (5/4 - 1/372)|V|$ . That inequality may not be best possible, but we conjecture that  $f_t(G) \leq 7|V|/6$  is.
- (iii) for  $\delta \geq 3$  and  $k = 3$  we prove  $f_t(G) \leq 3|V|/2$  and this inequality is best possible.
- (iv) for  $\delta \geq 3$  and  $k = 2$  the inequality  $g_t(G) \leq 3|V|/4$  holds and is best possible.

**Keywords:** domination; total domination ; partitioned graph; hypergraph

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# 1 Notation

By  $G = (V, E)$  we denote a *graph*  $G$  with *vertex* set  $V = V(G)$  and *edge* set  $E = E(G)$ . The *order* of  $G$  is  $|V(G)| = n$ . For  $x \in V(G)$  we denote by  $N_G(x)$  the set of *neighbours* to  $x$  and  $N_G[x] = \{x\} \cup N_G(x)$ . Indices may be omitted if clear from context. The *degree* of  $x$  is  $d_G(x) = |N_G(x)|$ , the number of neighbours to  $x$ . We let  $\delta(G) = \delta$  denote the *minimum degree* in  $G$  and  $\Delta(G) = \Delta$  the *maximum degree*. A *hypergraph*  $H = (V, E)$  has *vertex set*  $V = V(H)$  and its set of *hyperedges*, or *edges* for short, is  $E = E(H)$ . Each hyperedge  $e$  is a subset of  $V$ ,  $e \subseteq V(H)$ . A vertex  $v$  is *incident* with an edge  $e$  if  $v \in e$ , the *degree* of  $v$  is the number of hyperedges in  $H$  containing  $v$ . We let  $\delta(H) = \delta$  denote the *minimum degree* in  $H$  and  $\Delta(H) = \Delta$  the *maximum degree*.  $H$  is *r-regular* if each vertex has degree  $r$ , i.e.  $d_H(x) = r$ , or equivalently,  $x$  is contained in precisely  $r$  edges.  $H$  is *k-uniform* if each hyperedge contains exactly  $k$  vertices. Two edges  $e_1$  and  $e_2$  are said to be *overlapping* if  $|V(e_1) \cap V(e_2)| \geq 2$ . Let  $Y \subseteq V(H)$  then  $E(Y)$  denotes all hyperedges,  $e$ , contained in  $Y$  (i.e.  $V(e) \subseteq Y$ ).

For a hypergraph  $H$  a *hitting set* or a *transversal*  $\mathcal{T}$  is a set of vertices  $\mathcal{T} \subseteq V(H)$  such that  $e \cap \mathcal{T} \neq \emptyset$  for each hyperedge  $e$  in  $E(H)$ , i.e. each edge  $e$  contains at least one vertex from  $\mathcal{T}$ .  $\mathcal{T}(H)$  denotes the minimum cardinality of a transversal for the hypergraph  $H$ . For sets  $S, T \subseteq V$ , in a graph  $G$  the set  $S$  *totally dominates*  $T$  if every vertex in  $T$  is adjacent to some vertex of  $S$ . The minimum number of vertices needed to totally dominate  $V$  is the *total domination number*  $\gamma_t(G)$ . For a subset  $S$  of  $V$  we let  $\gamma_t(G; S)$  denote the smallest number of vertices in  $G$  which totally dominates  $S$ . A *partition*  $V = (V_1, V_2, \dots, V_k)$  of  $V(G)$  into  $k$  disjoint sets,  $k \geq 2$ , has  $V = \bigcup_{i=1}^k V_i$ ,  $V_i \cap V_j = \emptyset$ ,  $1 \leq i < j \leq k$ . For a partition  $(V_1, V_2, \dots, V_k)$  of  $V$ , we define the following.

$$\begin{aligned} f_t(G; V_1, V_2, \dots, V_k) &= \gamma_t(G) + \gamma_t(G; V_1) + \gamma_t(G; V_2) + \dots + \gamma_t(G; V_k) \\ g_t(G; V_1, V_2, \dots, V_k) &= \gamma_t(G; V_1) + \gamma_t(G; V_2) + \dots + \gamma_t(G; V_k) \end{aligned}$$

We furthermore define  $f_t(G)$  and  $g_t(G)$  as follows.

$$\begin{aligned} f_t(G) &= \max\{f_t(G; V_1, V_2, \dots, V_k) \mid V_1, V_2, \dots, V_k \text{ is a partition of } V\} \\ g_t(G) &= \max\{g_t(G; V_1, V_2, \dots, V_k) \mid V_1, V_2, \dots, V_k \text{ is a partition of } V\} \end{aligned}$$

For further notation we refer to Chartrand and Lesniak [1].

# 2 Introduction

The theory of domination is outlined in two books by Haynes, Hedetniemi and Slater [5, 6]. A combination of domination and partitions is treated by Hartnell and Vestergaard [7], Seager [14], Tuza and Vestergaard [17], Henning and Vestergaard [11]. There has been an upsurge in the study of total domination. New results on total domination are given by Henning, Kang, Shan, Thomassé and Yeo in [10, 12, 15, 18]. In [9] Henning surveys recent results on total domination. Here we shall study total domination in partitioned graphs.

### 3 Bounds on $\gamma_t$

We summarize in Theorem 1 results found by Henning, Thomassé and Yeo.

**Theorem 1** *Let  $G$  be a connected graph with  $n$  vertices and minimum degree  $\delta(G) = \delta$ . Then*

*$\delta \geq 2$  implies  $\gamma_t(G) \leq 4n/7$  for  $G \notin \{C_3, C_5, C_6, C_{10}\}$  ([8, Corollary 6]).*

*$\delta \geq 3$  implies  $\gamma_t(G) \leq n/2$ . ([15]).*

*$\delta \geq 4$  implies  $\gamma_t(G) \leq 3n/7$  ([15]) and there exists some  $\epsilon > 0$  such that  $\gamma_t(G) \leq (3/7 - \epsilon)n$  for  $G \neq G_{14}$ , where  $G_{14}$  is an incidence bipartite graph of order 14 derived from the Fano plane ([19]).*

It is a conjecture that  $\delta \geq 5$  implies  $\gamma_t(G) \leq 4n/11$ .

Theorem 2 and Theorem 3 below, give conditions for equality in Theorem 1.

**Theorem 2** ([9, Theorem 29]) *Let  $G$  be a connected graph of order  $n > 14$  with  $\delta \geq 2$ . Then  $\gamma_t(G) = 4n/7$  if and only if  $G$  can be obtained from a connected graph  $F$  of order at least three by adding  $|V(F)|$  disjoint copies of  $C_6$ , one corresponding to each  $v \in V(F)$ , such that either  $v$  is joined by a new edge to a vertex in its corresponding  $C_6$  or by two new edges to two vertices at distance two apart in its corresponding  $C_6$ .*

The family  $\mathcal{G} \cup \mathcal{H}$  is constructed in [3] as follows. Take two copies  $a_1b_1a_2b_2 \dots a_kb_k$  and  $c_1d_1c_2d_2 \dots c_kd_k$ , of the path  $P_{2k}$ ,  $k \geq 2$ , and add edges  $a_id_i$ ,  $b_ic_i$  for  $i = 1, 2, \dots, k$ . From this the graph of order  $4k$  belonging to the infinite family  $\mathcal{G}$  is obtained by adding  $a_1c_1$  and  $b_kd_k$ , while the graph of order  $4k$  in  $\mathcal{H}$  is obtained by adding  $a_1b_k$  and  $c_1d_k$ . The generalized Petersen graph  $GP_{16}$  is obtained from two circuits  $u_1u_2u_3 \dots u_7u_8$  and  $v_1v_2v_3 \dots v_7v_8$  by addition of edges  $u_1v_1, u_2v_4, u_3v_7, u_4v_2, u_5v_5, u_6v_8, u_7v_3, u_8v_6$ .

**Theorem 3** ([12, Theorem 5]) *Let  $G$  be a connected graph with  $\delta(G) \geq 3$ . Then  $\gamma_t(G) = n/2$  if and only if  $G \in \mathcal{G} \cup \mathcal{H}$  or  $G = GP_{16}$ .*

### 4 $f_t$ for $k$ -partitioned graphs with $\delta \geq 2$

We have that  $f_t$  increases with the number of partition classes, i.e.,  $f_t(G; V_1, V_2, \dots, V_k) \leq f_t(G; V_1, V_2, \dots, V_k, V_{k+1})$ . Therefore we get a weaker inequality if we partition  $V$  into more than two classes. That is demonstrated in Theorem 4 below.

**Theorem 4** *Let  $G$  be a connected graph of order  $n$  with  $\delta(G) \geq 2$  and  $G \notin \{C_3, C_5, C_6, C_{10}\}$ . Let  $V(G) = V$  be partitioned into  $k$  classes,  $k \geq 2$ . Then  $f_t(G) \leq 11n/7$ .*

**If  $k = 2$  then  $f_t(G) \leq 3n/2$ . Equality holds if and only if  $G$  is a circuit of length zero modulo four,  $G = C_{4t}$ ,  $t \geq 1$ .**

If  $k = 3$  then  $f_t(G) \leq 11n/7$ . For  $n > 14$  equality holds if and only if  $G$  can be obtained from a circuit or a path of order at least three by joining each of its vertices by one edge to disjoint copies of  $C_6$ .

If  $k \geq 4$  then  $f_t(G) \leq 11n/7$  and for  $n > 14$  equality holds if and only if  $\Delta(G) \leq k$  and  $G$  can be obtained from a connected graph  $F$  having order at least three and  $g_t(F) = |V(F)|$  by adding disjoint copies of  $C_6$ , one corresponding to each  $v \in V(F)$ , such that either  $v$  is joined by a new edge to one vertex in its corresponding  $C_6$  or by two new edges to two vertices at distance two apart in its corresponding  $C_6$ .

**Proof.** By Theorem 1 we have  $\gamma_t(G) \leq 4n/7$  and assigning to each vertex its own class dominator we have  $g_t(G) \leq n$ . Therefore  $f_t(G) = \gamma_t(G) + g_t(G) \leq 11n/7$ . The result for  $k = 2$  is proven by Frendrup, Henning and Vestergaard in [4, Theorem 2]. For  $k \geq 3$  the equality  $f_t(G) = 11n/7$  implies  $\gamma_t(G) = 4n/7$  and  $g_t(G) = n$  and therefore  $G$  has the structure described in Theorem 2. Since  $g_t(G) = n$  each subgraph  $H$  of  $G$  must satisfy  $g_t(H) = |V(H)|$  and further  $\Delta(G) \leq k$ . Let  $H_1$  be the graph obtained from a circuit  $C_6 : v_1 v_2 \dots v_6$  by adding a new vertex  $x$  and the edge  $xv_1$  and let  $H_2 := H_1 + xv_3$ . Observe for  $k = 3$  that  $g_t(H_1) = |V(H_1)|$  (obtainable from partitioning  $x, v_1, v_2, \dots, v_6$  into classes indexed 1122133 or 1221133) while  $g_t(H_2) < |V(H_2)|$ . For  $k \geq 4$  we can easily show that  $g_t(H_i) = |V(H_i)|$ ,  $i = 1, 2$ . This proves for  $k \geq 3$  that  $f_t(G) = 11n/7$  implies  $G$  has the structure described in this theorem. Conversely, assume first that  $k = 3$  and that  $G$  is obtainable as a disjoint union of  $H_1$ 's with edges added between the vertices named  $x$ , so they span  $F$ , where  $F$  is a path or circuit. We must exhibit a partition of  $V(G)$  proving that  $f_t(G) = 11n/7$ , i.e. that  $g_t(G) = |V(G)|$ . It is easy to find a partition  $V'_1, V'_2, V'_3$  of  $V(F)$  such that  $g_t(F) = |V(F)|$ . If  $k = 3$  we can extend this partition to all the  $H_1$ 's such that the following holds, which proves that  $g_t(G; V'_1, V'_2, V'_3) = n$ .

- $N(x) = N_F(x) \cup \{v_1\}$  contains at most one vertex from each  $V'_1, V'_2, V'_3$  (just put  $v_1$  in the partition set which doesn't contain any of the two vertices in  $N_F(x)$ ).
- $N(v_1) = \{x, v_2, v_6\}$  contains one vertex from each  $V'_1, V'_2, V'_3$  (just put  $v_2$  and  $v_6$  in the partition sets such that this holds).
- $N(v_3), N(v_5) \subset \{v_2, v_4, v_6\}$ , which contains one vertex from each  $V'_1, V'_2, V'_3$  (just put  $v_4$  in the same set as  $x$ ).
- $N(v_2), N(v_4), N(v_6) \subset \{v_1, v_3, v_5\}$ , which contains one vertex from each  $V'_1, V'_2, V'_3$  (just put  $v_3$  and  $v_5$  in the partition sets such that this holds).

Assume next that  $k \geq 4$ . Then a vertex  $x \in F$  may belong to a unit  $H_1$  or  $H_2$ . Again there is a partition  $V'_1, V'_2, \dots, V'_k$  of  $V(F)$  such that  $g_t(F) = |V(F)|$  and similarly to above we can extend this partition to all of  $G$ , such that the neighbourhood of every vertex in  $G$  contains at most one vertex from any partition set. The details are left to the reader. This proves that  $g_t(G) = n$ .  $\square$

## 5 $g_t$ for two-partitioned graphs with $\delta \geq 3$

Chvátal and McDiarmid [2] and Tuza [16] independently established the following result about transversals in hypergraphs (see also Thomassé and Yeo [15] for a short proof of this result).

**Theorem 5** ([2, 16, 15]) *If  $H$  is a hypergraph with all edges of size at least three, then  $\mathcal{T}(H) \leq (|V(H)| + |E(H)|)/4$ .*

**Theorem 6** *Let  $G$  be a graph of order  $n$  with  $\delta \geq 3$  and let  $G$  be partitioned into two classes. Then  $g_t(G) \leq 3n/4$ .*

**Proof.** From the two-partitioned graph  $G$ , we define for  $i = 1, 2$ ,  $H_i$  to be the hypergraph on  $n$  vertices and  $m_i$  edges where  $V(H_i) = V(G)$  and the hyperedges of  $H_i$  are the sets of neighbourhoods of class  $i$  vertices. In other words,  $e \in E(H_i)$  precisely if, for some vertex  $v$  in  $V_i$ ,  $e = N_G(v)$ . Each edge in  $H_i$  has at least three vertices because  $\delta(G) \geq 3$ . In  $G$  we see that a set  $\mathcal{T}_i$  of vertices totally dominates  $V_i$  if and only if  $\mathcal{T}_i$  is a transversal of  $H_i$ . Applying Theorem 5 to  $H_1$  and  $H_2$  separately we obtain transversals  $\mathcal{T}_i$  of  $H_i$ ,  $i = 1, 2$ , satisfying

$$|\mathcal{T}_1| \leq \frac{m_1+n}{4} \quad |\mathcal{T}_2| \leq \frac{m_2+n}{4}.$$

Since  $m_1 + m_2 = n$  we obtain  $|\mathcal{T}_1| + |\mathcal{T}_2| \leq \frac{m_1+n}{4} + \frac{m_2+n}{4} = \frac{3n}{4}$ . This proves Theorem 6.  $\square$

An example of graphs with equality  $g_t(G) = 3n/4$  is given in the next section.

## 6 An infinite family of graphs extremal for Theorem 6

We have the following theorem.

**Theorem 7** *For each integer  $r \geq 1$  there exists a connected bipartite graph  $G_r$  of order  $n = 16r$  with  $\delta(G_r) = 3$  which for  $V(G_r)$  partitioned into two classes has  $g_t(G_r) = 3|V(G_r)|/4$  and  $f_t(G_r) \geq 9|V(G_r)|/8$ .*

**Proof.** We define the graph  $G_r$  as follows. Define the vertex set of  $G_r$  to be  $V(G_r) = W_r \cup A_r \cup B_r$ , where

$$\begin{aligned} W_r &= \{w_0, w_1, w_2, \dots, w_{8r-1}\} \\ A_r &= \{a_0, a_1, a_2, \dots, a_{4r-1}\} \\ B_r &= \{b_0, b_1, b_2, \dots, b_{4r-1}\} \end{aligned}$$

We define the edge set of  $G_r$  such that the following holds, for all  $i \in \{0, 1, 2, \dots, r-1\}$  (where  $b_{-1} = b_{4r-1}$  by definition):

$$\begin{aligned} N(w_{8i}) &= \{a_{4i}, a_{4i+1}, b_{4i}\} & N(w_{8i+1}) &= \{a_{4i}, a_{4i+1}, b_{4i}\} \\ N(w_{8i+2}) &= \{a_{4i}, a_{4i+2}, b_{4i}\} & N(w_{8i+3}) &= \{a_{4i+1}, a_{4i+2}, b_{4i-1}\} \\ N(w_{8i+4}) &= \{a_{4i+2}, b_{4i+1}, b_{4i+2}\} & N(w_{8i+5}) &= \{a_{4i+3}, b_{4i+1}, b_{4i+2}\} \\ N(w_{8i+6}) &= \{a_{4i+3}, b_{4i+1}, b_{4i+3}\} & N(w_{8i+7}) &= \{a_{4i+3}, b_{4i+2}, b_{4i+3}\} \end{aligned}$$

We now assume  $r \geq 1$  is fixed, and therefore omit the subscripts of the above sets and graph. Define  $V_1$  and  $V_2$  as follows.

$$\begin{aligned} V_1 &= A \cup \bigcup_{i=0}^{r-1} \{w_{8i+1}, w_{8i+2}, w_{8i+3}, w_{8i+5}\} \\ V_2 &= B \cup \bigcup_{i=0}^{r-1} \{w_{8i}, w_{8i+4}, w_{8i+6}, w_{8i+7}\} \end{aligned}$$

We will now show that if  $S_i$  is a set such that every vertex in  $V_i$  has a neighbour in  $S_i$ , then  $|S_i| \geq 3|V(G)|/8$ , for  $i = 1, 2$ . This would imply that  $f_t(G) \geq 9|V(G)|/8$  and  $g_t(G) \geq 6|V(G)|/8$  when  $k = 2$  (as clearly the above would also imply that  $\gamma_t(G) \geq 3|V(G)|/8$ ). From Theorem 6 follows that  $g_t(G) = 3|V(G)|/4$ .

Let  $S_1$  be a set that totally dominates  $V_1$  (i.e. every vertex in  $V_1$  has a neighbour in  $S_1$ ). As  $w_{8i+5}$  has a neighbour in  $S_1$  we note that  $|S_1 \cap \{a_{4i+3}, b_{4i+1}, b_{4i+2}\}| \geq 1$ , for all  $i = 0, 1, \dots, r-1$ . As  $w_{8i+1}$ ,  $w_{8i+2}$  and  $w_{8i+3}$  all have a neighbour in  $S_1$  we note that  $|S_1 \cap \{a_{4i}, a_{4i+1}, a_{4i+2}, b_{4i}, b_{4i-1}\}| \geq 2$ , for all  $i = 0, 1, \dots, r-1$  (recall that  $b_{-1} = b_{4r-1}$ ). As the above sets are all disjoint we note that  $|S_1 \cap (A \cup B)| \geq 3|A \cup B|/8$ .

As  $a_{4i+3}$  has a neighbour in  $S_1$  we note that  $|S_1 \cap \{w_{8i+5}, w_{8i+6}, w_{8i+7}\}| \geq 1$ , for all  $i = 0, 1, 2, \dots, r-1$ . As  $a_{4i}$ ,  $a_{4i+1}$  and  $a_{4i+2}$  all have a neighbour in  $S_1$  we note that  $|S_1 \cap \{w_{8i}, w_{8i+1}, w_{8i+2}, w_{8i+3}, w_{8i+4}\}| \geq 2$ , for all  $i = 0, 1, 2, \dots, r-1$ . As the above sets are all disjoint we note that  $|S_1 \cap W| \geq 3|W|/8$ . This implies the desired result for  $S_1$ .

The fact that if  $S_2$  totally dominates  $V_2$ , then  $|S_2| \geq 3|V(G)|/8$  is proved analogously to above. We now just need to show that  $G$  is connected. Let  $Q_i = \{a_{4i}, a_{4i+1}, a_{4i+2}, a_{4i+3}, b_{4i}, b_{4i+1}, b_{4i+2}, b_{4i+3}\}$  and let  $P_i = \{w_{8i}, w_{8i+1}, \dots, w_{8i+7}\}$  for all  $i = 0, 1, 2, \dots, r-1$ . Note that  $G[P_i \cup Q_i]$  is connected. As the edges  $w_{8i+3}b_{4i-1}$ , for all  $i = 0, 1, 2, \dots, r-1$  connects  $P_i$  with  $Q_{i-1}$  ( $Q_{-1} = Q_{r-1}$ ) we are done.  $\square$

## 7 $f_t(G)$ for two-partitioned graphs with $\delta \geq 3$

Let  $G$  be a graph of order  $n$  with  $\delta(G) \geq 3$  and let its vertices be partitioned into two sets.

From Theorems 1 and 6 it follows immediately that  $f_t(G) = \gamma_t(G) + g_t(G) \leq n/2 + 3n/4 = 5n/4$  when  $\delta(G) \geq 3$ . We shall in Theorem 8 below prove a slightly stronger result and later pose an even stronger conjecture.

The following result is known (see for example [13]).

**Lemma 1** ([13]) *If  $G$  is a 3-regular graph, then there exists a matching  $M$  in  $G$ , such that  $|M| \geq \frac{7}{16}|V(G)|$ .*

**Lemma 2** *Let  $H$  be a 2-regular 3-uniform hypergraph with no two edges overlapping. Then  $\mathcal{T}(H) \leq \frac{|V(H)| + |E(H)|}{4} - \frac{|V(H)|}{24}$ .*

**Proof.** Let  $H$  be a 2-regular 3-uniform hypergraph with no overlapping edges. Define the graph  $G_H$  as follows  $V(G_H) = E(H)$  and  $E(G_H) = \{e_1e_2 : |V(e_1) \cap V(e_2)| = 1\}$ . As there are no overlapping edges and  $H$  is 2-regular and 3-uniform, we note that  $G_H$  is a 3-regular graph. By Lemma 1, there exists a matching  $M$  in  $G_H$ , such that  $|M| \geq \frac{7}{16}|V(G_H)|$ .



If  $e_1 e_2 \in M$ , then by the definition of  $G_H$  we note that  $V(e_1) \cap V(e_2) = \{x_{e_1 e_2}\}$  for some  $x_{e_1 e_2} \in V(H)$ . Let  $X = \{x_f \mid f \in M\}$  and note that  $2|M|$  edges in  $H$  contain a vertex from  $X$  (as  $M$  was a matching). Let  $X'$  be a set of vertices of order  $|E(H)| - 2|M|$  containing a vertex from every edge in  $H$ , which does not contain a vertex from  $X$ . Note that  $X \cup X'$  is a transversal of  $H$  of order  $|M| + (|E(H)| - 2|M|)$ . By the above bound on  $|M|$  we get the following, as  $3|E(H)| = \sum_{x \in V(H)} d(x) = 2|V(H)|$ .

$$\begin{aligned} \mathcal{T}(H) &\leq |E(H)| - |M| \leq |E(H)| - \frac{7}{16}|E(H)| \\ &= \frac{|E(H)|}{4} + \frac{5|E(H)|}{16} = \frac{|E(H)|}{4} + \frac{5}{16} \times \frac{2|V(H)|}{3} \\ &= \frac{|V(H)| + |E(H)|}{4} - \frac{|V(H)|}{24} \end{aligned}$$

□

**Lemma 3** *Let  $H$  be a 3-uniform hypergraph, where multiple edges are allowed. For each edge and vertex in  $H$  we assign a non-empty subset of  $\{0, 1, 2\}$ . Let this subset be denoted by  $L(q)$  for all  $q \in V(H) \cup E(H)$ . Let  $H_i$  be the 3-uniform hypergraph containing vertex-set  $V_i = \{v : i \in L(v) \text{ and } v \in V(H)\}$  and edge-set  $E_i = \{e : i \in L(e) \text{ and } e \in E(H)\}$ , for  $i = 0, 1, 2$ . Let  $Y \subseteq V(H)$  be arbitrary and assume that the following holds.*

- (a):  $\Delta(H_1), \Delta(H_2) \leq 2$
- (b):  $\Delta(H - E(Y)) \leq 4$ .
- (c): *There are no overlapping edges in  $H_i$ ,  $i \in \{1, 2\}$ .*
- (d): *If  $e \in E(H) - E(Y)$ , then  $0 \in L(e)$  and  $|L(e)| \geq 2$ .*

*This implies that the following holds.*

$$\sum_{i=0}^3 \mathcal{T}(H_i) \leq \left( \sum_{i=0}^3 \frac{|V_i| + |E_i|}{4} \right) - \frac{|V(H_0) \cap V(H_1) \cap V(H_2) \setminus N_H[Y]|}{372}$$

**Remark.** We assume here in Lemma 3 that the assignment of a set  $L(q)$  to each  $q$  is done such that  $H_0, H_1, H_2$  really are hypergraphs, i.e., such that each hyperedge in  $E_i$  consists of vertices from  $V_i$ ,  $i = 0, 1, 2$ . This requirement will be satisfied in the proof of Theorem 8 where the lemma is applied.

**Proof.** Assume that the lemma is false, and that  $H$  is a counterexample with minimum  $|E_0| + |E_1| + |E_2|$ . Clearly  $|E_0| + |E_1| + |E_2| > 0$ , as otherwise  $\sum_{i=0}^3 \mathcal{T}(H_i) = 0$ . For simplicity we will use the following notation:

$$\begin{aligned} T^* &= \sum_{i=0}^3 \mathcal{T}(H_i) \\ S^* &= \sum_{i=0}^3 \frac{|V_i| + |E_i|}{4} \\ V^* &= V(H_0) \cap V(H_1) \cap V(H_2) \end{aligned}$$

We recall that  $H$  was assumed to be a “minimal” counterexample to  $T^* \leq S^* - (|V^* \setminus N_H[Y]|)/372$ . We will now prove a few claims, which end in a contradiction, thereby proving the lemma. For  $H$  the left hand side of the inequality,  $\ell$ , and the right hand side of the inequality,  $r$ , in Lemma 3 satisfies  $\ell > r$ . We shall construct smaller  $H'$  which also satisfies (a)-(d) and which therefore has  $\ell' \leq r'$  by the minimality of  $H$ .  $H'$  is to be constructed

such that there exist  $\alpha \leq \beta$  for which  $\ell - \alpha \leq \ell'$  and  $r' \leq r - \beta$ . Those inequalities combine to give the desired contradiction  $\ell \leq r$ .

*Claim A:* If we add a vertex to  $Y$ , then  $N[Y]$  does not increase by more than 9 vertices.

*Proof of Claim A:* This follows from the fact that  $H$  is 3-uniform and  $\Delta(H - E(Y)) \leq 4$ , by (b) in the statement of the lemma.

*Claim B:* There is no  $e = \{v_1, v_2, x\} \in E_i$ , such that  $d_{H_i}(v_1) = d_{H_i}(v_2) = 1$  and  $d_{H_i}(x) = 2$ , for  $i = 0, 1, 2$ .

*Proof of Claim B:* Assume that there is such an edge  $e = \{v_1, v_2, x\} \in E_i$ . Let  $e' = \{w_1, w_2, x\}$  be the other edge in  $H_i$  containing  $x$ . Now delete  $v_1, v_2, x, e$  and  $e'$  from  $H_i$  and add  $\{v_1, v_2, x, w_1, w_2\}$  to  $Y$ . Note that (a)-(d) still hold and that  $T^*$  decreases by 1 as we simply add  $x$  to any transversal in the new  $H_i$  in order to get a transversal in the old  $H_i$ . By Claim A the set  $N[Y]$  does not increase by more than 45 vertices. As  $V^*$  does not decrease by more than 3 vertices and  $S^*$  decreases by  $5/4$ , we are done by the “minimality” of  $H$  (as  $\alpha = 1 \leq 5/4 - 48/372 = \beta$  in the argument above Claim A).

*Claim C:* There is no  $e = \{x, v_1, v_2\} \in E_i$ , such that  $d_{H_i}(v_1) = d_{H_i}(v_2) = 2$  and  $d_{H_i}(x) = 1$ , for  $i = 1, 2$ .

*Proof of Claim C:* Assume that there is such an edge  $e = \{x, v_1, v_2\} \in E_i$ . Let  $e_1 = \{w_1, w_2, v_1\}$  be the other edge in  $H_i$  containing  $v_1$  and let  $e_2 = \{u_1, u_2, v_2\}$  be the other edge in  $H_i$  containing  $v_2$ . As there are no overlapping edges in  $H_i$  (by (c) in the statement of the lemma) we note that  $e_1 \neq e_2$  and  $|\{w_1, w_2, u_1, u_2\}| \geq 3$ . Let  $S$  be any subset of  $\{w_1, w_2, u_1, u_2\}$  such that  $|S| = 3$ . We now separately consider the cases when addition of  $S$  as a new hyperedge to  $H_i$  causes overlapping edges in  $H_i$ , and when it doesn't.

Assume that adding  $S$  to  $E_i$  does not cause overlapping edges in  $H_i - e_1 - e_2$ . Now delete  $x, v_1, v_2, e, e_1$  and  $e_2$  from  $H_i$  and add the edge  $S$  to  $H_i$  (and  $H$ ). Furthermore add  $\{x, v_1, v_2, w_1, w_2, u_1, u_2\}$  to  $Y$ . Note that (a)-(d) still hold. If  $T'$  is a transversal in the new  $H_i$  then due to the edge  $S$  we either have  $\{u_1, u_2\} \cap T' \neq \emptyset$ , in which case  $T' \cup \{v_1\}$  is a transversal in the old  $H_i$  or  $\{w_1, w_2\} \cap T' \neq \emptyset$ , in which case  $T' \cup \{v_2\}$  is a transversal in the old  $H_i$ . Therefore  $T^*$  decreases by at most one. By Claim A we have that  $N[Y]$  does not increase by more than 63 vertices. As  $V^*$  does not decrease by more than 3 and  $S^*$  decreases by  $5/4$ , we are done by the “minimality” of  $H$  (as  $1 \leq 5/4 - 66/372$ ).

So now assume that the above addition of  $S$  would cause overlapping edges in  $H_i - e_1 - e_2$ . This can only happen if there is an edge  $e' \in E_i$  such that  $|S \cap V(e')| \geq 2$ . Note that by (a) the degree in  $H_i$  is two for all vertices in  $S \cap V(e')$  (they only lie in  $S$  and  $e'$ ). Now delete the vertices  $\{x, v_1, v_2\} \cup (S \cap V(e'))$  from  $H_i$  and delete the edges  $e, e_1, e_2$  and  $e'$  from  $H_i$  (do not add the edge  $S$  to  $H_i$ ). Furthermore add  $\{x, v_1, v_2, w_1, w_2, u_1, u_2\} \cup (V(e') - S)$  to  $Y$ . Note that (a)-(d) still hold. By a similar argument to above we note that  $T^*$  decreases by at most two. By Claim A we see that  $N[Y]$  does not increase by more than 72 vertices. As  $V^*$  does not decrease by more than 6

and  $S^*$  decreases by at least  $9/4$ , we are done by the “minimality” of  $H$  (as  $2 \leq 9/4 - 78/372$ ).

*Claim D:* There is no  $e = \{x, v_1, v_2\} \in E_0$ , such that  $d_{H_0}(v_1) = d_{H_0}(v_2) = 2$  and  $d_{H_0}(x) = 1$  and  $|N_{H_0}[V(e)]| \geq 6$ .

*Proof of Claim D:* Assume that there is such an edge  $e = \{x, v_1, v_2\} \in E_0$ . Let  $e_1 = \{w_1, w_2, v_1\}$  be the other edge in  $H_0$  containing  $v_1$  and let  $e_2 = \{u_1, u_2, v_2\}$  be the other edge in  $H_0$  containing  $v_2$ . If  $e_1 = e_2$ , then  $|N_{H_0}[V(e)]| \leq 4$ , a contradiction. So assume that  $e_1 \neq e_2$ . As  $|N_{H_0}[V(e)]| \geq 6$  we note that  $|\{w_1, w_2, u_1, u_2\}| \geq 3$ . We are now done analogously to Claim C.

*Claim E:*  $\Delta(H_1), \Delta(H_2) \leq 1$ .

*Proof of Claim E:* Assume that  $\Delta(H_1) \geq 2$ . By (a) we have  $\Delta(H_1) = 2$ . By Claim B and Claim C we note that there is a 2-regular component,  $R$ , in  $H_1$ . There are no overlapping edges in  $R$  by (c). By Lemma 2 there is a transversal  $T_R$  in  $R$  of order at most  $(|V(R)| + |E(R)|)/4 - |V(R)|/24$ . So delete all edges and vertices in  $R$  and add all vertices in  $R$  to  $Y$ . By Claim A we have that  $N[Y]$  increases by at most  $9|V(R)|$  vertices. We now have a contradiction to the “minimality” of  $H$ , as  $|V(R)|/24 \geq 9|V(R)|/372$ . Analogously we can show that  $\Delta(H_2) \leq 1$ .

*Claim F:* Assume  $e_1, e_2 \in E(H_0)$  overlap and  $e_i = (x_1, x_2, u_i)$  for  $i = 1, 2$ , where  $u_1 \neq u_2$ . If  $d_{H_0}(x_1) = d_{H_0}(x_2) = 2$ , then there is an edge  $e' \in E(H_0)$  such that  $\{u_1, u_2\} \subseteq V(e')$ .

*Proof of Claim F:* Let  $e_1$  and  $e_2$  be defined as in the Claim, and assume that there is no edge  $e' \in E(H_0)$  such that  $\{u_1, u_2\} \subseteq V(e')$ . Delete  $e_1, e_2, x_1, x_2$  and  $u_1$  from  $H_0$ . For every edge,  $e''$ , in  $H_0$  that contains  $u_1$ , delete  $e''$  and add the edge  $(e'' - \{u_1\}) \cup \{u_2\}$  instead. Furthermore add  $\{x_1, x_2, u_1, u_2\}$  and  $V(e'')$  from all transformed edges, to  $Y$ . As there is at most 4 edges containing  $u_1$  in  $H_0 - E(Y)$  we note that  $Y$  increases by at most 10 (the neighbours of  $u_1$  in  $H_0 - E(Y)$  and  $\{u_1, u_2\}$ ). Therefore  $V^* - N[Y]$  decreases by at most  $3 + 90$ , by Claim A. We also note that  $S^*$  decreases by  $5/4$ .

We now show that  $T^*$  decreases by at most one. If  $u_2 \in T'$  then  $T' \cup \{u_1\}$  is a transversal in the old  $H_0$ . If  $u_2 \notin T'$  then  $T' \cup \{x_1\}$  is a transversal in the old  $H_0$ . As (a)-(d) still holds after the above operations, we have a contradiction to the “minimality” of  $H$ , as  $1 \leq 5/4 - 93/372$ .

*Definition G:* Let  $x \in V^* - N[Y]$  be arbitrary. The vertex  $x$  exists since otherwise we would be done by Theorem 5.

*Claim H:*  $d_{H_1}(u) = d_{H_2}(u) = 1$  for all  $u \in N_{H_0}[x]$ , where  $x$  is defined in Definition G.

*Proof of Claim H:* Assume that  $u \in N_{H_0}[x]$  has  $d_{H_2}(u) = 0$  or  $u \notin V(H_2)$ , which are the only possibilities for  $u$ , if  $d_{H_2}(u) \neq 1$  (by Claim E). If  $u \in V(H_2)$  and  $d_{H_2}(u) = 0$ , then delete  $u$  from  $V(H_2)$ . We are now done as  $T^*$  is unchanged,  $S^*$  decreases by  $1/4$  and  $V^* - N[Y]$  does not decrease by more than one. So we may assume that  $u \notin V(H_2)$ . Since  $x \in V^*$  we note that  $x \in V(H_1)$  and  $x \in V(H_2)$ , which by the above argument implies that  $d_{H_1}(x) = d_{H_2}(x) = 1$  and  $u \neq x$ . Let  $e_1 = \{x, u, q\}$  be the edge in  $H_1$  (and

$H_0$ ) containing  $u$  and  $x$ . Let  $e_2$  be the edge in  $H_2$  (and  $H_0$ ) that contains  $x$ . Note that  $d_{H_0}(x) = 2$  and  $d_{H_0}(u) = 1$ . If  $d_{H_0}(q) = 1$  then we are done by Claim B. So  $d_{H_0}(q) \geq 2$ . However as any edge containing  $q$  must also lie in  $H_1$  or  $H_2$ , as  $q \notin Y$ , we note that  $d_{H_0}(q) = 2$ . Let  $e_q$  be the edge in  $H_2$  that contains  $q$ . Note that  $e_q \neq e_2$ , by Claim F. As  $e_q$  and  $e_2$  do not intersect we note that  $|N_{H_0}[V(e)]| = 7 \geq 6$ , so we are done by Claim D.

*Claim I:* Let  $e_1 \in E_1$  and  $e_2 \in E_2$  be the edges containing  $x$  (defined in Definition G). They exist by Claim H. Then  $V(e_1) \cap V(e_2) = \{x\}$ .

*Proof of Claim I:* Assume for the sake of contradiction that  $|V(e_1) \cap V(e_2)| \geq 2$ . If  $|V(e_1) \cap V(e_2)| = 3$ , then we delete  $e_1$  from  $H_0$  and add  $V(e_1)$  to  $Y$ . This contradicts the "minimality" of  $H$ , as  $T^*$  remains unchanged,  $S^*$  decreases by  $1/4$  and  $N[Y]$  increases from Claim A by at most 27. Therefore assume that  $|V(e_1) \cap V(e_2)| = 2$ . Let  $e_1 = \{x, v, w\}$  and let  $e_2 = \{x, v, y\}$  where  $w \neq y$ . As  $d_{H_0}(x) = d_{H_0}(v) = 2$ , there is an edge,  $e'$ , in  $H_0$  such that  $\{w, y\} \subseteq V(e')$ , by Claim F. However  $e' \notin E(H_1)$  and  $e' \notin E(H_2)$  by Claim E. This is however a contradiction to (d), as  $w, y \notin Y$ .

*Claim J:* We now obtain a contradiction.

*Proof of Claim J:* Let  $e_1 \in E_1$  and  $e_2 \in E_2$  be the edges containing  $x$  (defined in Definition G). They exist by Claim H and  $V(e_1) \cap V(e_2) = \{x\}$ , by Claim I. Let  $e_1 = \{x, v_1, v_2\}$  and let  $e_2 = \{x, w_1, w_2\}$ . Let  $e'_1$  be the edge in  $H_1$  containing  $w_1$  and let  $e''_1$  be the edge in  $H_1$  containing  $w_2$  (they exist by Claim H). Let  $e'_2$  be the edge in  $H_2$  containing  $v_1$  and let  $e''_2$  be the edge in  $H_2$  containing  $v_2$  (they exist by Claim H).

If  $e'_1 = e''_1$ , then  $V(e'_1) \cap V(e_2) = \{w_1, w_2\}$  and  $e'_1 = \{w_1, w_2, r\}$  for some  $r \in V(H_0)$ . By Claim F, there is an edge in  $H_0$  that contains  $x$  and  $r$ . But this is a contradiction, as neither  $e_1$  or  $e_2$  contain  $r$ , by Claim H. Therefore  $e'_1 \neq e''_1$ . Analogously we can show that  $e'_2 \neq e''_2$ .

We now delete  $e_1, e'_1, e''_1$  from  $H, H_0$  and  $H_1$ . Delete  $e_2, e'_2, e''_2$  from  $H, H_0$  and  $H_2$ . Delete  $V(e_1) \cup V(e'_1) \cup V(e''_1)$  from  $V(H_1)$  and delete  $V(e_2) \cup V(e'_2) \cup V(e''_2)$  from  $V(H_2)$ . Delete  $V(e_1) \cup V(e_2)$  from  $H$  and  $H_0$ . Let  $S_1$  be any subset of size three in  $V(e'_1) \cup V(e''_1) - \{w_1, w_2\}$  and let  $S_2$  be any subset of size three in  $V(e'_2) \cup V(e''_2) - \{v_1, v_2\}$ . Add the edges  $S_1$  and  $S_2$  to  $H$  and  $H_0$ . Finally add all vertices in  $V(e'_1) \cup V(e''_1) \cup V(e'_2) \cup V(e''_2) - \{w_1, w_2, v_1, v_2, x\}$  to  $Y$ .

We first show that  $T^*$  decreases by at most 8. It is clear that the transversal size drops by three in both  $H_1$  and  $H_2$ . So assume that  $T'$  is a transversal of the new  $H_0$ . As in the proof of Claim C we note that one of the three edges  $e_1, e'_2, e''_2$  are already covered by a vertex in  $T'$  (due to  $S_2$ ) and the other two edges can be covered by one additional vertex. Similarly by adding one more vertex to  $T'$  we can make sure that  $e_2, e'_1, e''_1$  are all covered. Therefore the transversal size drops by at most two in  $H_0$ .

Note that  $S^*$  drops by  $33/4$  as we delete 9 vertices in each of  $H_1$  and  $H_2$  and we delete 5 vertices in  $H_0$ . We also delete three edges in each of  $H_1$  and  $H_2$  and six edges in  $H_0$ . But we also add two edges in  $H_0$ .

$N[Y]$  increases by at most 72 vertices by Claim A, as  $|V(e'_1) \cup V(e''_1) \cup V(e'_2) \cup V(e''_2) - \{w_1, w_2, v_1, v_2, x\}| \leq 8$ . As  $V^*$  decreases by at most 13, we note that  $V^* - N[Y]$  decreases by at most 85. We note that (a)-(d) still

holds after the above operations. We therefore have a contradiction to the "minimality" of  $H$ , as  $8 \leq 33/4 - 85/372$ .  $\square$

**Theorem 8** *If  $G$  is a graph with  $\delta(G) \geq 3$  and two partition classes then  $f_t(G) \leq (\frac{5}{4} - \frac{1}{372})|V(G)|$ .*

**Proof.** Let  $G$  be any graph with  $\delta(G) \geq 3$  and let  $(W_1, W_2)$  be a partition of  $V(G)$ . Define the hypergraph  $H_G$ , such that  $V(H_G) = V(G)$  and  $E(H_G)$  is obtained by selecting for each  $v \in V(G)$  one set of three vertices from  $N_G(v)$  to form a hyperedge.  $E(H_G) = \{e_v : v \in V(G)\}$ ,  $e_v = \{x_v, y_v, z_v\} \subseteq N_G(v)$ . Furthermore for every hyperedge,  $e \in E(H_G)$  let  $L(e)$  be the set  $\{0, i\}$  if  $v \in W_i$ . For reasons which will be clear later we let  $L(v) = \{0, 1, 2\}$  for every  $v \in V(H_G)$ . Let  $H_i$  be the 3-uniform hypergraph containing vertex-set  $V_i = \{v : i \in L(v) \text{ and } v \in V(H)\}$  and edge-set  $E_i = \{e : i \in L(e) \text{ and } e \in E(H)\}$ , for  $i = 0, 1, 2$ . Note that a transversal of  $H_0$  corresponds to a total dominating set in  $G$  and a transversal of  $H_i$  ( $i \in \{1, 2\}$ ) corresponds to a total dominating set in  $G$  of the set  $W_i$ . Therefore we would be done if we could show that  $\mathcal{T}(H_0) + \mathcal{T}(H_1) + \mathcal{T}(H_2) \leq (\frac{5}{4} - \frac{1}{372})|V(G)|$ . Let  $Y$  be an empty set. We note that  $|E_1| + |E_2| = |E_0| = |V_0| = |V_1| = |V_2| = |V(H_0) \cap V(H_1) \cap V(H_2) \setminus N_H[Y]| = |V(G)|$  and therefore the inequality above is equivalent to

$$(*) \quad \sum_{i=0}^3 \mathcal{T}(H_i) \leq \left( \sum_{i=0}^3 \frac{|V_i| + |E_i|}{4} \right) - \frac{|V(H_0) \cap V(H_1) \cap V(H_2) \setminus N_H[Y]|}{372}$$

For simplicity we will use the following notation:

$$\begin{aligned} T^* &= \sum_{i=0}^3 \mathcal{T}(H_i) \\ S^* &= \sum_{i=0}^3 \frac{|V_i| + |E_i|}{4} \\ V^* &= V(H_0) \cap V(H_1) \cap V(H_2) \end{aligned}$$

We will now do a few transformations on  $H, H_0, H_1, H_2$ .

*Transformation 1:* While there is some vertex  $x \in V(H)$  with  $d_{H_0}(x) \geq 5$  (or equivalently  $d_H(x) \geq 5$ ), delete  $x$  and all edges incident with  $x$  from  $H$  (and therefore also from  $H_0, H_1$  and  $H_2$ ).

*Claim A:* If  $(*)$  holds for the resulting hypergraphs, then it also holds for our original hypergraphs.

*Proof of Claim A:* We note that  $T^*$  drops by at most three, as we may place  $x$  in the transversal of the new  $H_i$ 's in order to get transversals in the old  $H_i$ 's. We note that  $S^*$  decreases by at least  $13/4$ , as we delete  $x$  from  $H_0, H_1, H_2$  and 5 edges from  $H_0$  plus a total of 5 edges from  $H_1$  and  $H_2$ . As  $V^*$  decreases by one and  $N_H[Y] = \emptyset$  remains unchanged, we are done.

*Transformation 2:* While there is a vertex  $x \in V(H)$  with  $d_{H_1}(x) \geq 3$ , delete  $x$  and all edges incident to  $x$  from  $H_0$  and  $H_1$ . Also delete these edges from  $H$  (but do not delete  $x$  or any edges incident to  $x$  in  $H_2$ ). If  $d_{H_2}(x) = 0$  then delete  $x$  from  $H_2$  (i.e. delete 2 from  $L(x)$ ). If  $d_{H_2}(x) > 0$  then note that  $d_{H_2}(x) = 1$  (as we have performed transformation 1 as long as we could) and put  $N_{H_2}[x]$  in  $Y$ .

*Claim B: If (\*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.*

*Proof of Claim B:* We note that  $T^*$  drops by at most two, as we may place  $x$  in the transversal of the new  $H_0$  and  $H_1$  in order to get transversals in the old  $H_0$  and  $H_1$ . We note that  $S^*$  decreases by at least  $9/4$ , as we delete 3 edges and 1 vertex from  $H_0$  and  $H_1$  and we either delete a vertex in  $H_2$  or 4 edges from  $H_0$ . As  $V^*$  decreases by one and  $N_H[Y]$  increases by at most 21 (as  $\Delta(H) \leq 4$ , after Transformation 1), we are done.

*Transformation 3:* While there is a vertex  $x \in V(H)$  with  $d_{H_2}(x) \geq 3$ , then do the following. Delete  $x$  and all edges incident to  $x$  from  $H_0$  and  $H_2$ . Also delete these edges from  $H$  (but do not delete  $x$  or any edges incident to  $x$  in  $H_1$ ). Furthermore delete any vertices in  $H_2$ , which get degree zero by the above transformation. If  $d_{H_1}(x) = 0$  then delete  $x$  from  $H_1$ . If  $d_{H_1}(x) > 0$ , then we put  $N_{H_1}[x]$  in  $Y$ .

*Claim C: If (\*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.*

*Proof of Claim C:* We note that  $T^*$  drops by at most two, as we may place  $x$  in the transversal of the new  $H_0$  and  $H_2$  in order to get transversals in the old  $H_0$  and  $H_2$ . Let's count any edge,  $e$ , in  $H_1$ , which does not lie in  $H_0$  as contributing  $1 + |V(e) \cap V(H_0)|/3$  to the sum  $S^*$ . We note that there are no such edges when we start the transformation 3's.

We note that  $S^*$  now decreases by at least  $25/12$ , because of the following. For every edge containing  $x$  in  $H_2$ , which does not lie in  $H_0$  there is a vertex of degree one in the edge, due to the above transformations. Therefore we either delete an edge in  $H_0$  or a vertex in  $H_2$  for each of the edges containing  $x$  in  $H_2$ . As we also delete the edges in  $H_2$  and the vertex  $x$  in  $H_0$  and  $H_2$  we note that  $S^*$  drops by at least  $8/4$ . So if  $d_{H_1}(x) = 0$  then  $S^*$  decreases by at least  $9/4$  as claimed. If  $d_{H_1}(x) > 0$  and the edge,  $e$ , containing  $x$  in  $H_1$  also lies in  $H_0$ , then we are done as we delete an extra edge in  $H_0$  and the edge left in  $H_1$  is counted as at most  $1 + 2/3$ . If  $d_{H_1}(x) > 0$  and the edge,  $e$ , containing  $x$  in  $H_1$  does not lie in  $H_0$ , then we decrease the value of  $e$  by  $1/3$  as  $1 + |V(e) \cap V(H_0)|/3$  decreases. This shows that  $S^*$  decreases by at least  $25/12$ .

As  $V^*$  decreases by one and  $N[Y]$  increases by at most 21 (as  $\Delta(H) \leq 4$ , after Transformation 1), we are done.

*Transformation 4:* If  $e_1, e_2 \in E(H_i)$  and  $|V(e_1) \cap V(e_2)| \geq 2$  for some  $i \in \{1, 2\}$ , then we do the following.

If  $|V(e_1) \cap V(e_2)| = 3$ , then if  $e_1, e_2 \in E_0$  we delete  $e_2$  from both  $H_0$  and  $H_i$ . If  $e_j \notin E_0$  ( $j \in \{1, 2\}$ ) then we delete  $e_j$  from  $H_i$  (in this case  $V(e_j) \subseteq Y$ ). So now assume that  $|V(e_1) \cap V(e_2)| = 2$  and  $e_1 = (u_1, x, y)$  and  $e_2 = (u_2, x, y)$ , where  $u_1 \neq u_2$ ,

If  $d_{H_i}(u_1) = d_{H_i}(u_2) = 2$ , then by the above transformations we note that  $e_1, e_2 \in E_0$ . We now add a new vertex  $q$  to  $H$ ,  $H_0$  and  $H_i$ . We delete  $e_1$  and  $e_2$  from  $H$ ,  $H_i$  and  $H_0$  and add the edges  $\{q, x, y\}$  to  $H$ ,  $H_i$  and  $H_0$ .

If  $d_{H_i}(u_j) = 1$ , for some  $j \in \{1, 2\}$ , then do the following. Delete  $e_1, e_2$  and the vertices  $\{u_j, x, y\}$  from  $H_i$ . Add the vertices  $\{u_1, u_2, x, y\}$  to  $Y$ .

*Claim D: If (\*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.*

*Proof of Claim D:* In the case when  $|V(e_1) \cap V(e_2)| = 3$  we note that  $T^*$  remains unchanged,  $S^*$  decreases by  $1/4$  and  $V^* - N[Y]$  remains unchanged. We are now done with this case.

In the case when  $d_{H_i}(u_1) = d_{H_i}(u_2) = 2$ , we note that  $T^*$ ,  $S^*$  and  $V^*$  remain unchanged and  $N[Y]$  can only grow by adding  $q$  to it, but  $q \notin V^*$ . We also note that the above transformation decreases the number of edges in  $H_i$ , so it cannot continue indefinitely. We are now done with this case.

In the case when  $d_{H_i}(u_j) = 1$ , we note that  $T^*$  decreases by at most one,  $S^*$  decreases by  $5/4$ ,  $V^*$  decreases by at most three and  $N[Y]$  increases by at most 24 (In  $H - e_1 - e - 2$  we note that  $u_1$  and  $u_2$  have degree at most 3 while  $x$  and  $y$  have degree at most 2). As  $1/4 \geq 27/372$  we are done with this case.

*Claim E:  $\Delta(H_1), \Delta(H_2) \leq 2$  and  $\Delta(H - E(Y)) \leq 4$  and there are no overlapping edges in  $H_i$ ,  $i \in \{1, 2\}$ .*

*Proof of Claim E:* The fact that  $\Delta(H_1), \Delta(H_2) \leq 2$  follow from Transformations 2 and 3. As  $\Delta(H) \leq 4$  after Transformation 1 and no other transformation increases  $\Delta(H)$ , we note that  $\Delta(H - E(Y)) \leq \Delta(H) \leq 4$ . There are no overlapping edges in  $H_i$ ,  $i \in \{1, 2\}$  due to Transformation 4.

*Claim F: If  $e \in E(H) - E(Y)$ , then  $0 \in L(e)$  and  $|L(e)| \geq 2$ .*

*Proof of Claim F:* This was true before Transformation 1 as it was true for all edges. Transformation 1 clearly does not change this property. In Transformation 2, we only keep an edge,  $e$ , in  $H_i$ , where  $i \in \{1, 2\}$  but delete it in  $H_0$  if we put  $V(e)$  in  $Y$ . So the above still holds after Transformation 2. Analogously it also holds after Transformation 3. It is not difficult to check that it also holds after Transformation 4 (note that the above property holds for the edge we might add to  $H$  in Transformation 4).

We now see that (\*) holds due to Lemma 3. That implies the theorem.

□

## 8 Possible strengthening of Theorem 8

No graph extremal for Theorem 8 is known and probably an inequality  $f_t(G) \leq \alpha|V(G)|$  can be obtained for some  $\alpha$  smaller than  $\frac{5}{4} - \frac{1}{372}$ . Certainly  $\alpha$  must be at least  $9/8$ , that is demonstrated by the graphs of section 6.

There is a graph of order 12 having  $f_t(H_{12}) = 7n/6$ , namely  $H_{12}$  from the family  $\mathcal{H}$  defined after Theorem 2, with the two  $P_6$ 's as its partition classes. Unless we, e.g., demand that the order of the graphs be large,  $H_{12}$  shows that we cannot get a better inequality than the following conjecture.

**Conjecture 1** *Let  $G$  be a graph of order  $n$  with  $\delta \geq 3$  and let  $V(G)$  be partitioned into two classes. Then  $f_t(G) \leq 7n/6$ .*

## 9 Three partition classes

**Theorem 9** *Let  $G$  be a graph of order  $n$  with  $\delta \geq 3$  and let  $V(G)$  be partitioned into three classes. Then  $f_t(G) \leq 3n/2$ .*

*For arbitrarily large  $n$ ,  $n \equiv 0 \pmod{6}$ , there exist graphs  $G_n$  with  $g_t(G_n) = n$ ,  $\gamma_t(G_n) = n/3$ ,  $f_t(G) = 4n/3$ .*

**Proof.** By Theorem 1 we have that  $\gamma_t(G) \leq n/2$ , and  $g_t(G) \leq n$  holds trivially, so by addition we get  $f_t(G) \leq 3n/2$  as desired.

Assume a graph  $G$  has  $g_t(G) = n$ . Then  $\Delta(G) \leq 3$  and as  $\delta(G) \geq 3$ ,  $G$  is cubic. Since each vertex has three neighbours, one in each partition class, we see for each  $i = 1, 2, 3$ , that vertices in class  $V_i$  span a matching in  $G$ .

Listing the 3 neighbours to each  $V_i$ -vertex we count each vertex of  $G$  once, so  $3|V_i| = n$  giving  $|V_1| = |V_2| = |V_3| = n/3$ .

Each  $V_1$ -vertex is adjacent to precisely one  $V_2$ -vertex and that has no other  $V_1$ -neighbour, so there is a perfect matching of  $V_1V_2$ -edges and analogously  $G$  contains perfect matchings of  $V_1V_3$ - and  $V_2V_3$ -edges.

One partition class  $V_i$  totally dominates  $G$  so  $\gamma_t(G) \leq n/3$ . In fact,  $\gamma_t(G) = n/3$  because each vertex in  $G$  can totally dominate at most its three neighbours.

Following the steps above, it is now easy for  $n \equiv 0 \pmod{3}$  to construct a graph  $G_n$  with  $g_t(G_n) = n$ . This graph has  $f_t(G_n) = \gamma_t(G_n) + g_t(G_n) = 4n/3$ .  $\square$

We do not know if there, for  $k = 3, \delta \geq 3$ , are graphs  $G$  with  $4n/3 < f_t(G) \leq 3n/2$ , but we pose the following conjecture.

**Conjecture 2** *There exists some positive  $\epsilon$  such that the following holds. If  $G$  is a graph with  $\delta(G) \geq 3$  and the vertices of  $G$  are partitioned into three classes, then  $f_t(G) \leq (3/2 - \epsilon)|V(G)|$ .*

**Theorem 10** *Let  $G$  be a graph of order  $n$  with  $\delta \geq 3$ , partitioned into at least four classes. Then  $f_t(G) \leq 3n/2$  and there exists an infinite family of graphs with  $f_t(G) = 3n/2$ .*

**Proof.** The inequality is proven as in Theorem 9. For a graph with  $f_t(H) = 3n/2$ , consider a graph  $H \in \mathcal{H}$  ( $\mathcal{H}$  is defined after Theorem 2). Let  $v_1, v_2, \dots, v_{n/2}$  and  $u_1, u_2, \dots, u_{n/2}$  be two disjoint paths in  $H$  such that  $\{v_1u_2, v_2u_1, v_1v_{n/2}, u_1u_{n/2}\} \subseteq E(H)$ . Let  $V_1, V_2, V_3, V_4$  be a partition of  $H$  such that  $l(v_1), l(v_2), \dots, l(v_{n/2}), \dots = 1, 2, 3, 4, 1, 2, 3, 4, \dots$  and  $l(u_1), l(u_2), \dots, l(u_{n/2}), \dots = 4, 3, 2, 1, 4, 3, 2, 1, \dots$  where  $l(x) = i$  if  $x \in V_i$ , then  $f_t(H; V_1, V_2, V_3, V_4) = 3n/2$ .  $\square$

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