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# Rate-distortion in closed-loop LTI systems

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***Abstract*—We consider a networked LTI system subject to an average data-rate constraint in the feedback path. We provide upper bounds to the minimal source coding rate required to achieve mean square stability and a desired level of performance. In the quadratic Gaussian case, an almost complete rate-distortion characterization is presented.**

## I. Introduction

This paper focuses on the interplay between average data-rate constraints (in bits per sample) and stationary performance for a networked control system comprising a noisy LTI plant and an average data-rate constraint in the feedback path. In such a setup, the results of [8] guarantee that it is possible to find causal encoders and decoders such that the resulting closed loop system is mean square stable, if and only if the average data-rate is greater than the sum of the logarithm of the absolute value of the unstable plant poles. This result has been extended in several directions (see, e.g., [7], [9]). However, when performance bounds subject to average data-rate constraints are sought, there are relatively fewer results available. Indeed, to our knowledge, there are no computable characterizations of the optimal encoding policies in networked control scenarios [1], [3], [5], [9], [13].

In this note, we present upper and lower bounds on the minimal average data-rate that allows one to attain a given performance level (as measured by the stationary variance of the plant output). From a source-coding perspective, we are aiming at characterizing the rate-distortion function in closed-loop systems. This extends beyond causal rate-distortion theory [2] due to being subject to a stability constraint. Our results exploit a framework for networked control system design subject to average data-rates developed in [10], [11].

## II. Problem Setup

Consider the NCS of Figure 1, where $P$ is an LTI plant with state $x \in \mathbb{R}^{n_x}$ and initial state $x_o$, $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}$ is a sensor output, $e \in \mathbb{R}^{n_e}$ is a signal related to closed loop performance, and $d \in \mathbb{R}^{n_d}$ is a disturbance. We assume that $(x_o, d)$ are jointly second-order and Gaussian (with finite entropies). The feedback path in Figure 1 comprises a delay-free noiseless digital channel, a causal encoder whose output $y_c$ is a sequence of binary words, and a causal decoder. The average data-rate across the channel is defined as

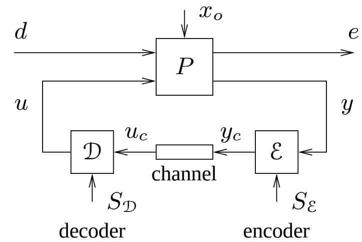


Fig. 1. Networked control system.

$$\mathcal{R} \triangleq \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} R(i), \tag{1}$$

where $R(i)$ referes to the expected length (in nats) of $y_c(i)$.

We do not restrict the complexity of the encoder or the decoder *a priori*, and only assume them to be causal, and to have access to independent side information $S_{\mathcal{E}}$ and $S_{\mathcal{D}}$. Our aim is characterizing

$$\mathcal{R}(D) \triangleq \inf_{\sigma_e^2 \leq D} \mathcal{R}, \tag{2}$$

where $\sigma_e^2 \triangleq \operatorname{trace}\{P_e\}$, $P_e$ is the stationary variance matrix of $e$, $D > 0$ is a desired level of performance, and the optimization is carried out with respect to all causal encoders $\mathcal{E}$ and decoders $\mathcal{D}$ that render the resulting NCS (asymptotically) mean square stable (MSS), i.e., that render $(x, u, d)$ jointly second-order and asymptotically wide-sense stationary processes.

## III. An Information-Theoretic lower bound on average data-rates

*Theorem 3.1:* Consider the NCS of Figure 1. Under suitable assumptions,

$$\mathcal{R} \geq I_\infty(y \to u) \geq I_\infty(y_G \to u_G), \tag{3}$$

where $I_\infty(\alpha \to \beta)$ denotes the mutual information rate [6] between $\alpha$ and $\beta$, and $(y_G, u_G)$ are jointly Gaussian processes with the same second order statistics as $(y, u)$. ■

Thus, in order to bound $\mathcal{R}(D)$ from below, it suffices to minimize the directed mutual information rate that would appear across the source coding scheme, when all signals in the loop are jointly Gaussian.

*Lemma 3.1:* Suppose that $(y^k, u^k)$ in Fig. 1 are second order and jointly Gaussian random sequences. Then $u^k$ can be constructed from $y^k$ as

$$u(i) = L_i(y^i, u^{i-1}) + s(i), \quad i = 1, \ldots, k \tag{4}$$

where, for each $i = 1, \ldots, k$, $s(i)$ is a zero-mean Gaussian random variable such that $s(i) \perp\!\!\!\perp (u^{i-1}, y^{i-1}, s^{i-1})$, and

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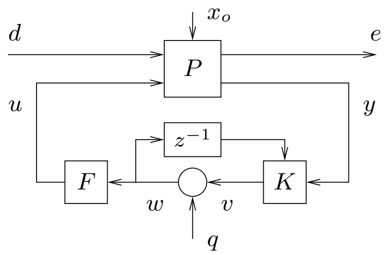


Fig. 2. NCS that arises when, in Figure 1, the encoder $\mathcal{E}$ and decoder $\mathcal{D}$ form a linear source coding scheme.

where $L_i : \mathbb{R}^{i\times(i-1)} \to \mathbb{R}$ is a linear operator such that $L_i(y^i, u^{i-1})$ is the minimum mean-square error estimator of $u(i)$ given $(y^i, u^{i-1})$. ■

We conclude from the above that, for a given performance level $D$, the minimum of $I_\infty(y_G \to u_G)$ over all causal encoders and decoders is achievable by an encoder/decoder pair which behaves as a linear system plus additive white Gaussian noise $s^k$ such that $s(i) \perp\!\!\!\perp (y^i, u^{i-1})$, $\forall i$.

## IV. Lower and upper Bounds on $\mathcal{R}_D$

We next define the class of *linear source coding schemes*, which are capable of yielding a relationship between $y$ and $u$ of the form given by (4).

*Definition 4.1:* A source coding scheme is said to be linear if and only if, when used around a noiseless digital channel, is such that its input $y$ and output $u$ are related via

$$u = Fw, \quad w = q + v, \quad v = K\,\mathrm{diag}\left\{z^{-1}, 1\right\}\begin{bmatrix} w \\ y \end{bmatrix}, \tag{5}$$

where $v$ and $w$ are auxiliary signals, $q$ is a second-order zero-mean i.i.d. sequence, both $F$ and $K$ are proper LTI systems, and $q$ is independent of $(x_o, d)$. ■

When a linear source coding scheme is used in the NCS of Figure 1, the LTI feedback system of Figure 2 arises.

*Lemma 4.1:* Consider the NCS of Figure 1 and assume that the encoder $\mathcal{E}$ and the decoder $\mathcal{D}$ form a linear source coding scheme. Under suitable assumptions, $I_\infty(y \to u) = I_\infty(v \to w)$ and

$$\frac{1}{4\pi}\int_{-\pi}^{\pi} \log \frac{S_w(\mathrm{e}^{j\omega})}{\sigma_q^2}\, d\omega \le I_\infty(v \to w), \tag{6}$$

where $S_w$ is the stationary power spectral density of $w$ and $\sigma_q^2$ is the variance of the auxiliary noise $q$. ■

Linear source coding schemes have sufficient degrees of freedom to allow one to whiten $w$ without compromising optimality. Thus, our results lead to:

*Theorem 4.1:* Consider the NCS of Figure 1 under suitable assumptions. Define, with reference to the feedback scheme of Figure 2, the infimal signal-to-noise ratio function

$$\gamma(D) \triangleq \inf_{\sigma_e^2 \le D} \frac{\sigma_v^2}{\sigma_q^2}, \tag{7}$$

where $\sigma_\alpha^2$, $\alpha \in \{v, q, e\}$, is the stationary variance of $\alpha$ in Figure 2, and the optimization is carried out with respect to all $\sigma_q^2 \in \mathbb{R}^+$ and all proper LTI filters $F$ and $K$ which render the feedback system of Figure 2 internally stable and well-posed. Then:

$$\frac{1}{2}\log\Big(1 + \gamma(D)\Big) \le \mathcal{R}(D). \tag{8}$$

Moreover, there exists a linear source coding scheme such that

$$\mathcal{R}(D) < \frac{1}{2}\log\Big(1 + \gamma(D)\Big) + \frac{1}{2}\log\left(\frac{2\pi e}{12}\right) + \log 2. \tag{9}$$

■

Theorem 4.1 characterizes the minimal average data-rate that guarantees a given stationary performance level, in terms of $\gamma(D)$, i.e., in terms of the minimal SNR that guarantees the desired performance level in a related LTI architecture. Interestingly, the upper bound in (9) is valid even if one removes the assumption of $(x_o, d)$ being Gaussian

To find $\gamma(D)$, one can resort to the results in [4]. A case where an explicit solution is available is when $D \to \infty$, i.e., when only stabilization is sought. In that case, it follows from Theorem 4.1 and [12] that

$$\gamma(\infty) = \left(\prod_{i=1}^{n_p} |p_i|^2\right) - 1, \tag{10}$$

where $p_1, \ldots, p_{n_p}$ are the unstable poles of $P$. If one uses (10) in (8) and (9), then one recovers, within a modest gap, the absolute minimal average data-rate compatible with stability derived in [8].

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