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Correction to “Packetized Predictive Control of Stochastic Systems over Bit-Rate Limited Channels with Packet Loss”

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Abstract—We correct the results in Section V of the above mentioned manuscript.

In [1], we showed that a particular class of networked control system (NCS) with quantization, i.i.d. dropouts and disturbances can be described as a Markov jump linear system of the form

\[
\theta_{k+1} = \tilde{A}(d_k)\theta_k + \tilde{B}(d_k)\nu_k, \quad (1)
\]

where

\[
\theta_k \triangleq \begin{bmatrix} w_k \\ b_{k-1} \end{bmatrix} \in \mathbb{R}^{n+N}, \quad \nu_k \triangleq \begin{bmatrix} u_k \\ n_k \end{bmatrix} \in \mathbb{R}^{m+N}
\]

and \(\{d_k\}_{k \in \mathbb{N}}\) is a Bernoulli dropout process, with

\[
\text{Prob}(d_k = 1) = p \in (0,1).
\]

Throughout [1] we showed that properties of the NCS can be conveniently stated in terms of the expected system matrices

\[
\bar{A} = \begin{bmatrix} A & \nu \end{bmatrix}, \quad B = \begin{bmatrix} B_u & B_n(p) \end{bmatrix}
\]

and the matrix \(\tilde{A} = \begin{bmatrix} A & \nu \end{bmatrix} - A \cdot \tilde{A}(0).\) Unfortunately, Theorem 4 in Section V-A of [1] is incorrect. For white disturbances \(\{w_k\}_{k \in \mathbb{N}}\) can be accommodated by using standard state augmentation techniques; see, e.g., [2].

**Theorem 4:** Suppose that (1) is MSS and AWSS and that \(\{w_k\}_{k \in \mathbb{N}}\) is white with \(\sigma_w^2 = \text{tr} R_w(0).\) Define

\[
\mathcal{F}(z) \triangleq (zI - \bar{A}(p))^{-1}
\]

\[
\mathcal{C}(p) \triangleq (\sigma_w^2/m)B_uB_u^T + (\sigma_n^2/N)(1-p)\mathcal{E} \in \mathbb{R}^{(n+N) \times (n+N)},
\]

where \(\text{see [1, Sec.2] for definitions}\)

\[
\mathcal{E} \triangleq \frac{B_u(p)B_u(p)^T}{(1-p)^2} = \begin{bmatrix} B_1e_1^T(\Psi^T\Psi)^{-1} & B_1e_1^T(\Psi^T\Psi)^{-1} \\ B_1e_2^T(\Psi^T\Psi)^{-1} & B_1e_2^T(\Psi^T\Psi)^{-1} \end{bmatrix},
\]

(3)

Then, the spectral density of \(\{\theta_k\}_{k \in \mathbb{N}}\) is given by

\[
S_{\theta}(e^{j\omega}) = \mathcal{F}(e^{j\omega})((\sigma_w^2/m)\mathcal{K}_w\mathcal{K}_w^T + (\sigma_n^2/N)\mathcal{K}_n\mathcal{K}_n^T)\mathcal{F}^T(e^{-j\omega}),
\]

where \(R_w(0)\) solves the following linear matrix equation:

\[
R_w(0) = \bar{A}(p)R_w(0)\bar{A}(p)^T + (1-p)\tilde{A}R_w(0)\tilde{A}^T + \mathcal{C}(p),
\]

(5)

Proof: See the appendix.

To further elucidate the situation, we note that (5) is linear and that its solution can be stated as the linear combination

\[
R_w(0) = (\sigma_w^2/m)R_w(0) + (\sigma_n^2/N)R_n(0),
\]

where \(R_w(0)\) and \(R_n(0)\) satisfy

\[
R_w(0) = \bar{A}(p)R_w(0)\bar{A}(p)^T + (1-p)\tilde{A}R_w(0)\tilde{A}^T + B_wB_w^T
\]

\[
R_n(0) = \bar{A}(p)R_n(0)\bar{A}(p)^T + (1-p)\tilde{A}R_n(0)\tilde{A}^T + (1-p)\mathcal{E}.
\]

Therefore, the distortion \(D\) defined by (52) in [1] is given by

\[
D \triangleq \text{tr}(\tilde{Q}R_w(0)) + \lambda[0 \ e_1^T]R_n(0)[0 \ e_1^T]^T,
\]

where \(\tilde{Q}\) is given in terms of the Kronecker product

\[
\tilde{Q} \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \mathcal{Q}.
\]

Thus, \(D = \alpha \sigma_n^2 + \beta,\) with

\[
\alpha = (1/N)\text{tr}(\tilde{Q}R_w(0)) + (\lambda/N)[0 \ e_1^T]R_n(0)[0 \ e_1^T]^T
\]

\[
\beta = (\sigma_n^2/m)\text{tr}(\tilde{Q}R_n(0)) + (\lambda \sigma_w^2/m)[0 \ e_1^T]R_n(0)[0 \ e_1^T]^T.
\]

The above expressions replace Lemma 11 of [1].

To derive a noise-shaping model, (6) can be substituted into into (4) to provide

\[
S_{\theta}(e^{j\omega}) = \mathcal{F}(e^{j\omega})((\sigma_w^2/m)\mathcal{K}_w\mathcal{K}_w^T + (\sigma_n^2/N)\mathcal{K}_n\mathcal{K}_n^T)\mathcal{F}^T(e^{-j\omega}),
\]

where \(\mathcal{K}_w\) and \(\mathcal{K}_n\) are obtained from the factorizations

\[
\mathcal{K}_w\mathcal{K}_w^T = B_wB_w^T + (1-p)\tilde{A}R_w(0)\tilde{A}^T
\]

\[
\mathcal{K}_n\mathcal{K}_n^T = (1-p)\mathcal{E} + p\tilde{A}R_n(0)\tilde{A}^T
\]

If we define

\[
\mathcal{H}(z) \triangleq \begin{bmatrix} I & 0 \end{bmatrix} \mathcal{F}(z),
\]

then the above provides the noise-shaping model depicted in Fig. 2. The latter replaces Fig. 2 and Corollary 1 of [1].

**Remark 1:** We would like to emphasize that Theorem 4 can also be proven by adapting results in [3]–[5]. However, the noise shaping interpretation in Fig. 2 does not explicitly need an additional noise term to quantify second-order dropout effects, as opposed to what is done in [3]–[5].

The upper bound on the coding rate provided by Theorem 5 in [1] is also no longer correct, since it relied upon \(R_0(0)\). The new Theorem 5 is provided below:

**Theorem 5:** For any \(1 \leq N \in \mathbb{N},\) the minimum bit-rate \(R\) of \(\tilde{u}_k\) satisfies

\[
R(D) \leq \frac{1}{2} \log_2 \left( \frac{\text{det}(I + (N/\sigma_n^2)R_0(0))}{N} \right) + \frac{N}{2} \log_2 \left( \frac{\hat{e}^2}{6} \right) + 1,
\]

where

\[
R_0(0) = \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix} R_0(0) \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix}^T.
\]

Proof: Follows immediately from (73) in [1] by omitting the last step where \(R_0(0)\) was written in terms of \(R_0(0)\) and (50) was used.

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By using results in [6, Sec.5], the covariance matrix

$$
\begin{align*}
\lim_{\sigma^2 \to \infty} R(D) & \leq \frac{1}{2} \log_2 \left( \det(I + [\Gamma \ 0]R_0(0)[\Gamma \ 0]^T) \right) \\
& + \frac{N}{2} \log_2 \left( \frac{\pi e}{6} \right) + 1,
\end{align*}
$$

expression, which is positively bounded away from zero and replaces (58) in [1].

**Remark 2:** By using results in [6, Sec.5], the covariance matrix $R_0(0)$ can be expressed explicitly in terms of Kronecker products and matrix inversions. Specifically, let

$$
G \triangleq A(p) \otimes A(p)^T + p(1-p)\tilde{A} \otimes \tilde{A}^T
$$

and let $c \in \mathbb{R}^{(n+N)^2}$ be the vectorized version of the matrix $C(p)$ given in (2). Then, the vectorized version of $R_0(0)$ is simply given by $r = (I - G)^{-1}c$. Using this approach, it is straightforward to numerically evaluate the rate and distortion in (7).

We finalize this note by revisiting the NCS considered in Section V-C of [1]. Fig. 3 illustrates the rate and distortion trade-off for different horizon lengths and a fixed packet loss probability $p = 0.0085$. It may be noticed that the distortion can be reduced by using a longer horizon length in addition to increasing the bit-rate. Fig. 4 shows that when the packet-loss probability increases, it is necessary to use a larger horizon length to guarantee stability and thereby reduce the distortion.

**REFERENCES**


**APPENDIX**

**Proof of Theorem 4**

Since $\{\nu_k\}_{k \in \mathbb{N}_0}$ is white and thus $\mathbb{E}\{\theta_k \nu_k^T\} = 0$, the system recursion (1) provides

$$
\begin{align*}
\mathbb{E}\{\theta_{k+1} \theta_{k+1}^T\} &= \mathbb{E}\{\tilde{A}(d_k)\theta_k \theta_k^T \tilde{A}(d_k)^T\} + \mathbb{E}\{\tilde{B}(d_k)\nu_k \nu_k^T \tilde{B}(d_k)^T\}.
\end{align*}
$$

Therefore, by conditioning on $d_k$ and using the law of total expectation, we obtain:

$$
\begin{align*}
\mathbb{E}\{\theta_{k+1} \theta_{k+1}^T\} &= \mathbb{E}\{\tilde{A}(d_k)\theta_k \theta_k^T \tilde{A}(d_k)^T\} | d_k = 1) \\
&+ (1-p)\mathbb{E}\{\tilde{A}(d_k)\theta_k \theta_k^T \tilde{A}(d_k)^T\} | d_k = 0) \\
&+ p\mathbb{E}\{\tilde{B}(d_k)\nu_k \nu_k^T \tilde{B}(d_k)^T\} | d_k = \{1\} \\
&+ (1-p)\mathbb{E}\{\tilde{B}(d_k)\nu_k \nu_k^T \tilde{B}(d_k)^T\} | d_k = \{0\} \\
&= p\tilde{A}(1)\mathbb{E}\{\theta_0 \theta_0^T\} \tilde{A}(1)^T + (1-p)\tilde{A}(0)\mathbb{E}\{\theta_0 \theta_0^T\} \tilde{A}(0)^T \\
&+ p\tilde{B}(1)\mathbb{E}\{\nu_0 \nu_0^T\} \tilde{B}(1)^T + (1-p)\tilde{B}(0)\mathbb{E}\{\nu_0 \nu_0^T\} \tilde{B}(0)^T,
\end{align*}
$$

where we have used the fact that $\{d_k\}_{k \in \mathbb{N}_0}$ is Bernoulli and $\nu_k$ and $\theta_k$ are independent of $d_k$. Direct algebraic manipulations allow us to
rewrite the above as
\[ E\{\theta_{k+1}\theta_k^T\} = A(p)E\{\theta_k\theta_k^T\}A(p)^T + p(1-p)\tilde{A}E\{\theta_k\theta_k^T\}\tilde{A}^T + C(p). \quad (8) \]

In a similar way, one can derive that
\[ E\{\theta_{k+\ell+1}\theta_k^T\} = E\{(\tilde{A}(d_{k+\ell})\theta_{k+\ell} + \tilde{B}(d_{k+\ell})\nu_{k+\ell})\theta_k^T\} \]
\[ = E\{\tilde{A}(d_{k+\ell})\theta_{k+\ell}\theta_k^T\} + E\{\tilde{B}(d_{k+\ell})\nu_{k+\ell}\theta_k^T\} \]
\[ = A(p)E\{\theta_{k+\ell}\theta_k^T\} + B(p)E\{\nu_{k+\ell}\theta_k^T\} \]
\[ = A(p)E\{\theta_{k+\ell}\theta_k^T\}, \quad \forall \ell \in \mathbb{N}_0, \quad (9) \]
since \( \nu_k \) is white and \( \theta_k \) and \( \theta_{k+\ell} \) are independent of \( d_{k+\ell} \) for non-negative values of \( \ell \). Equation (9) gives the explicit expression
\[ E\{\theta_{k+\ell}\theta_k^T\} = A(p)^\ell E\{\theta_k\theta_k^T\}, \quad \forall \ell \in \mathbb{N}_0. \quad (10) \]

Since the system is AWSS, we have \( \lim_{\ell \to \infty} E\{\theta_{k+1}\theta_{k+1}^T\} = R_0(0) \), the stationary covariance matrix of \( \{\theta_k\}_{k \in \mathbb{N}_0} \). By (8) and results in [7], [8], the latter is given by the solution to (5).

On the other hand, in steady state, (10) gives that the covariance function
\[ R_0(\ell) = A(p)^\ell R_0(0), \quad \forall \ell \in \mathbb{N}_0. \quad (11) \]
Consequently, the positive real part of the spectrum of \( \{\theta_k\}_{k \in \mathbb{N}_0} \) is given by
\[ S_\theta^+(z) = \frac{1}{2} R_0(0) + \sum_{\ell=1}^{\infty} R_0(\ell)z^{-\ell} \]
\[ = (1/2)I + A(p)(zI - A(p)^{-1})R_0(0), \]
where we have used the fact that, by assumption, (1) is MSS and AWSS, thus \( A(p) \) is Schur (see Lemma 4 in [1]) and the geometric series
\[ \sum_{n=0}^{\infty} (A(p)z^{-1})^n = (I - A(p)z^{-1})^{-1}. \]

Since \( \{\theta_k\}_{k \in \mathbb{N}_0} \) is AWSS, its spectrum satisfies [9]
\[ S_\theta(z) = S_\theta^+(z) + (S_\theta^+(z^{-1}))^T \]
\[ = R_0(0) + A(p)(zI - A(p)^{-1})R_0(0) \]
\[ + R_0(0)(z^{-1}I - A(p)^{-1})^{-1}R_0(0) \]
\[ + A(p)R_0(0)(z^{-1}I - A(p)^{-1})^{-1}R_0(0)A(p)^T, \]

Therefore, we have
\[ (zI - A(p))S_\theta(z)(z^{-1}I - A(p))^T \]
\[ = (zI - A(p))R_0(0)(z^{-1}I - A(p))^T \]
\[ + (zI - A(p))A(p)(zI - A(p)^{-1}R_0(0)(z^{-1}I - A(p))^T \]
\[ + (zI - A(p))R_0(0)(z^{-1}I - A(p)^{-1}R_0(0)(z^{-1}I - A(p))^T \]
\[ = (zI - A(p))^\ell R_0(0)(z^{-1}I - A(p))^T \]
\[ + A(p)R_0(0)(z^{-1}I - A(p))^T + (zI - A(p))R_0(0)A(p)^T, \]

since \( (zI - A(p))A(p)(zI - A(p)^{-1} = A(p) \). Thus,
\[ F^{-1}(z)S_\theta(z)F^{-T}(z^{-1}) \]
\[ = (zR_0(0) - A(p)R_0(0))(z^{-1}I - A(p))^T + z^{-1}A(p)R_0(0) \]
\[ - A(p)R_0(0)A(p)^T + zR_0(0)A(p)^T - A(p)R_0(0)A(p)^T \]
\[ = R_0(0) - z^{-1}A(p)R_0(0) - zR_0(0)A(p)^T \]
\[ + A(p)R_0(0)A(p)^T + z^{-1}A(p)R_0(0) - A(p)R_0(0)A(p)^T \]
\[ = R_0(0) - A(p)R_0(0)A(p)^T, \]
and (5) establishes (4). \( \square \)