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Schiøler, Henrik; Leth, John-Josef; Gholami, Mehdi

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Stability of Randomly Switched Diffusions

Henrik Schioler, John Leth and Mehdi Gholami

***Abstract*— This paper provides a sufficient criterion for ϵ-moment stability (boundedness) and ergodicity for a class of systems comprising a finite set of diffusions among which switching is governed by a continuous time Markov chain. Stability/instability properties for each separate subsystem are assumed to be quantified by a Lyapunov function candidate and an associated growth rate equation. For the set of Lyapunov functions a compatibility criterion is assumed to be fulfilled bounding the ratio between pairs of Lyapunov functions. The established criterion is shown to be equivalent to an exact criterion for the almost sure convergence of an associated process bounding moments of the process under study. Examples are provided to illustrate the use of the established criterion.**

***Index Terms*— stochastic system; switching diffusion; stability**

I. Introduction

Randomly Switched Systems, (RSS) with Piecewise Deterministic Processes (PDP) [1] as a subclass, denotes a class of systems where system state evolves in time according to one among a finite set of smooth dynamics selected by a discrete mode switching process. RSS have been suggested for modeling within various fields such as finance, population dynamics, manufacturing, and fault tolerant control [2].

In [3] general sufficient conditions for the existence and uniqueness of stationary distributions of PDPs are provided through a basic result of [4]. Markov Jump Linear Systems (MJLS) as studied in [5], [6] and [7] is the special case of RSS where smooth dynamics are linear and switching is governed by a continuous time Markov chain with discrete state space. In [5], [6] noise free dynamics are assumed as the basis of analytical results although [5] more broadly suggests discrete modes governed by linear stochastic differential equations, i.e. linear diffusions. The work [8] provides sufficient conditions for almost sure convergence of 2nd order MJLS based on projections to the unit circle and the existence and uniqueness of a stationary distribution of the projected process. The work [9] provides general sufficient conditions for convergence in distribution of RSS expressed in terms of switched Stochastic Differential Equations, i.e. so called Switched Diffusion Processes (SDP). SDPs are also studied in [10], [11] and [12] where the two former treats stability criteria for systems comprising both stable and unstable modes and the latter stability criteria for switching among stable systems allowing discontinuous jumps of the continuous state at mode switching instants. In [13] Linear Jump Systems are analyzed for stability under deterministic bounds for the number of jumps within time intervals of defined lengths.

A great variety of definitions of stochastic stability exist as surveyed in e.g. [14] and [15]. For the above mentioned studies, stochastic stability unanimously implies some kind of stochastic convergence to an equilibrium state. In [12] results are given in the shape of exponential *p*th moment convergence to 0, generally implying convergence in probability. In [15] stability results for fault tolerant control systems are given in terms of mean square exponential convergence to 0, whereas in [16] 3 different definitions of mean square stochastic stability are considered. In [10] the concept of input-to-state stability (ISS) is extended to a probabilistic setting through expectation and applied to SDP.

We find stability definitions based on stochastic convergence insufficient for many practically appearing probabilistic models since such models frequently include a driving (process) noise component preventing even stable systems from convergence. Such a term is included in [12], where the underlying assumptions however force the noise component to vanish at the equilibrium, as also indicated by the examples provided. Although theoretically appealing such assumptions seem unrealistic for practical cases, where stable operation is characterized by stationary random fluctuations around the equilibrium

Thus stability definitions based on stationary or ergodic behavior as studied in [3] seem more suitable from a practical point of view. For systems with a compact state space the main criteria are of mixing type, i.e. systems almost never map proper subsets into themselves, which for Markovian systems corresponds to irreducibility, where all states are mutually reachable. For systems with non-compact state spaces, the main questions is that of stability, whereas the mixing/irreducibility property is mostly taken as a prerequisite. Ergodicity of non compact systems is studied in [4] and [17] for discrete and continuous time respectively.

In this paper we give sufficient conditions pth moment boundedness and ergodicity for switched diffusion processes (SDP), where a Lyapunov function candidate has been identified for each subsystem as in [10]. Our approach is similar to that of [18] in their treatment of 1st order jump linear systems (JLS). Our contribution distinguishes from [3] and [9] in the more specific setup yielding more specific and operational results. It contrasts to [8] in its generality to nth order systems and to [18] be including state jumps at switching moments allowing its application to multiple Lyapunov analysis.

The section to follow provides mathematical prerequisites and main definitions. This is followed by the analytical

All authors are with the Department for Electronic Systems, Aalborg University, Aalborg, Denmark henrik@es.aau.dk,jjl@es.aau.dk,mehdi@cs.aau.dk

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results section, where a number of lemmas are given along with the main results expressed as theorems. The analytical results section is preceded by a section devoted to numerical results of simulations comprising a practically illustrative control system with unstable faulty modes. Results for the method suggested in this paper are compared to results from neighboring methods suggested in [8] and [16]. Finally conclusions and discussions are provided along with suggestions for the direction of future research.

II. Mathematical prerequisites

We generally have a state space $\mathbb{R}^n \times \mathcal{P}$ and a state (x, σ) evolving in continuous time, i.e: $(x, \sigma) : \Omega \times \mathbb{R} \to \mathbb{R}^n \times \mathcal{P}$, where Ω is an appropriate probability space. The process x is interpreted as the continuous state of the system, whereas the switching process σ indicates the discrete state, i.e. normal and faulty modes. A finite number of distinct discrete states is assumed, i.e. $\mathcal{P}$=$\{1, .., M\}$. We generally assume that the pair (x, σ) constitutes a continuous time Markov chain, where the switching process is assumed to be governed by an independent Markov chain.

The treatment is based on system dynamics in the shape of a Stochastic Differential Equation (SDE), where the noise part is modeled as a Brownian motion. For each discrete mode $p = \sigma(t)$ an SDE is defined, i.e.

$$dx = f_p(x)dt + g_p(x)dw \tag{1}$$

where w is a vector of n independent standard Brownian motions and $f_p : \mathbb{R}^n \to \mathbb{R}^n$ and $g_p : \mathbb{R}^n \to \mathbb{R}^n$ are appropriate mappings satisfying suitable smoothness conditions to ensure unique continuous solutions to (1), see e.g. [19].

We say that a functional $V : \mathbb{R}^n \to \mathbb{R}_+$ is a Lyapunov function candidate if it is continuous differentiable, $V(x) \geq 0$ and $V^{-1}([0, C])$ are compact sets for all $C \geq 0$.

We assume for each $p \in \mathcal{P}$ that there exists a real number λ_p and a Lyapunov function candidate $V_p : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\nabla V_p(x)f_p(x) \leq \lambda_p V_p(x) \tag{2}$$

with ∇V denoting the gradient of V. For $\lambda_p < 0$, (2) ensures stability of the associated deterministic system, $dx = f_p(x)dt$. We generally do not assume $\lambda_p < 0$, since we allow instability for some discrete states.

For a sufficiently smooth V (twice continuous differentiable will do) we conclude from Itô's chain rule that the process $V(x(t))$ has the stochastic differential

$$dV = \nabla V f_p dt + g_p^T H_V g_p dt + \nabla V g_p dw \tag{3}$$

where H_V denotes the Hessian of V and all terms are to be evaluated at $x(t)$. Hence we get

$$\frac{d}{dt}E(V(x(t))|x(t) = x) = \nabla V(x)f_p(x)+g_p(x)^T H_V(x)g_p(x)$$

In the sequel we assume that $g_p^T(x(t))H_{V_p}(x(t))g_p(x(t))$ is globally bounded above by some positive constant K_p and that (2) holds for every $x \in \mathbb{R}^n$. Hence

$$\frac{d}{dt}E[V_p(x(t))] \leq \lambda_p E[V_p(x(t))] + K_p \tag{4}$$

Note that (4) holds for any initial distribution of x, hence it holds conditionally for $x(0) = \bar{x}$.

We shall in the sequel assume, as in [10], that the Lyapunov function candidates V_p are compatible, i.e. a real number $\mu > 1$ exists such that

$$V_{p'}(x) \leq \mu V_p(x), \quad \forall p, p' \in \mathcal{P},\ \forall x \in \mathbb{R}^n$$

A. Switching process

The switching process σ governs the choice of smooth dynamics for the continuous state. The evolution of σ is specified through an infinitesimal generator matrix $Q = \{q_{ij}\}$ defined by

$$P(\sigma(t + h) = j|\sigma(t) = i) = hq_{ij} + O(h)$$

for $j \neq i$ and

$$P(\sigma(t + h) = i|\sigma(t) = i) = 1 - hq_{ii} + O(h)$$

We say that the process σ is communicating (irreducible) *iff* for any $i \neq j$ a sequence of distinct states $\{k_1, ..k_p\}$ exist such that $i = k_1, j = k_p$ and $q_{k_s,k_{s+1}} > 0$. In looser terms the process is communicating, if any state is reachable from any other state in finite time with a probability greater than zero. If a process is communicating the generator matrix Q has an eigenvalue of multiplicity 1 at 0 and an accompanying one-dimensional eigenspace of solutions to $0 = \pi Q$. All other eigenvalues of Q fall in the left complex half plane. The unique probabilistic (left) eigenvector π is called the stationary distribution of the chain. A finite communicating chain is ergodic, i.e.

$$P_*(\lim_{t\to\infty} \frac{1}{t}\int_0^t I_{\sigma(t)=i}dt = \pi_i) = 1, \quad \forall i$$

where P_* is the probability distribution induced on the space of realizations through the specification of transition rates. We shall in the sequel assume σ to be irreducible.

III. Stochastic stability

Stability properties of $x(t)$ needs to be established in the context of stochastic stability, where a variety of inter related definitions exist. Most definitions are based on associated definitions of convergence, i.e. convergence in probability, convergence in mean/moment and almost sure convergence as listed below.

$$\lim_{t\to\infty} P(|x(t)| > \delta) = 0,\ \forall \delta > 0$$
$$\lim_{t\to\infty} E(|x(t)|^\epsilon) = 0$$
$$P(\lim_{t\to\infty} x(t) = 0) = 1$$

The two latter cannot be ordered in strength they both imply the former. As mentioned in the introduction most practically appearing stochastic systems fail to converge to any equilibrium. In this case stability definitions should reflect, not convergence, but stationary behavior. An immediate definition based on ϵ-moment can be stated as; for each initial distribution there exists $\epsilon > 0$ and $K > 0$ such that

$$E(|x(t)|^\epsilon) \leq K, \quad \forall t \geq 0 \tag{5}$$

We say that a system is ϵ-moment-stable if (5) is fulfilled. In many works focus has been put on 2nd moment stability. However second moment stability is generally a stronger requirement than ϵ-moment stability for $\epsilon < 2$, so focusing on second moment (mean and variance) may lead to overly pessimistic analysis and designs in cases where no specific requirements regarding second moment stability have been put forth.

More generally we define a system to be *moment-stable in the wide sense* (MSWS) if for any initial distribution there exists $K > 0$ and a Lyapunov function candidate V such that

$$E(V(x(t))) \leq K, \quad \forall t \geq 0 \tag{6}$$

Alternatively stable stationary behavior could be defined in terms of the statistics generated by process realizations as implied by ergodicity. That is, a stationary probability measure Φ exists such that for all measurable subsets A

$$P(\lim_{t\to\infty} \frac{1}{t}\int_0^t I_{x(\tau)\in A} d\tau = \Phi(A)) = 1 \tag{7}$$

If both (5) and (7) are fulfilled we have through the Markov inequality for $A = \{|x| \leq C\}$

$$\Phi(A) \geq 1 - \frac{K}{C^\epsilon}$$

Hence the process $x(t)$ has to be closer to 0 than C, infinitely often and visits this neighborhood with an average frequency above $1 - \frac{K}{C^\epsilon}$ over the long run. In the sequel we consider a discrete time skeleton $\{x(n\Delta)\}_{n\in\mathbb{N}}$ in regards to ergodicity, i.e.

$$P(\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} I_{x(n\Delta)\in A} = \Phi(A)) = 1 \tag{8}$$

IV. Stability Analysis

The analysis is based on the definition of a dominating process U, for which stability criteria are given. Since U is an approximation from above, the presented criteria can only be sufficient. Analysis is initially performed conditioned on a specific realization $\bar{\sigma}$ of the switching process σ and the initial condition $x(0) = \bar{x}$ and subsequently turned into an unconditional result through expectation over the probability space of realizations and initial conditions.

Let the sequence $\{t_j\}$ be the transition instants of a particular realization $\bar{\sigma}$ of the switching process σ, such that $\bar{\sigma}(t) = p_i$ for $t \in [t_i, t_{i+1})$. We define the conditional expectation $E_{\bar{\sigma}}$ by

$$E_{\bar{\sigma}}[\cdot] = E[\cdot|\sigma = \bar{\sigma}, x(0) = \bar{x}]$$

Hence for $t \in [t_i, t_{i+1})$ we have from (4) that

$$\frac{d}{dt}E_{\bar{\sigma}}(V_{p_i}(x(t))) \leq \lambda_{p_i} E_{\bar{\sigma}}[V_{p_i}(x(t))] + K_{p_i}$$

Now for the process W defined by

$$\frac{d}{dt}W(t) = \lambda_{p_i} W(t) + K_{p_i}, \quad t \in [t_i, t_{i+1}) \tag{9}$$

$$W(t_{i+1}) = \mu W(t_{i+1}^-), \quad \mu > 1 \tag{10}$$

$$W(0) = V_{\bar{\sigma}(0)}(\bar{x}) \tag{11}$$

we have the following two lemmas whose proof we leave to the reader.

Lemma 1:

$$E_{\bar{\sigma}}[V_{\sigma(t)}(x(t))] \leq W(t), \ \forall t \geq 0 \tag{12}$$

Let $W_{min} = \min_{\{p \ | \ \lambda_p < 0\}}\{-K_p/\lambda_p, V_{\bar{\sigma}(0)}(x)\}$. It is then easily proved that $W(t) \geq W_{min}$ for all $t > 0$ if $W(0) \geq W_{min}$. Thus for $W(0) \geq W_{min}, \ t \in [t_i, t_{i+1})$ and $0 \leq \epsilon < 1$

$$\begin{aligned}\frac{d}{dt}W^\epsilon(t) &= \epsilon W^{\epsilon-1}(t)(\lambda_{p_i} W(t) + K_{p_i}) \\ &\leq \epsilon\lambda_{p_i} W^\epsilon(t) + \epsilon K_{p_i} W_{min}^{\epsilon-1}\end{aligned}$$

Now let $\bar{K} = \max_p\{K_p\}, \ \bar{\kappa} = \epsilon\bar{K}W_{min}^{\epsilon-1}$, and define the process U by

$$\begin{aligned}\frac{d}{dt}U(t) &= \epsilon\lambda_{p_i} U(t) + \bar{\kappa}, \quad t \in [t_i, t_{i+1}) \\ U(t_{i+1}) &= \mu^\epsilon U(t_{i+1}^-) \\ U(0) &= W^\epsilon(0)\end{aligned}$$

Lemma 2: For $\epsilon > 0$

$$U(t) \geq W^\epsilon(t), \quad \forall t \geq 0 \tag{13}$$

Moreover, we obtain the following result which is proven in appendix.

Lemma 3: Define the processes γ_p by $\gamma_p(t) = I_{\bar{\sigma}(t)=p} U(t)$ then

$$\begin{aligned}\frac{d}{dt}E[\gamma_l(t)] = \epsilon\lambda_l E[\gamma_l(t)] + \mu^\epsilon \sum_{j\neq l} q_{jl} E[\gamma_j(t)] \\ + E[\gamma_l(t)]q_{ll} + \kappa_l(t)\end{aligned} \tag{14}$$

where $0 \leq \kappa_l(t) \leq \bar{\kappa}$. Or more compactly

$$\frac{d}{dt}E[\gamma(t)] = \Lambda E[\gamma(t)] + \kappa(t), \tag{15}$$

where $\Lambda = \mu^\epsilon Q + \epsilon\text{Diag}(\lambda_1, .., \lambda_M) + (1 - \mu^\epsilon)\text{Diag}(Q)$, $\gamma(t) = [\gamma_1(t), .., \gamma_M(t)]^T$ and $\kappa(t) = [\kappa_1(t), .., \kappa_M(t)]^T$.

From (12) and (13) we then get

$$E_{\bar{\sigma}}^\epsilon[V_{\sigma(t)}(x(t))] \leq U(t) \tag{16}$$

which yields

$$E[E_{\bar{\sigma}}^\epsilon[V_{\sigma(t)}(x(t))]] \leq E[U(t)] = \sum_p E[\gamma_p(t)] \tag{17}$$

A. Moment stability in the wide sense for continuous time

We note that (17) implies, that if $E[\gamma_l(t)] \to 0$ for $t \to \infty$ and all $l \in \mathcal{P}$ then $E(E_{\bar{\sigma}}^\epsilon[V_{\sigma(t)}(x(t))]) \to 0$ for $t \to \infty$. However, due to $\kappa_l(t)$ in (14) convergence to zero is not possible. Still boundedness of $\gamma_l(t)$ leads to a bounded ϵ-moment of $E_{\bar{\sigma}}[V_{\sigma(t)}(x(t))]$. Moreover, Jensen's inequality gives for $0 < \epsilon < 1$

$$E(E_{\bar{\sigma}}^\epsilon[V_{\sigma(t)}(x(t))]) \geq E(E_{\bar{\sigma}}[V_{\sigma(t)}^\epsilon(x(t))]) = E[V_{\sigma(t)}^\epsilon(x(t))]$$

which in turn leads to the boundedness of $E[V_{\sigma(t)}^\epsilon(x(t))]$ and in fact moment stability in the wide sense (MSWS) of (x, σ).

Algebraically, stability of (15) is determined by the eigenvalues of the matrix $\Lambda = \Lambda(\mu, \lambda_1, .., \lambda_M, \epsilon)$, where $\Lambda_{jl} = \mu^\epsilon q_{jl}$ for $j \neq l$ and $\Lambda_{ll} = \epsilon\lambda_l + q_{ll}$. Note that $\Lambda(\mu, \lambda_1, .., \lambda_M, 0) = Q$ has (for an irreducible σ) an eigenvalue at 0 of multiplicity 1, and all other eigenvalues in the left complex plane.

A sufficient criterion for the existence of an $\epsilon > 0$ such that $\Lambda(\mu, \lambda_1, .., \lambda_M, \epsilon)$ is stable is that the root locus of Λ for positive ϵ takes the root at 0 to the left half plane. Defining $\Gamma = \frac{1}{\mu^\epsilon}\Lambda$ we obtain that stability properties of Γ are equivalent to those of Λ and

$$\Gamma = Q + \frac{\epsilon}{\mu^\epsilon} diag(\lambda_1, .., \lambda_M) + \frac{1-\mu^\epsilon}{\mu^\epsilon} diag(Q)$$

Let $D(s, \epsilon)$ be the determinant of $sI - \Gamma$ then the implicit function theorem gives the following sufficient stability criterion

$$\frac{D_\epsilon(0,0)}{D_s(0,0)} > 0 \tag{18}$$

which through the Jacobi formula for determinant derivatives, lemma 1 to 3 and [18, lemma 3.10], leads to the first main result which is proven in the Appendix

Theorem 1: Assume that

$$E[E^\delta_{\bar\sigma}[V_{\sigma(0)}(x(0))]] < \infty, \quad \forall \delta \in]0,1[$$

and

$$\sum_{i=1}^{M} \pi_i(\lambda_i - log(\mu)q_{ii}) < 0 \tag{19}$$

then $0 < \epsilon < 1$ and $K < \infty$ exists such that

$$E[V^\epsilon_{\sigma(t)}(x(t))] < K, \quad \forall t \geq 0$$

B. Ergodicity

In this section we establish sufficient criteria for ergodicity to accompagne the previously established stability criteria. Ergodicity is generally based on two main characteristics; recurrence and irredicibility, where the former is in fact a stability property and the latter is concerned with mutual reachability within the state space. We conjecture that irredicibilty of the switched process (x, σ) is inherited from the irredicibility of σ and at least one of the subprocesses (1) indexed by $p \in \mathcal{P}$. Since this paper is devoted to the study of stability we omit further treatment of irredicibility and refer the reader to e.g. [20] for nessesary and sufficient criteria for irreducibility of diffusion processes.

It is readily recognized that the derivation of (15) is carried out without reference to the initial distribution of $(x(0), \sigma(0))$ so that it in fact holds for $(x(0), \sigma(0))$ concentrated on a single element $(\bar x, \bar\sigma)$ or in other words conditional on the initial conditions $x(0) = \bar x$ and $\sigma(0) = \bar\sigma$, i.e.

$$\frac{d}{dt}E[\gamma(t)|x(0) = \bar x, \sigma(0) = \bar\sigma] = \Lambda E[\gamma(t)|x(0) = \bar x, \sigma(0) = \bar\sigma] + \kappa(t)$$

Let $\bar\lambda > \sup\{Re(\lambda) \mid \lambda \text{ eigenvalue of } \Lambda\}$. Then using the fact that there exists a constant L such that

$$|\exp(\Lambda\ t)E[\gamma(0)|x(0) = \bar x, \sigma(0) = \bar\sigma]| \leq L\exp(\bar\lambda t)|E[\gamma(0)|x(0) = \bar x, \sigma(0) = \bar\sigma]|, \ \forall t \geq 0$$

with $|\cdot|$ denoting the 1-norm, we conclude that a $\Delta > 0$, $0 < \alpha < 1$ and $\kappa > 0$ exist so that

$$|E[\gamma(\Delta)|x(0) = \bar x, \sigma(0) = \bar\sigma]| \leq \alpha|E[\gamma(0)|x(0) = \bar x, \sigma(0) = \bar\sigma]| + \kappa$$

Now assume that $|\gamma(0)| \geq (\kappa + c)/(1-\alpha)$ for a positive constant c. Then

$$|E[\gamma(\Delta)|x(0) = \bar x, \sigma(0) = \bar\sigma]| \leq |E[\gamma(0)|x(0) = \bar x, \sigma(0) = \bar\sigma]| - c \tag{20}$$

and since by construction

$$\begin{aligned} |E[\gamma(t) \mid x(0) = \bar x, \sigma(0) = \bar\sigma]| &= E[|\gamma(t)| \mid x(0) = \bar x, \sigma(0) = \bar\sigma] \\ &= E[U(t) \mid x(0) = \bar x, \sigma(0) = \bar\sigma] \end{aligned}$$

inequality (20) becomes

$$\begin{aligned} E[U(\Delta)|x(0) = \bar x, \sigma(0) = \bar\sigma] &\leq E[U(0)|x(0) = \bar x, \sigma(0) = \bar\sigma] - c \\ &= V^\epsilon_{\bar\sigma}(\bar x) - c \end{aligned}$$

For $0 < \epsilon < 1$, Jensen's inequality and (16) then yields

$$\begin{aligned} E[U(t)|x(0) =&\bar x, \sigma(0) = \bar\sigma] \\ &\geq E(E^\epsilon_{\bar\sigma}(V_{\sigma(t)}(x(t)))|x(0) = \bar x, \sigma(0) = \bar\sigma) \\ &\geq E(V^\epsilon_{\sigma(t)}(x(t)))|x(0) = \bar x, \sigma(0) = \bar\sigma) \end{aligned}$$

so that finally

$$E(V^\epsilon_{\sigma(\Delta)}(x(\Delta)))|x(0) = \bar x, \sigma(0) = \bar\sigma) \leq V^\epsilon_{\bar\sigma}(\bar x) - c$$

Through [4, Theorem 5.1] we obtain the second main result

Theorem 2: Assume the process $(x(t), \sigma(t))$ to be irreducible and (19) is fulfilled then the discrete time process $(x(n\Delta), \sigma(n\Delta))$ is ergodic.

C. Relation to almost sure convergence

For $K_{p_i} = 0$, consider the dominating process W defined by (9) to (11). Then

$$W(t) = W(0)\ exp(\int_0^t \lambda_{\sigma(t)}dt)\mu^{N(t)}$$

where $N(t)$ denotes the number of state switches in $[0, t]$. The expected sojourn time T_i in every state i is inversely proportional to the sum of rates out of that state, i.e. $T_i = 1/\sum_{j\in\mathcal{P}, j\neq i} q_{ij} = -1/q_{ii}$. For an ergodic chain the average fraction of time in state i approaches π_i. Thus the average number of returns to state i within $[0, t]$ is $\pi_i\ t/T_i = -\pi_i\ tq_{ii}$ such that $N(t)$ approaches $-t\ \sum_{i\in\mathcal{P}} \pi_i\ q_{ii}$.

Now by taking logarithms

$$log(W(t)) = log(W(0)) + \int_0^t \lambda_{\sigma(t)} dt + log(\mu)N(t)$$

and recalling that for an ergodic chain $\int_0^t \lambda_{\sigma(t)} dt \approx t \sum_{i\in\mathcal{P}} \pi_i \lambda_i$ we obtain that (19) is also an exact criterion for almost sure convergence of W to 0.

V. EXAMPLES

The example comprises a linearized model of a car backing with an attached trailer The state x of the system is the angle between the directional vectors of car and trailer and the input is the angle α between directional vectors of car and front wheels i.e. the steering angle. Due to road and tire imperfections a random term is included in the system dynamics to give the following diffusion equation

$$dx = (x - \alpha)dt + dw \tag{21}$$

where w is a standard Brownian motion.

It is assumed that steering angle α is set by a reference R through a servo mechanism comprising 1st order dynamics, i.e.

$$\frac{d}{dt}\alpha = -C\alpha + R \tag{22}$$

where $C > 0$ is a design parameter. The loop is closed through a proportional feedback, i.e. $R = K\ x_m$, where the control gain K is also a design parameter and x_m is the measurement of the trailer angle, i.e. $x_m = x + N$, where $N(t)$ is an independent standard Gaussian measurement noise. We assume a simplistic fault model comprising 2 discrete states: $\{0, 1\}$, where the former indicates the fault free situation and the latter a faulty situation in which the measurement of the system state is not available. In the latter case we simplistically set $R = 0$ with resulting system poles $s = \{1, -C\}$. We set design parameters $K = 4$ and $C = 2$. For both states the resulting continuous time dynamics are linear with matrices A_0 and A_1, where a continuous time Markov process switches between the two discrete system state.

$$A_0 = \begin{bmatrix} 1 & -1 \\ 4 & -2 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix}$$

Selecting Lyapunov functions V_0 and V_1 is a non trivial task, for which the systematic study is postponed to future research. It is readily shown that Lyapunov function candidates may be found such that

$$\frac{d}{dt}V_d \leq 2\lambda_d V_d \tag{23}$$

where λ_d is the largest real part of eigenvalues of A_d. Thus quadratic Lyapunov function candidates may be found for which

$$\frac{d}{dt}V_0 \leq -1V_0 \quad \text{and} \quad \frac{d}{dt}V_1 \leq 2V_1.$$

In this case we choose $V_d(x) = x^t Q_d x$ and solve the following linear matrix inequalities for Q_d

$$Q_d \geq 0 \quad \text{and} \quad A^t Q_d + Q_d A - 2\lambda_d Q_d \leq 0$$

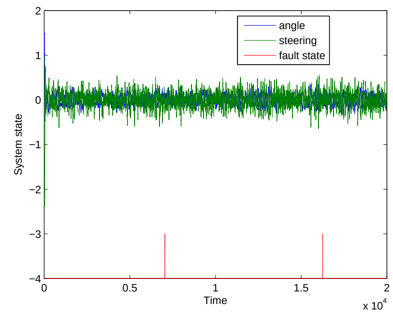

Fig. 1. Simulation result for $T_1 = 0.2$ and $\pi_1 = 0.001$, $E(V_{\sigma(t)}(x(t))) < \infty$.

using the YALMIP [21] library for MATLAB, i.e.

```
Q_d = sdpvar(2,2);
F=[Q_d >= 0, A'*Q_d+Q_d*A-2*lambda_d*Q_d<=0]
solvesdp(F)
```

A least conservative value for μ may be found by solving the generalized eigenproblem

$$Q_0 v_i = \mu_i Q_1 v_i.$$

Then the least conservative μ can be found as

$$\max_i\{\mu_i, \frac{1}{\mu_i}\} = \max\{13.6231, 37.8529, \\ 1/13.6231, 1/37.8529\} = 37.8529.$$

We may however scale Q_0 without altering λ_0 and λ_1 in (23). The scaling factor yielding the smallest value of μ is $1/\sqrt{max_i\{\mu_i\}min_i\{\mu_i\}} = 0.044$, yielding $\mu = 1.6669$.

We parametrize the switching process through the stationary error probability $\pi_1 = q_{01}/(q_{01} + q_{10})$ and the error state sojourn time $T_1 = 1/q_{10}$, with q_{01} and q_{10} the two transition rates.

We present simulation results for 3 different situations; one where $E(V_{\sigma(t)}(x(t))) < \infty$ (figure (1)), one where $E(V^\epsilon_{\sigma(t)}(x(t)))$ can be proven finite only for $\epsilon << 1$ (figure (2)) and finally one where no moments $E(V^\epsilon_{\sigma(t)}(x(t)))$ can be proven finite (figure (3)).

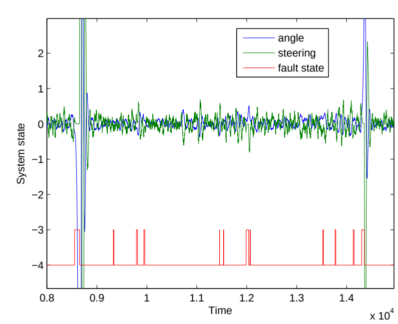

Fig. 2. Simulation result for $T_1 = 1.6$ and $\pi_1 = 0.03$, $E(V^\epsilon_{\sigma(t)}(x(t))) < \infty$.

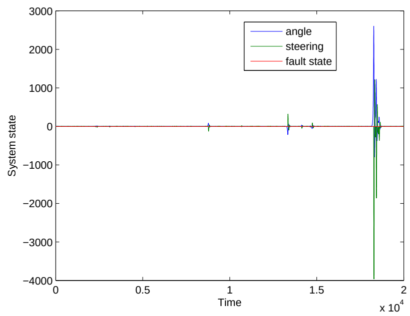

Fig. 3. Simulation result for $T_1 = 1.6$ and $\pi_1 = 0.3$, no guaranteed stable moments.

It seems clear from figures 1 and 3 that, whereas the former exhibits evident stable behavior the latter is obviously unstable. The intermediate case depicted in 2 would be harder to categorize by inspection. In such a case the theoretical results obtained in this paper could resolve the dispute.

A. *Comparison of results*

Application of the method proposed in [8] would require

$$(A_{2,2}(i) - A_{1,1}(i))^2 + 4A_{2,1}(i)A_{1,2}(i) < 0 \quad \text{for} \quad i \in \{0, 1\}$$

which is not fulfilled in this case. An immediate reason for this, is that the criterion proposed in [8] additionally sanctions irreducibility, which is left out of consideration here.

A comparison of results may also be conducted with the exact criteria proposed in [16] for so called *Stochastic Stability* (SS) and *Mean Stochastic Stability* (MSS), which are both strongly related to second moment stability. SS is achieved *iff*

$$E[\int_0^\infty |x(t)|^2 dt | x_0, \sigma_0] \leq T(x_0, \sigma_0)$$

for all initial conditions x_0, σ_0, whereas MSS is equivalent to

$$\lim_{t \to \infty} E[|x(t)|^2 | x_0, \sigma_0] = 0$$

for all initial conditions. Thus SS and MSS both refer to the behaviour of 2nd moments. Disregarding noise in the example above yields a system of the type defined in equation (2.1) of [16] for which both SS and MSS are equivalent to the following coupled LMIs being feasible for symmetric and positive definite matrices $K_i, i \in \mathcal{P}$

$$A(i)^T K_i + K_i A(i) + \sum_j q_{ij} K_j < 0 \tag{24}$$

as specified in theorems 2 and 4 of [16]. Application of (24) to the example above has been conducted also with the YALMIP tool. Only the first parameter setting, i.e. $T_1 = 0.2$ and $\pi_1 = 0.001$ yields feasible LMIs, corresponding well to the fact that in this case our method guarantees a finite second moment, i.e. $E(V_{\sigma(t)}(x(t))) \leq \infty$. Theorem 7 of [16] treats the noisy case. However as specified in equation (2.50) of [16] the noise intensity is scaled by state value, which is not consistent with the example presented above.

VI. Conclusion

A sufficient criterion for ϵ-moment stability (boundedness) and ergodicity has been established for a class of systems comprising finite set of diffusions among which switching is governed by a continuous time Markov chain. For each separate diffusion stability/instability properties are assumed to be quantified by a Lyapunov function and an associated growth rate equation. For the discrete set of Lyapunov functions a compatibility criterion is assumed to be fulfilled. The established sufficient criterion is shown to be equivalent to an exact criterion for the almost sure convergence of a dominating process. Examples are provided to illustrate the use of the established criterion.

It may be argued that since the established criterion is only sufficient it may provide overly pessimistic conclusions. However the equivalence to the exact criterion for the dominating process indicates the possibility of tightness for special cases. Another source of conservatism is the compatibility criterion for Lyapunov functions applied. This criterion may be refined such that a separate criterion is expressed for each pair of discrete neighboring modes in the transition graph. Finally the stability criteria established should not stand alone in the analysis of practically appearing systems.

Analysis of the eigenvalues of the matrix Λ could be use for identifying a particular $\epsilon > 0$ for which moment stability is guaranteed. This particular value indicates the nature of the resulting marginal state distribution, i.e. the power of its tail. Such results are valuable in the qualitative assessment of stability properties. The refinement of the compatibility criterion and the use of tail powers for stability assessment define important directions for future research.

VII. Appendix

A. *Proof of lemma* (3)

$$\begin{aligned}\gamma_l(t+h) = I_{\sigma_{t+h}=l}[h \sum_j \epsilon \lambda_j I_{\sigma_t=j} + \mu^\epsilon \sum_{j \neq l} I_{\sigma_t=j} \\ + I_{\sigma_t=l}]U(t) + I_{\sigma_{t+h}=l} h \epsilon \bar{K} W_{min}^{\epsilon-1}\end{aligned}$$

taking expected values (i.e. averaging over the space of switching process realizations) and neglecting higher order truncation errors, gives

$$
\begin{aligned}
E[\quad \gamma_l(t+h) \quad] \\
= \quad & h\sum_j \epsilon\lambda_j E[I_{\sigma_{t+h}=l} I_{\sigma_t=j} U(t)] \\
+ \quad & \mu^\epsilon \sum_{j\neq l} E[I_{\sigma_{t+h}=l} I_{\sigma_t=j} U(t)] \\
+ \quad & E[I_{\sigma_{t+h}=l} I_{\sigma_t=l} U(t)] + E[I_{\sigma_{t+h}=l}] h\epsilon \bar{K} W_{min}^{\epsilon-1} \\
= \quad & h(h\sum_{j\neq l} \epsilon\lambda_j q_{jl} E[I_{\sigma_t=j} U(t)] \\
+ \quad & \lambda_l (1-h\sum_{j\neq l} q_{lj}) E[I_{\sigma_t=l} U(t)]) \\
+ \quad & h\mu^\epsilon \sum_{j\neq l} q_{jl} E[I_{\sigma_t=j} U(t)] \\
+ \quad & (1-h\sum_{j\neq l} q_{lj}) E[I_{\sigma_t=l} U(t)] \\
+ \quad & E[I_{\sigma_{t+h}=l}] h\epsilon \bar{K} W_{min}^{\epsilon-1}
\end{aligned}
$$

$$
\begin{aligned}
\approx \quad & h\epsilon\lambda_l E[I_{\sigma_t=l} U(t)] + h\mu^\epsilon \sum_{j\neq l} q_{jl} E[I_{\sigma_t=j} U(t)] \\
+ \quad & (1-h\sum_{j\neq l} q_{lj}) E[I_{\sigma_t=l} U(t)] + E[I_{\sigma_{t+h}=l}] h\epsilon \bar{K} W_{min}^{\epsilon-1}
\end{aligned}
$$

so that subtracting $E[I_{\sigma_t=l} U(t)]$ and taking limits for $h \to 0$ gives

$$
\begin{aligned}
\frac{d}{dt} E[\gamma_l(t)] = \epsilon\lambda_l E[\gamma_l(t)] + \mu^\epsilon \sum_{j\neq l} q_{jl} E[\gamma_j(t)] \\
+ E[\gamma_l(t)] q_{ll} + \lim_{h\to 0} E[I_{\sigma_{t+h}=l}] \epsilon \bar{K} W_{min}^{\epsilon-1}
\end{aligned}
$$

B. Proof of Theorem (1)

From the Jacobi formula

$$
D_\epsilon = -tr(adj(sI-\Gamma)\frac{\partial\Gamma}{\partial\epsilon}),\ D_s = tr(adj(sI-\Gamma))
$$

such that $D_\epsilon(0,0) = tr(adj(Q)\frac{\partial\Gamma}{\partial\epsilon})$ and $D_s(0,0) = -tr(adj(Q))$ with $\frac{\partial\Gamma}{\partial\epsilon}$ evaluated at $(0,0)$. Since $\frac{\partial\Gamma}{\partial\epsilon}$ is a diagonal matrix it follows that

$$
D_\epsilon(0,0) = \sum_{i=1}^M M_{ii}(\frac{\partial\Gamma}{\partial\epsilon})_{ii},\ D_s(0,0) = -\sum_{i=1}^M M_{ii}
$$

where M_{ii} are the diagonal minors of Q. From [18, lemma 3.10], there is a positive constant c so that

$$
M_{ii} = (-1)^{M-1} c\pi_i
$$

hence

$$
\frac{D_\epsilon(0,0)}{D_s(0,0)} = -\sum_{i=1}^M \pi_i(\frac{\partial\Gamma}{\partial\epsilon})_{ii}
$$

Next for $s = \epsilon = 0$

$$
(\frac{\partial\Gamma}{\partial\epsilon}) = diag(\lambda_1,..,\lambda_M) - log(\mu)diag(Q)
$$

which yields

$$
\frac{D_\epsilon(0,0)}{D_s(0,0)} = -\sum_{i=1}^N \pi_i(\lambda_i - log(\mu)q_{ii})
$$

Thus from lemmas (1) to (3) and (18) the theorem follows.

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