# Model Reduction of Hybrid Systems

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AALBORG UNIVERSITY

Model Reduction of Hybrid Systems
PhD thesis

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# |Preface and Acknowledgments

This thesis is submitted in partial fulfillment of the requirements for the Doctor of Philosophy in Automation and Control at Aalborg University. The work has been carried out in the period from February 2007 to February 2010 under supervision of Professor Rafael Wisniewski.

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# |Abstract:

High-Technological solutions of today are characterized by complex dynamical models. A lot of these models have inherent hybrid/switching structure. Hybrid/switched systems are powerful models for distributed embedded systems design where discrete controls are applied to continuous processes. Hybrid systems are also an important modeling class for nonlinear systems because a wide variety of nonlinearities are either piecewise-affine (e.g., a saturated linear actuator characteristic) or can be approximated as hybrid systems. The complexity of verifying and assessing general properties of hybrid systems, designing controllers and implementations is very high so that the use of these models is limited in applications where the size of the state space is large. To cope with complexity, model reduction is a powerful technique.

This thesis presents methods for model reduction and stability analysis of hybrid/switched systems. Methods are designed to approximate hybrid/switched systems to low order models which adequately describe the behavior of the switched systems. Three frameworks for model reduction of switched systems are proposed which are based on the notion of the generalized gramians. Generalized gramians are the solutions to the observability and controllability Lyapunov inequalities. In the first framework the projection matrices are found based on the common generalized gramians. This framework preserves the stability of the original switched system for all switching signals while reducing the subsystems of the switched systems. The first framework is computationally efficient due to the construction of a single projection for all subsystems. This framework is used for switched controller reduction and it is shown that the stability of the closed loop system is guaranteed to be preserved for arbitrary switching signal. To compute the common generalized gramians linear matrix inequalities (LMI's) need to be solved. These LMI's are not always feasible. In order to solve the problem of conservatism, the second framework is presented. In this method the projection matrices are constructed based on the convex combinations of the generalized gramians. However this framework is less conservative than the first one, it does not guarantee the stability for all switching signals. The stability preservation is studied for this reduction technique. The third framework for model reduction of switched systems is based on the switching generalized gramians. The reduced order switched system is guaranteed to be stable for all switching signal in this method. This framework uses stability conditions which are based on switching quadratic Lyapunov functions which are less

conservative than the stability conditions based on common quadratic Lyapunov functions. The stability conditions which are used for this method are very useful in model reduction and design problems because they have slack variables in the conditions. Similar conditions for a class of switched nonlinear systems are derived in this thesis. The results are used for output feedback control of switched nonlinear systems. Model reduction of piecewise affine systems is also studied in this thesis. The proposed method is based on the reduction of linear subsystems inside the polytopes. The methods which are proposed in this thesis are applied to several numerical examples.

# 1. | Introduction

In today's technological world with a growing demand for methods to model, to analyze, and to understand complex dynamical systems, this thesis presents methods for model reduction and stability analysis of hybrid/switched systems. Methods are designed to extract reduced order models that adequately describe the behavior of the switched systems.

This chapter describes the motivation of this work followed by an overview of the state of the art within model reduction, stability of switched systems and model reduction of hybrid /switched systems. The focus of the overview is on the methods which are related to the contributions of this work. The outline of this thesis is presented at the end of this chapter.

## 1.1 Motivation

The ever-increasing need for accurate mathematical modeling of physical as well as artificial processes leads to models of high complexity. These complex models are usually developed to understand the system, for diagnostic purposes or to be used for control design. To maintain tractability, efficient computational prototyping tools are required to replace such complex models by simpler models which capture their dominant characteristics. This need arises partly from limited computational, accuracy, or memory capabilities. Due to this fact, model reduction methods have become increasingly popular over the last two decades [1]-[3]. Such methods are designed to obtain a reduced order state space model that adequately describes the behavior of the system in question.

A lot of real world systems have inherent hybrid and switched structure. Hybrid systems include both continuous-time and discrete-event components, and are an important modeling class for many applications. For example, the dynamics of many industrial processes can be modeled as evolving in continuous-time at the lower level (of the physical system or artificial processes) being driven by discrete-event logical components that impose mode switching at

the higher level. Switched linear systems and Piecewise-affine (PWA) systems are special classes of hybrid systems. A switched linear system is a hybrid system which consists of several linear subsystems and a rule that orchestrates the switching among them. This class of hybrid systems are accurate enough to represent many practical engineering systems with complex dynamics [4]. In Piecewise-affine (PWA) the continuous dynamics within each discrete mode are affine and the mode switching always occurs at very specific subsets of the state space that are known a-priori. PWA systems are also an important modeling class for nonlinear systems because a wide variety of nonlinearities are either piecewise-affine (e.g., a saturated linear actuator characteristic) or can be approximated as piecewise-affine functions. Due to the ubiquitous nature of hybrid systems, there is a growing demand for methods to model, analyze, and to understand this class of dynamical systems.

In a lot of fields, some or all of the subsystems of the hybrid systems are large complex mathematical models. On the other hand, most of the methods that are proposed so far for control and analysis in hybrid and switched systems theory are suffering from high computational burden when dealing with large-scale dynamical systems. Because of the weakness of standard model reduction techniques in dealing directly with hybrid structure without sacrificing essential features and also pressing needs for efficient analysis and control of large-scale dynamical hybrid and switched systems, it is necessary to study model reduction of hybrid and switched systems in particular.

The reduction method should respect hybrid structure and preserve main features while approximates a hybrid dynamical system. Stability is an essential property that has to be preserved in the process of model reduction [5]. This motivates the research community to study stability conditions and to develop stability conditions which are more convenient to be preserved in the reduction process. This problem is more challenging for hybrid/switched systems.

Model reduction of hybrid/switched systems finds applications in different areas. These include simulation and control of large-scale structures, circuit simulation and synthesis, reachability analysis, safety verification, and simulation and control of micro-electro-mechanical systems, to name but a few.
In the sequel an overview of the results on model reduction, stability of switched systems and in particular model reduction of switched systems is presented.

## 1.2 State of the Art and Background

The focus of this thesis is on model reduction frameworks for hybrid systems which possibly preserve the stability of original hybrid systems in the process of reduction. Model reduction methods and stability of hybrid/switched systems are important areas in this work. In the following an overview on model reduction methods with more stress on gramian based reduction techniques is presented. The classes of hybrid systems which are of interest in this thesis are presented along with some stability results in the literature. Finally, the state of art of model reduction of hybrid/switched system is described.

## 1.2.1 Model Order Reduction

Model reduction is an approach to overcome the problem of complexity in dynamical systems. The goal in model order reduction is to produce a low dimensional system that has almost the same response characteristics as the original system with less storage requirements and lower evaluation time. The resulting reduced model might be used to replace the original system as a component in a larger simulation or to design a low dimensional controller suitable for real time applications.

Over the past two decades, model reduction methods have become increasingly popular [1]-[3]. The largest group of model order reduction algorithms applies to linear systems.

Consider a dynamical system with realization:

$$G := (A, B, C, D), \tag{1.1}$$

where $G$ is transfer matrix with associated state-space representation:

$$\begin{cases} \eta x(t) = Ax(t) + Bu(t), & x(t) \in \mathbb{R}^n, \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{1.2}$$

where $\eta$ is either the derivative operator $\eta f(t) = \dfrac{df(t)}{dt}$, $t \in \mathbb{R}$ or the shift $\eta f(t) = f(t+1)$, $t \in \mathbb{Z}$.

The goal of model reduction is to approximate dynamical systems described by (1.2) with:

$$\begin{cases} \eta \hat{x}(t) = A_r \hat{x}(t) + B_r u(t), & \hat{x}(t) \in \mathbb{R}^k, \\ \tilde{y}(t) = C_r \hat{x}(t) + D_r u(t), \end{cases} \tag{1.3}$$

where $k < n$ and the outputs $y(t)$ and $\tilde{y}(t)$ are close to each other in some sense. Furthermore, system properties, like stability should be preserved and reduction procedure should be computationally stable and efficient.

Model reduction techniques for linear systems or more precisely linear time invariant (LTI) systems are divided into two broad categories, namely moment matching based techniques and singular value decomposition (SVD) based methods.

In moment matching based methods, the focus is on matching the coefficients of the Taylor (or McLaurin) series of the transfer matrix. Given the expansion of original transfer matrix around $s_0$ :

$$G(s_0 + s) = m_0 + m_1 s + m_2 s^2 + m_3 s^3 + \cdots ,$$   (1.4)

where $m_l$ are the moments. The reduced order model is obtained by matching $k$ moments of the original model and the reduced model with the form:

$$G_r(s_0 + s) = m_{r0} + m_{r1} s + m_{r2} s^2 + m_{r3} s^3 + \cdots ,$$   (1.5)

i. e. $m_{ri} = m_i$ , $i = 1, 2, ..., k$ , where $k < n$ .

Moment matching based methods like Krylov subspace method can be implemented iteratively, which are numerically very efficient, but these do not automatically preserve stability and have no guaranteed error bound for approximation[1],[2],[6].

The second group of model reduction algorithms is the family of SVD-based reduction methods. Unlike moment matching based method, SVD-based reduction methods have guaranteed error bounds for approximation, and guarantee that the stability of the original system will be preserved in the reduced order model [1][2].

One of the model reduction methods from this category that is well-known is balanced model reduction first introduced in [7] and later in [8].To apply balanced reduction, first the system is transformed to a basis where the states which are difficult to control are simultaneously difficult to observe. This is achieved by simultaneously diagonalizing the controllability and the observability gramians, which are solutions to the reachability and the observability Lyapunov equations. The reduced model is obtained by truncating the least observable and controllable states in the balanced structure.

Gramians for continuous time systems are given from the following Lyapunov equations:

$$AP + PA^* + BB^* = 0,$$
$$A^*Q + QA + C^*C = 0,$$   (1.6)

and for discrete time systems from:

$$APA^* - P + BB^* = 0,$$
$$A^*QA - Q + C^*C = 0,$$
(1.7)

For stable $A$, controllable $(A, B)$ and observable $(A, C)$ these equations have a unique positive definite solutions $P$ and $Q$, called the controllability and observability gramians. In balanced reduction, first the system is transformed to the balanced structure in which gramians are equal and diagonal:

$$P = Q = diag(\sigma_1 I_{k_1}, ..., \sigma_q I_{k_q}),$$
$$\sum_{j=1}^{q} k_j = n,$$
(1.8)

where $\sigma_i > \sigma_{i+1}$ and they are called Hankel singular values.

The reduced model is easily obtained by truncating the states which are associated with the set of the least Hankel singular values. Applying the method to stable, minimal $G$, if we keep all the states associated with $\sigma_m (1 \leq m \leq r)$, by truncating the rest, the reduced model $G_r$ will be minimal and stable and satisfies[1][2]:

$$\|G - G_r\|_\infty \leq 2 \sum_{j=r+1}^{q} \sigma_j .$$
(1.9)

Balanced truncation has been improved from different viewpoints and several model reduction techniques have been proposed based on the idea of balanced truncation [2].The balancing methods introduced above try to approximate the full-order model $G$ over all frequencies. However, in many applications we are only interested in a certain frequency range. This problem leads to the so-called frequency weighted balancing method [1],[2]and [9]-[13]. In this method input weight $W_i(s)$ and output weights $W_o(s)$ are used in the reduction process to make the weighted error:

$$\|W_i(s)(G(s) - G_r(s))W_o(s)\|_\infty ,$$
(1.10)

small.

The frequency weighted balanced which was first proposed in [9][10] provides more accurate approximation compared to ordinary balanced truncation. Stability of the reduced order model is not guaranteed in case of a two-sided weighting for this method. To tackle the stability problem a new technique was proposed in [11] which uses only strictly proper

weights. The method was modified in [12] to allow for proper weights. A new frequency weighted balancing method to modify Enns' method in [13]. The method guarantees stability and yields a simple error bound.

In many cases the input and output weights are not given. Instead the problem is to reduce the model over a given frequency range [1],[2]. This problem can be attacked directly by balanced reduction within frequency bound, which was first proposed in [14] and then modified in [2] to preserve the stability of the original system and provide an error bound for approximation. In this method, the controllability gramian $P(\omega_1, \omega_2)$ and observability gramians $Q(\omega_1, \omega_2)$ within frequency range $[\omega_1, \omega_2]$ are defined as:

$$
\begin{aligned}
P(\omega_1, \omega_2) &= P(\omega_2) - P(\omega_1), \\
Q(\omega_1, \omega_2) &= Q(\omega_2) - Q(\omega_1),
\end{aligned}
\tag{1.11}
$$

where:

$$
\begin{aligned}
P(\omega) &:= \frac{1}{2\pi} \int_{-\omega}^{\omega} (Ij\theta - A)^{-1} BB^*(-Ij\theta - A^*)^{-1} d\theta, \\
Q(\omega) &:= \frac{1}{2\pi} \int_{-\omega}^{\omega} (-Ij\theta - A^*)^{-1} C^*C(Ij\theta - A)^{-1} d\theta.
\end{aligned}
\tag{1.12}
$$

Similar method for model reduction of linear systems within bounded frequency was proposed in [15] and the modification on this method was discussed in [16].

In several applications we are interested in approximating the dynamical system within a specified time interval. This problem was addressed in [14]. Controllability gramian $P[t_1, t_2]$ and observability gramians $Q[t_1, t_2]$ in this method are defined as:

$$
\begin{aligned}
P[t_1, t_2] &:= \int_{t_1}^{t_2} e^{A\tau} BB^T e^{A^T\tau} d\tau, \\
Q[t_1, t_2] &:= \int_{t_1}^{t_2} e^{A\tau} C^T C e^{A^T\tau} d\tau,
\end{aligned}
\tag{1.13}
$$

Similar to the frequency domain approach, this method is modified in [2] to preserve the stability of original system and to provide for approximation error bound. The balanced model reduction within specified time interval is more accurate than ordinary balanced reduction technique within the time interval which the method is applied.

The methods which we have surveyed so far all keep the absolute approximation error small. However there are reduction methods which are trying to keep the relative error small rather than absolute error. The balanced stochastic truncation approach belongs to the family of relative error methods [17], which is an SVD-based algorithm. In contrast to absolute error

methods like the ordinary balanced truncation, the balanced stochastic truncation method has the main advantage in  provision of a uniform approximation of the frequency response of the original system over the whole frequency domain, and particularly, in preservation of phase information [2]. For example, for a minimum-phase original system, the balanced stochastic approximation is also minimum-phase. This is not generally true for the absolute error methods [18].

Let $G(s)$ be MIMO square transfer matrix with a minimal sate space realization described in (1.2). If $D$ is nonsingular it is possible to compute the left spectral factor $\psi(s)$ of $G(s)G^T(-s)$ satisfying:

$$\psi^T(-s)\psi(s) = G(s)G^T(-s).$$

(1.14)

The state space realization of $G$ is called a balanced stochastic realization if:

$$W_c^G = W_o^\psi = diag(\sigma_1 I_{k_1},...,\sigma_q I_{k_q}),$$

$$\sum_{j=1}^{q} k_j = n,$$

(1.15)

where $W_c^G$ is the controllability gramian of $G(s)$, the matrix $W_o^\psi$ is the observability gramian of $\psi(s)$ and $\sigma_i$ is the $i^{th}$ Hankel singular value of the stable part of the so-called "phase matrix" $F(s) = (\psi^T(-s))^{-1}G(s)$. The singular values in (1.15) are ordered decreasingly [2],[18],[22]. The rest of the algorithm is the same as ordinary balanced reduction. The bound for relative error for this reduction method is obtained as:

$$\left\|G^{-1}(G-G_r)\right\|_\infty \leq \prod_{i=r+1}^{q} \frac{1+\sigma_i}{1-\sigma_i} - 1 .$$

(1.16)

Similar to frequency domain balanced truncation, frequency-domain balanced stochastic truncation was proposed to improve the accuracy [19][20].

One of the methods that have been developed based on balanced model reduction is the method based on the generalized gramian [21]. In this method, Lyapunov inequalities (rather than Lyapunov equations) are solved to compute generalized gramians. The physical interpretations of generalized gramians are similar to ordinary gramians. Generalized gramians were used to devise a technique for structure preserving model reduction methods

in [22]. Generalized Gramians for continuous time systems are given by the solutions of the Lyapunov inequalities:

$$AP_g + P_g A^* + BB^* \leq 0,$$
$$A^* Q_g + Q_g A + C^* C \leq 0,$$

$$(1.17)$$

and for discrete time systems by:

$$AP_g A^* - P_g + BB^* \leq 0,$$
$$A^* Q_g A - Q_g + C^* C \leq 0.$$

$$(1.18)$$

This linear matrix inequality (LMI) approach to the model reduction problem is particularly useful when some structures need to be preserved in the process of model reduction. Controller reduction is a typical example of this type of problems.

Most of the studies related to model reduction presented so far have been devoted to linear case and just few methods have been proposed for nonlinear cases. Nonlinear model reduction in the general case is a challenge. The most popular methods are proper orthogonal decomposition (POD) method, trajectory piecewise linearization (TPWL) reduction technique and nonlinear balanced model reduction methods. In TPWL, nonlinear model is approximated by piecewise linear model along a trajectory. In this method for different simulating input, different simulating trajectories are obtained which leads to different reduced order models[23],[24].In POD, we need to compute snapshots from the nonlinear system. Singular value decomposition (SVD) is made from the correlation matrix and then a global basis is derived for projection [25]-[28]. Two approaches have been proposed as nonlinear extension of balanced model reduction. The first approach is based on empirical gramians. The empirical observability and controllability gramians are derived from several snapshot series with representative inputs and initial values. The global basis for reduction is derived by balancing these empirical gramians[29][30]. The second method for nonlinear balanced model reduction is based on controllability and observability function. These functions can be obtained from PDEs of Lyapunov and Hamilton-Jacobi type. Singular value functions which are nonlinear state-space extension of the Hankel singular values in the linear case play an important role in this method[31][32]. Snapshots or input information are not needed in this reduction method but it is computationally very expensive. One of the nonlinear reduction methods which is similar to linear balanced model reduction is the

method which was proposed in[33][34]. This method has been devised for model reduction of so-called $\Phi$-systems and is based on diagonally dominant generalized gramians. $\Phi$-systems are a class of discrete-time switched nonlinear systems of the form:

$$\begin{cases} x(k+1) = A\Phi(x(k)) + Bu, \\ y(k) = C\Phi(x(k)) + Du, \end{cases} \tag{1.19}$$

where $x(k) \in \mathbb{R}^n$ is the state, $y(k) \in \mathbb{R}^p$ is the output and $u(k) \in \mathbb{R}^m$ is input. $A$, $B$, $C$ and $D$ are matrices of appropriate dimensions. Furthermore:

$$\Phi(x(k)) := \begin{bmatrix} \bar{\Phi}(x_1(k)) \\ \bar{\Phi}(x_2(k)) \\ \vdots \\ \bar{\Phi}(x_n(k)) \end{bmatrix}, \tag{1.20}$$

where:

$$\bar{\Phi} \in OL := \left\{ \phi : \mathbb{R} \to \mathbb{R} \mid \forall s,t \in \mathbb{R}, \ |\phi(s) + \phi(t)| \leq |s+t| \right\}. \tag{1.21}$$

In some literature, systems with this description have been called $\sigma$-systems [35],[36]. The standard saturation and the hyperbolic tangent (popular activation function in neural networks) are examples of this type of nonlinearities [33]-[39]. The discrete-time recurrent artificial neural network is a special case of $\Phi$-systems [35]-[38]. Furthermore, results related to this class of nonlinear systems have potential applications in the classical problems related to uncertain nonlinearities such as Lur'e systems [39].

Gramians which are used for reduction of such systems are the solutions to Lyapunov inequalities:

$$\begin{aligned} APA^* - P + BB^* &\leq 0, \\ A^*QA - Q + C^*C &\leq 0, \end{aligned} \tag{1.22}$$

where $P$ and $Q$ are positive diagonal dominant matrices i. e. :

$$\begin{cases} P = P^* > 0, \\ |p_{ii}| > \sum_{j \neq i} |p_{ij}|, \quad \forall i, \end{cases} \tag{1.23}$$

$$\begin{cases} Q = Q^* > 0, \\ |q_{ii}| > \sum_{j \neq i} |q_{ij}|, \quad \forall i. \end{cases} \tag{1.24}$$

The procedure to find the reduced-order model is similar to the balanced truncation for a linear system using generalized gramians. Main difference is that, the balanced-like model needs to be transformed back in the end of procedure. It is because the original system by no means is input–output equivalent to balanced-like $\Phi$-systems in general. Therefore, it cannot be said to be a balanced realization of the original $\Phi$-systems and simply truncating the former does not result in an approximant of the latter. For this reason this model reduction method is only described as the "model reduction via a balanced system" rather than the "balanced truncation". More details are discussed in [33][34].

## 1.2.2 Hybrid Systems and Stability

Hybrid systems are complex systems which exhibit both discrete event dynamics as well as continuous dynamics. Hybrid dynamics provide a convenient framework for modeling systems in broad range of applications. Some examples are: continuous motions which may be interrupted by collisions, electrical circuits with switches or diodes, and control of chemical reactions by valves and pumps. In this part, mathematical definitions for some important classed of hybrid systems are presented followed by some results from stability theory for hybrid systems. Stability is an important property which should be preserved under reduction. Therefore stability conditions for switched systems play an important role in this thesis.

### Hybrid Systems

Switched systems and piecewise affine systems are two important classes of hybrid systems. A switched system is composed of a family of dynamical systems and a rule that governs the switching among them:

$$\begin{cases} \eta x(t) = F_\sigma(x(t), u(t)), & x(t) \in \mathbb{R}^n, \\ y(t) = G_\sigma(x(t), u(t)), \end{cases} \tag{1.25}$$

where $\eta$ is either the derivative operator $\eta f(t) = \dfrac{df(t)}{dt}$ , $t \in \mathbb{R}$ or the shift $\eta f(t) = f(t+1)$ , $t \in \mathbb{Z}$ , $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^p$ is the output, $u(t) \in \mathbb{R}^m$ is the input,

and $\sigma : \mathbb{R}^{\geq 0} \to K \subset \mathbb{N}$ is the switching signal that is a piecewise constant map of the time. K is the set of discrete modes.

Switched linear systems are accurate enough to represent many practical engineering systems with complex dynamics. They are relatively easy to handle compared to nonlinear switched systems and many powerful tools from linear analysis are applicable to cope with them. A switched linear system is mathematically described by:

$$\Sigma : \begin{cases} \eta x(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t), \\ y(t) = C_{\sigma(t)} x(t) + D_{\sigma(t)} u(t), \end{cases} \tag{1.26}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^p$ is the output, $u(t) \in \mathbb{R}^m$ is the input, and $\sigma : \mathbb{R}^{\geq 0} \to K \subset \mathbb{N}$ is the switching signal that is a piecewise constant map of the time. K is the set of discrete modes. For each $i \in K$, $A_i$, $B_i$, $C_i$, $D_i$ are matrices of appropriate dimensions. The indicator function is defined as:

$$\zeta_i(t) := \begin{cases} 1, & \text{when the switched system is described} \\ & \text{by the i}^{th} \text{ mode matrices } (A_i, B_i, C_i, D_i) \\ 0, & \text{otherwise.} \end{cases} \tag{1.27}$$

The switched system (1.26) can also be written as the following using indicator function:

$$\Sigma : \begin{cases} \eta x(t) = \sum_{i=1}^{|K|} \zeta_i (A_i x(t) + B_i u(t)), \\ y(t) = \sum_{i=1}^{|K|} \zeta_i (C_i x(t) + D_i u(t)). \end{cases} \tag{1.28}$$

One of the other important classes of hybrid systems which has been studied extensively in the literature is a class of piecewise affine systems. Piecewise-affine (PWA) systems are a special class of hybrid systems in that the continuous dynamics within each discrete mode are affine and the mode switching always occurs at subsets of the state space that are known a-priori. PWA systems are also an important modeling class for nonlinear systems because a wide variety of nonlinearities are either piecewise-affine or can be approximated by piecewise-affine functions. This class is equivalent to many other classes of hybrid systems

such as mixed logical dynamical systems, linear complementary systems, and max-min-plus scaling systems and thus form a very general class of linear hybrid systems [41].

Let $J$ be a finite index set and $\#J$ is cardinality of $J$. A polyhedral set $P$ in $\mathbb{R}^n$ is the intersection of a family of closed half spaces $H_j^- = \{x \in \mathbb{R}^n \mid \langle x, N_j \rangle \leq a_j\}$ for $N_j \in \mathbb{R}^n$ and $a_j \in \mathbb{R}$ ,where $j \in J$ and $\langle .,. \rangle$ is scalar product in $\mathbb{R}^n$, i.e. $P := \bigcap_{j \in J} H_j^-$. The polyhedral set $P$ can be expressed by the inequality (1.29) to be understood components wise:

$$P = \{x \in \mathbb{R}^n \mid Nx \leq a\} , \tag{1.29}$$

where $N = [N_1^T \dots N_{\#J}^T]^T$ , $a = [a_1 \dots a_{\#J}]^T$ .

Let $K = \{P_j \mid j \in I\}$ be a polyhedral Complex with the index set $I$.

$$|K| := \bigcup_{j \in I} P_j \subseteq \mathbb{R}^n .$$

Let $E$ be any polyhedral set ($\mathbb{R}^n$ inclusively). A piecewise linear partition of $E$ is a polyhedral complex $K$ such that $E = |K|$. The elements of $K$ are called cells.

$$K_i := \{P \in K \mid \dim(P) = i\} .$$

Piecewise affine systems are dynamical systems in full dimensional cells $K_n$ of linear partition associated to a quadruple $(E, K, U, S)$, where $E$ is a polyhedral set (a polytope) in $\mathbb{R}^n$ , $K$ is a piecewise affine partition of $E$ , $U$ is a polyhedral set (of admissible inputs) in $\mathbb{R}^m$ , and $S = \{s_p : P \in K_n\}$ is a family of piecewise affine systems:

$$s_p : \begin{cases} \dot{x} = A_p x + B_p u + a_p, \\ y = C_p x + D_p u. \end{cases} \tag{1.30}$$

## Stability of Switched Systems

Over the last decade problem of stability analysis for hybrid/switched systems has received a lot of attention [42],[43]. The stability issues of such switched systems give rise to several interesting and challenging mathematical problems and include several interesting phenomena. Two interesting observations are:

- when all the subsystems are exponentially stable, the switched systems may have divergent trajectories for certain switching signals [44], [45].
- one may switch between unstable subsystems to make the switched system exponentially stable [44], [45].

The stability of switched systems depends not only on the dynamics of each subsystem but also on the properties of switching signals. Therefore, the results in the context of stability analysis for switched systems can be divided into two types of problems. One is the stability analysis of switched systems with arbitrary switching signal and the other one is stability analysis of switched system with restricted switching signal.

## Stability Analysis for Switched Systems: Arbitrary Switching Signal

In a lot of applications, switching information and therefore switching signal is not available. The stability of such systems needs to be studied without any restriction on the switching signal.

The existence of a common quadratic Lyapunov function for all subsystems assures the quadratic stability of the switched system. The problem of finding a common quadratic Lyapunov function can be expressed as linear matrix inequality (LMI). To find a common quadratic Lyapunov function, Lyapunov inequalities need to be solved. There exists a common quadratic Lyapunov function for linear switched system, if there exists:

$$P = P^* > 0, \qquad P \in \mathbb{R}^{n \times n}, \tag{1.31}$$

such that

$$A_i P + A_i^* P < 0, \tag{1.32}$$

for the continuous-time systems, or

$$A_i^* P A_i - P < 0, \tag{1.33}$$

for discrete-time cases are satisfied. A common quadratic Lyapunov function is:

$$V(x) = x^* P x. \tag{1.34}$$

Note that quadratic stability is a special class of exponential stability, which implies asymptotic stability.

There have been several research works focused on the problem of determining algebraic conditions on the subsystems matrices to assure the existence of common quadratic Lyapunov function [45],[46]. For example, in [47], [48], a necessary and sufficient condition for the existence of a common quadratic Lyapunov function for the second order bimodal systems has been proposed. The results are based on stability of the pencil of matrices. The pencil of matrices is defined as:

$$\gamma_\alpha(A_1, A_2) = \alpha A_1 + (1 - \alpha) A_2 \ , \tag{1.35}$$

where $\alpha \in [0,1]$.

These conditions can be summarized in the following theorem.

**Theorem 1: [47]–[49]** Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Hurwitz. The following conditions are equivalent:

1. There exists a common quadratic Lyapunov function for the bimodal switched system with $A_1, A_2$ as the two subsystems matrices.

2. The matrix pencils $\gamma_\alpha(A_1, A_2)$ and $\gamma_\alpha(A_1, A_2^{-1})$ are Hurwitz.

3. The matrices $A_1 A_2$ and $A_1 A_2^{-1}$ do not have any negative real eigenvalues.

This necessary and sufficient condition is difficult to be generalized for higher dimensions. For a pair of $n$ dimensional matrices a necessary condition has been derived in the following theorem [48][50]:

**Theorem 2: [48],[50]** Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be Hurwitz. A necessary condition for existence of common quadratic Lyapunov function is that $A_1 \gamma_\alpha(A_1, A_2)$ and $A_1(\gamma_\alpha(A_1, A_2))^{-1}$ do not have any negative real eigenvalues for all $\alpha \in [0,1]$.

There are several results on necessary and sufficient conditions for stability of switched systems with more than two subsystems. In [51], a tensor condition was introduced as a necessary condition for the existence of a common quadratic Lyapunov function for a

switched system consisting of a finite number of $n^{th}$ order LTI systems. The tensor condition was shown to be necessary and sufficient when the switched system only contains a pair of subsystems. The problem of finding necessary and sufficient conditions for the existence of a common quadratic Lyapunov function for general cases of higher order and more than two modes remains an open problem[42].

It is worth mentioning that linear matrix inequality (LMI) for finding common quadratic Lyapunov function is computationally expensive to solve particularly for the large number of discrete modes. The existence of a common quadratic Lyapunov function is only a sufficient condition for the stability of switching systems and it is not necessary. There are examples of stable switched systems which do not have a common quadratic Lyapunov function [46] (Chapter 2). The existence of a common quadratic Lyapunov function for stability analysis could be conservative. To solve this problem, the conditions based on switched quadratic Lyapunov functions have received some attention [52],[53]. The switched quadratic Lyapunov functions are in the form:

$$V(x) = x^* P_{\sigma(t)} x \ ,$$

(1.36)

equivalently, using indicator function:

$$V(x(t)) = x(t)^* (\sum_{i=1}^{|K|} \zeta_i(t) P_i) \, x(t), \qquad x(t) \in \mathbb{R}^n \ .$$

(1.37)

To check the existence of the switched quadratic Lyapunov functions, linear matrix inequalities need to be solved. In the sequel, a theorem on stability analysis based on switched quadratic Lyapunov functions is proposed:

**Theorem 3** [52]: The following statements are equivalent:

1. There exists a Lyapunov function of the form (1.37) whose difference is negative definite, proving asymptotic stability of discrete-time switched system:

$$x(t+1) = \sum_{i=1}^{|K|} \zeta_i(A_i x(t)) \ .$$

(1.38)

2. There exist $|K|$ symmetric matrices, $P_1, P_2, ..., P_{|K|}$ satisfying:

$$\begin{bmatrix} P_i & A_i^* P_j \\ P_j A_i & P_j \end{bmatrix} > 0, \qquad \forall (i,j) \in K \times K. \tag{1.39}$$

Lyapunov function is given by:

$$V(x(t)) = x(t)^* (\sum_{i=1}^{|K|} \zeta_i(t) P_i) \, x(t), \qquad x(t) \in \mathbb{R}^n. \tag{1.40}$$

3. There exist $|\kappa|$ matrices $S_1, S_2, ..., S_{|K|}$ which are symmetric and $|\kappa|$ matrices, $G_1, G_2, ..., G_{|K|}$ ,satisfying:

$$\begin{bmatrix} G_i + G_i^* - S_i & G_i^* A_i^* \\ A_i G_i & S_j \end{bmatrix} > 0, \qquad \forall (i,j) \in K \times K. \tag{1.41}$$

Lyapunov function is given by:

$$V(x(t)) = x(t)^* (\sum_{i=1}^{|K|} \zeta_i(t) S_i^{-1}) \, x(t), \qquad x(t) \in \mathbb{R}^n. \tag{1.42}$$

These conditions are less conservative than the conditions based on common quadratic Lyapunov functions. These conditions are suitable for design problems because of the slack variables.

The conditions in Theorem 3 are only sufficient conditions. There are some results on necessary and sufficient conditions for stability of discrete-time switched systems. Theorem 4 presents a necessary and sufficient condition for asymptotic stability of discrete-time switched systems.

**Theorem 4 [54]:** A switched linear system $x(t+1) = A_{\sigma(t)} x(t)$ , where $A_{\sigma(t)} \in \{A_1, A_2, ..., A_N\}$ , is asymptotically stable under arbitrary switching signal if and only if there exists a finite integer $n$ , such that:

$$\left\| A_{i_1} A_{i_2} ... A_{i_n} \right\|_\infty < 1, \tag{1.43}$$

for all $n$ -tuple $A_{\sigma(t)} \in \{A_1, A_2, ..., A_N\}$ , where $j = 1, 2, ..., n$ .

**Stability Analysis for Switched Systems: Restricted Switching Signal**

In a lot of examples, the stability under arbitrary switching signal may fail while switched system is stable with restricted switching signal. Restriction on switching signal may arise naturally from the physical constraints of the system. In some cases, one may have some knowledge about possible switching logic in a switched system. This knowledge may imply restrictions on the switching signals. For example, there must exist a certain bound on the time interval between two successive switchings. With such kind of a priori knowledge about the switching signals, it is possible to derive stronger stability results for a given hybrid system. The goal is to provide an answer to the question regarding what restrictions should be put on the switching signals in order to guarantee the stability of switched systems. The restrictions on switching signals may be either time domain restrictions or state space restrictions. It can be shown that it is always possible to maintain stability when all the subsystems are stable and switching is slow enough, in the sense that dwell time is sufficiently large [55]. $\tau_d \in \mathbb{R}^+$ is called the dwell time of a switching signal if the time interval between any two consecutive switchings is no smaller than $\tau_d$.

If one occasionally have a smaller dwell time between switching, it does not affect the stability provided that does not occur too frequently. This concept is captured by the concept of "average dwell-time" in [56].

$\tau_a \in \mathbb{R}^+$ is called the average dwell time of a switching signal $\sigma(t)$, if:

$$N_\sigma(t, \tau) \leq N_0 + \frac{t - \tau}{\tau_a}, \qquad (1.44)$$

holds for all $t \geq \tau \geq 0$ and some scalar $N_0 \geq 0$, where $N_\sigma(t, \tau)$ denotes the number of mode switches of a given switching signal $\sigma(t)$ over the interval $(\tau, t)$. If all the subsystems are stable then the switched system remains stable provided that the average dwell time is sufficiently large [56].

The stability analysis with restricted switching signal has been also studied in the framework of multiple Lyapunov functions (MLF). The basic idea is that multiple Lyapunov or Lyapunov-like functions, which may correspond to each single subsystem or certain region in the state space, are concatenated together to produce a non-traditional Lyapunov function. Multiple Lyapunov functions may not be monotonically decreasing along the state trajectories, may have discontinuities and be piecewise differentiable [44], [57], [58].

There are several results on stability based on multiple Lyapunov functions. In [44], the Lyapunov-like function is decreasing when the corresponding mode is active and does not increase its value at each switching instant.

Less conservative results can be obtained, if the switching signals are restricted in such a way that, at every time when we switch from a certain subsystem, its corresponding Lyapunov-like function value is smaller than its value at the previous exiting time, then the switched system is asymptotically stable [59]. The Lyapunov-like function may increase its value during a time interval, only if the increment is bounded by certain kind of continuous functions [60].

In stability analysis based on the multiple Lyapunov function, it is important to construct suitable family of Lyapunov-like functions. Piecewise quadratic Lyapunov functions are suitable candidates because the conditions for their existence can be formulated as linear matrix inequality (LMI) problems.

Consider the state space partition with regions $\{\Omega_1, \Omega_2, \Omega_3, ..., \Omega_N\}$. These regions are defined a priori to restrict switching signals. The goal is to find LMI conditions for the existence of quadratic Lyapunov-like functions of the form of $V_i(x) = x^* P_i x$, assigned to each region $\Omega_i$. This Lyapunov-like function needs to satisfy the following conditions [61]:

1.  There exist constant scalars $\beta_i \geq \alpha_i > 0$, such that:

$$\beta_i \|x\|^2 \geq V_i(x) \geq \alpha_i \|x\|^2 > 0 \quad , \tag{1.45}$$

    hold for all $x \in \Omega_i$.

2.  $\dot{V}_i(x) < 0$, for all $x \neq 0, x \in \Omega_i$.


3.  $x^* P_j x \leq x^* P_i x$ for $x \in \Omega_{i,j} \subset \Omega_i \bigcap \Omega_j$. The region $\Omega_{i,j}$ stands for the states

    where the trajectory passes from $\Omega_i$ to $\Omega_j$.

These constrained linear matrix inequalities can be replaced by LMI conditions without constraints using a technique called $S$-procedure[62].

Assume that each region has a quadratic representation or can be approximated in quadratic form:

$$\Omega_i = \left\{ x \middle| \quad x^* Q_i x \geq 0 \right\}, \tag{1.46}$$

and regions $\Omega_{i,j}$ can be expressed or approximated by:

$$\Omega_{i,j} = \left\{ x \middle| \quad x^* Q_{i,j} x \geq 0 \right\}. \tag{1.47}$$

The following theorem is the result of $S$-procedure:

**Theorem 5 [63]:** A switched linear $\dot{x}(t) = A_\sigma x(t)$ is stable if there exist symmetric matrices $P_i$ and scalars $\alpha > 0$, $\beta > 0$, $\mu_i \geq 0$, $\nu_i \geq 0$, $\vartheta_i \geq 0$ and $\eta_{i,j} \geq 0$, such that:

$$\begin{cases} \alpha I + \mu_i Q_i \leq P_i \leq \beta I - \nu_i Q_i, \\ A_i^* P_i + P_i A_i + \vartheta_i Q_i \leq -I, \\ P_j + \eta_{i,j} Q_{i,j} \leq P_i. \end{cases} \qquad (1.48)$$

are satisfied.

**Stabilization**

The problem of stabilization of switched system is a basic problem in theory of stability for hybrid systems which received a lot of attention. Most of the work in this context has focused on quadratic stabilization. A switched system is called quadratically stabilizable if there exist switching signals which stabilize the switched system with a quadratic Lyapunov function. It is well-known that a necessary and sufficient condition for a bimodal switched linear system to be quadratically stabilizable is the existence of a stable convex combination of the two subsystems' matrices.

**Theorem 6: [64]** A bimodal switched linear system $\dot{x}(t) = A_i x(t)$, $i = 1, 2$, is quadratically stablizable if and only if there exist a stable matrix in the pencil of subsystems' matrices $\gamma_\alpha(A_1, A_2)$.

The generalization of this theorem for switched linear systems with more than two discrete modes is stated in the sequel.

**Theorem 7: [61]** A switched linear system $\dot{x}(t) = A_i x(t)$, $i = 1, 2, ..., N$, is quadratically stablizable if there exist a stable matrix in the convex combination of subsystems' matrices

$$A_\alpha = \sum_{i=1}^{N} \alpha_i A_i \text{ ,where } \sum_{i=1}^{N} \alpha_i = 1, \quad \alpha_i \in [0,1].$$

The existence of a stable convex combination matrix is only sufficient for switched linear systems with more than two modes. There are examples for which no stable convex combination matrix exists, yet the system is quadratically stabilizable for a particular switching signal [45].

## 1.2.3 Model Reduction of Switched/Hybrid Systems

Complexity involved in analysis and control of large-scale hybrid dynamical systems has motivated the study of model reduction for hybrid systems in the last few years [65]-[73]. However there are not so many research reported in the literature. The study of model reduction problem for switched systems of Markovian type was among the first efforts in this context [72].

A jump linear system of Markovian type is described as:

$$\Sigma : \begin{cases} x(k+1) = A(\eta(k))x(k) + B(\eta(k))u(k), \\ y(k) = C(\eta(k))x(k) + D(\eta(k))u(k), \end{cases} \tag{1.49}$$

where $\eta(k)$ is a discrete homogeneous Markov chain on $S = \{1, 2, ..., N\}$ with transition probability matrix $\Lambda = [\lambda_{i,j}(k)]_{i,j \in S}$, and $\lambda_{i,j}$ is defined as

$$\lambda_{i,j} = P(\eta(k+1) = j | \eta(k) = i) \quad , \tag{1.50}$$

where: $\sum_{j=1}^{N} \lambda_{i,j} = 1$.

In Markov jump systems, the transition probabilities of the jumping process are important and so far, almost all the issues on Markov jump system have been investigated assuming the knowledge of transition probabilities. The likelihood to obtain the complete knowledge on the transition probabilities is questionable and the cost is probably high [73]. Due to this fact other efforts for model reduction of hybrid systems focused on other types of dynamical systems rather than switched systems of Markovian type.

Piece-wise affine (PWA) systems are one of the important subclasses of hybrid systems. PWA systems consist of affine dynamical systems on polyhedral subspace of state-space. The reduction of affine systems on polytopes was studied in[66]. It was shown that the dimension of the state space can be affinely reduced due to non-observability if and only if a subspace of the classical unobservable subspace, characterized using the normal vectors of the exit facets, is nontrivial. Let $s_p$ be a finite-dimensional linear time invariant affine system on a full dimensional polytope $X \subset \mathbb{R}^N$ :

$$s_p : \begin{cases} \dot{x} = A_p x + B_p u + a_p, \\ y = C_p x + D_p u + b_p. \end{cases} \tag{1.51}$$

Let $F = \{F_i | \quad i = 1, 2, ..., k\}$ be the set of all exit facet of this system and assume

that $F_i = X \bigcap \left\{ x \in \mathbb{R}^N \left| n_i^T x = \alpha_i \right. \right\}$. For $i = 1, 2, ..., k$ :

$$W_i := \ker(n_i^T) \, ,$$

$$W := \bigcap_{i=1}^{k} W_i = \ker(N_k) \, ,$$

where: $N_k := (n_1, n_2, ..., n_k)^T \in \mathbb{R}^{k \times n}$ and finally:

$$V := \ker((C_p^T \left| N_k^T \right.), A^T (C_p^T \left| N_k^T \right.), ...., (A^T)^{N-1} (C_p^T \left| N_k^T \right.))^T \, . \tag{1.52}$$

It was shown in [66] that reduction of the dimension of the state polytope $X$ is possible, using an affine transformation, provided that $V \neq \{0\}$. This result provides an exact reduction. While exact reduction is very elegant, the class of systems for which this procedure applies is quite small. This method only considers observability for investigating the importance of states to discard.

The method presented in [65] deals with the abstraction of both continuous and discrete parts of hybrid dynamical systems, and uses balanced residualization for reduction of the continuous part. Application of the method to switched systems may not preserve stability, and non-elegant behavior may arise for general hybrid systems because of approximation error and possible guard/reset map overlap.

The problem of model reduction for discrete switched system is addressed in several papers [67]-[71]. In [67], two different approaches are proposed to solve this problem. The first approach casts the model reduction problem to a convex optimization problem, which solves the model reduction problem by using a linearization procedure. The second one, based on the cone complementarity linearization idea, casts the model reduction problem to a sequential minimization problem subject to linear matrix inequality constraints. Both approaches have their own advantages and disadvantages concerning conservatism and computational complexity. These optimization problems will be very hard (if not infeasible) to solve for a large scale system. Not only this method is restricted to discrete time switched systems, it does not provide any hint about the number of states which is suitable to retain before reduction. Similar methods have been developed for more general classes of discrete time switched systems in [68]-[71]. In [69] and [70], this problem is investigated for discrete-time switched systems under average dwell time switching rather than arbitrary switching.

## 1.3 Outline of the Thesis

This thesis is a collection of publications and it is divided into two parts; an introduction, overview of the contributions and the contributions themselves. Part one has already been begun with an introduction and state of the art in Chapter 1. The summery of

contributions will be presented in Chapter 2. Part one closes with some conclusions on the work and suggestions for future work in Chapter 3. The publications made during this PhD project which are appended in part two are listed below.

**Paper A** [74]: In this paper, a general method for model/controller order reduction of switched linear dynamical systems is presented. The proposed technique is based on the common generalized gramian framework for reduction. This paper is the extended version of the conference papers [75] and [76]. The method in this paper preserves that stability for all switching signal.

**Paper B** [77]: In this paper, a method for model order reduction of switched linear dynamical systems is presented. The method uses convex generalized gramian which is a convex combination of the generalized gramians. This framework is less conservative than the method in paper A. Stability preservation under model reduction is also studied for the method in this paper

**Paper C** [78]: A general framework for model order reduction of discrete-time switched linear dynamical systems is presented in this paper. This method is based on switching generalized gramians and it preserves the stability of switched systems under reduction.

**Paper D** [79]: Reduction of an affine system inside the polytope is studied in this paper. The challenges for approximate reduction of this class of systems are discussed.

**Paper E** [80]: In this paper a method for stability analysis for a class of switched nonlinear system is proposed. This method is motivated by needs for stability conditions which can be preserved under reduction and also are suitable for design problems.

**Paper F** [81]: Output feedback design for a class of switched nonlinear systems is discussed in this paper along with some improvements on stability results of the Paper E. This shows how the proposed stability conditions can be used for design problems.

# 2. | Summary of Contributions

The contributions of this thesis are in the form of several publications in the context of model reduction and stability analysis for hybrid systems. These contributions are highlighted in this chapter.

## 1.    Common Generalized Gramian Framework for Switched Model/ Controller Reduction

Paper A presents a general method for model/controller order reduction of switched linear dynamical systems. The proposed technique is based on the generalized gramian framework for model reduction. It is shown that different classical reduction methods can be developed into generalized gramian framework. Balanced reduction within specified frequency bound is developed within this framework. In order to avoid numerical instability and also to increase the numerical efficiency, generalized gramian based Petrov-Galerkin projection is constructed instead of the similarity transform approach for reduction. The framework is developed for switched controller reduction. To the best of our knowledge, there is no other reported result on switched controller reduction in the literature. The method preserves the stability under arbitrary switching signal for both model and controller reduction. Furthermore it is applicable to both continuous and discrete time systems for different classical gramian based reduction methods.  The performance of the proposed method is illustrated by numerical examples.

## 2.    Convex Generalized Gramian Framework for Model Reduction of Switched Systems

In paper B, a method for model order reduction of switched linear dynamical systems is presented which uses convex generalized gramian rather than common generalized gramian. Convex generalized gramian is a convex combination of the generalized gramians. It is shown that different classical reduction methods can be developed into the generalized gramian framework for model reduction of linear systems and further for the reduction of switched systems by construction of the convex generalized gramian. Balanced reduction

within specified frequency bound is taken as an example which is developed within this framework. This framework is less conservative than the method based on common generalized gramian. In this paper also in order to avoid numerical instability and to increase the numerical efficiency, convex generalized gramian based Petrov-Galerkin projection is constructed instead of the similarity transform approach for reduction. It is proven that the method preserves the stability of the original switched system at least for stabilizing switching signal. Some discussions on the coefficient of the vertices of the convex variables are presented. The method is illustrated by numerical examples.

### 3.  Model Reduction of Switched Systems based on switching gramian

A method for model order reduction of discrete time switched linear systems is presented in paper C. The proposed technique uses switching generalized gramians. It is shown that different classical reduction methods can be developed into the generalized gramian framework for model reduction of linear systems and further for the reduction of switched systems. Discrete time balanced reduction within specified frequency interval is taken as an example which is developed within this framework. It is proven that the proposed reduction framework preserves the stability of the original switched system. The method is less conservative compared to methods based on convex or common generalized gramians.

### 4.  On exact/approximate reduction of dynamical systems on linear partitions

Order reduction problem for dynamical systems living on piecewise linear partitions is addressed in paper D. This problem is motivated by analysis and control of hybrid systems. The technique presented is based on the transformation of affine dynamical systems inside the cells into a new structure and it can be applied for both exact reduction and also approximate model reduction. In this method both controllability and observability of the affine system inside the polytopes are considered for the reduction purpose. The framework is illustrated with a numerical example.

### 5.  Stability analysis for a class of switched nonlinear systems

Stability analysis for a class of switched nonlinear systems is addressed in this paper. Two linear matrix inequality (LMI) based sufficient conditions for asymptotic stability are proposed for switched nonlinear systems. These conditions are analogous counterparts for switched linear systems which are shown to be easily verifiable and suitable for design problems.

### 6.  Stability analysis for a class of switched nonlinear systems

The problem of output feedback control for a class of switched nonlinear systems is addressed in this paper. Two conditions for output feedback controller synthesis based on the proposed stability conditions are presented. These conditions are based on conditions for stability analysis for switched linear systems which are shown to be easily verifiable and suitable for design problems.

# 3. | Conclusions and Future Works

Methods for model reduction and stability analysis of hybrid/switched systems are presented in this thesis. Three frameworks for model reduction of switched systems are proposed which are based on the notion of the generalized gramians. Generalized gramians are the solutions to the observability and controllability Lyapunov inequalities. To compute the generalized gramians linear matrix inequalities (LMI's) need to be solved. In the frameworks based on common generalized gramian and switching generalized gramians the stability of the original switched system is guaranteed to be preserved for all switching signals while the switched system is reduced. Order reduction problem for dynamical systems living on piecewise linear partitions is also studied in this thesis. The conditions for stability which can be preserved under projection are important in model reduction of switched systems. Motivated by this, conditions for stability analysis and control of a class of switched nonlinear systems are proposed in this thesis. These conditions are suitable for design problems because they have slack variables. A technique for output feedback control design based on these conditions is proposed. The methods which are proposed in this thesis are applied to several numerical examples. There are several directions for further research in model reduction of hybrid systems. Some of these ideas are listed bellow:

- **Lyapunov equation based reduction of switched systems:** LMI based model reduction is computationally more expensive compared to the methods which require solving Lyapunov equations. Finding the generalized gramians for some classes of switched system by solving the Lyapunov equations rather than LMI's can reduce significant amount of computations. This can improve the proposed methods for model reduction of switched system for particular classes of switched systems.
- **Switched controller reduction: A** method for switched controller reduction based on switching generalized gramian can be developed. The method is similar to the method which has been proposed based on common generalized gramian for switched controller reduction and it will be less conservative.

- **Model reduction of switched $\Phi$ -systems:** Developing a framework to reduce switched nonlinear systems is an interesting topic. The idea is to devise a reduction framework based on the generalized gramians which uses the stability conditions which have recently proposed in this these.
- **Partitioning the state-space for reduction:** The idea is to partition the state-space such that the resulting hybrid system is reducible with suitable approximation error.

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# Contributions

**Paper A:** Generalized Gramian Framework for Model/Controller Order Reduction of Switched Systems

**Paper B:** Switched Systems Reduction Framework Based on Convex Combination of Generalized Gramians

**Paper C:** Model Reduction of Switched Systems Based on Switching Generalized Gramians

**Paper D:** On Exact/Approximate Reduction of Dynamical Systems Living on Piecwise linear Partition

**Paper E:** Stability Analysis for a Class of Switched Nonlinear Systems

**Paper F:** Output feedback control for a class of switched nonlinear systems

# Paper A

Generalized Gramian Framework for Model/Controller Order Reduction of
Switched Systems

Hamid Reza Shaker and Rafael Wisniewski

## 1. Introduction

The ever-increasing need for accurate mathematical modeling of physical as well as artificial processes for simulation and control leads to models of high complexity, arising from high order and/or complex nonlinearities. To maintain tractability, efficient computational prototyping tools are required to replace such complex models by simpler models which capture their dominant characteristics. Due to this fact, model reduction methods have become increasingly popular over the last two decades [1]-[3]. Such methods are designed to extract a reduced order state space model that adequately describes the behavior of the system in question.

Low order controllers are preferred over high order controllers due to i) relative ease of implementation, ii) relative lower requirements on hardware (i.e. cheaper, simpler and more easily available hardware), iii) relative computational efficiency, and iv) decreased effects of computational delay on closed-loop stability and performance. For these reasons controller order reduction has received considerable attention in recent years [4]-[7].

Most of the studies on order reduction to date have been devoted to linear systems. The few methods proposed for nonlinear systems are not strong in comparison to linear reduction methods.

On the other hand, most of the methods that are proposed so far for control and analysis of hybrid/switched systems suffer from high computational burden when dealing with large-scale dynamical systems. This motivated the study of model reduction for hybrid systems [8]-[17]. The model reduction problem for switched systems of Markovian type was studied in [17]. In Markov jump systems, the transition probabilities of the jumping process are important and so far, almost all the issues on Markov jump system have been investigated assuming the knowledge of transition probabilities. However, the likelihood to obtain the complete knowledge on the transition probabilities is questionable and the cost is probably high [18]. The method presented in [8] deals with the abstraction of both continuous and discrete parts of hybrid dynamical systems, and uses balanced residualization for reduction of the continuous part. Application of the method to switched systems may not preserve stability, and non-elegant behavior may arise for general hybrid systems because of approximation error and possible guard/reset map overlap. It was shown in [9] that the dimension of the state space can be affinely reduced due to non-observability if and only if a subspace of the classical unobservable subspace, characterized using the normal vectors of the exit facets, is nontrivial. This result does not provide a strong tool for reduction of affine systems because it is an exact reduction. While exact reduction is very elegant, the class of systems for which this procedure applies is quite small. This method only considers observability for investigating the importance of states to discard. The problem of model reduction for discrete switched system is addressed in several papers [10][14]-[16]. In [10], two different approaches are proposed to solve this problem. The first approach casts the

model reduction problem as a convex optimization problem, which solves the model reduction problem by using a linearization procedure. The second one, based on the cone complementarity linearization idea, casts the model reduction problem as a sequential minimization problem subject to linear matrix inequality constraints. Both approaches have their own advantages and disadvantages concerning conservatism and computational complexity. These optimization problems will be very hard (if not infeasible) to solve for a large scale system. Not only this method is restricted to discrete time switched systems, it does not provide any hints about the number of states which is suitable to retain before reduction. Similar methods have been developed for more general classes of discrete time switched systems in [13]-[16]. In [14] and [15], this problem is investigated for discrete-time switched systems under average dwell time switching.

In this paper, we propose a method for model reduction of switched systems which can be categorized as gramian based model reduction methods. The balanced model reduction introduced in [19] is one of the most common gramian based model reduction schemes.

To apply balanced reduction, the system is first represented in a basis where the states which are difficult to reach are simultaneously difficult to observe. This is achieved by simultaneously diagonalizing the reachability and the observability gramians, which are solutions to the reachability and the observability Lyapunov equations. Then, the reduced model is obtained by truncating the states which have this property. Balanced model reduction method is modified and developed from different viewpoints [1],[2]. One of the methods that are presented based on balanced model reduction is the method based on the generalized gramian [20]. In this method, Lyapunov inequalities (rather than Lyapunov equations) are solved to compute generalized gramians. The physical interpretations of generalized gramians are similar to ordinary gramians. Generalized gramians are used to devise a technique for structure preserving model reduction methods in [21].

We first show that the generalized method in [20] can be extended to various gramian based reduction methods. We also modify the method in [21] to avoid numerical instability and also to achieve higher efficiency by building Petrov-Galerkin projection based on generalized gramians. We propose a method based on the balanced model reduction within frequency bound in this framework. We generalize the framework to model reduction of switched systems by solving a system of Lyapunov inequalities to find common generalized gramian. We compute Petrov-Galerkin projection based on these common generalized gramians. There are several advantages in using common generalized gramians rather than ordinary gramians, mainly in stability preservation, structure preservation and computational efficiency. Methods based on ordinary gramians have to repeatedly compute the projection matrices for gramians of different subsystems. In our framework, we build the projection matrices once for all subsystems based on common generalized gramian which is computational more efficient. On the other, if we use ordinary gramians the stability of the original switched system is not guaranteed to be preserved under reduction. In our framework, the stability of the original switched system is preserved for arbitrary switching signal.

In this paper we also propose a technique for switched controller reduction based on the generalized gramian framework and Petrov-Galerkin projection which can be considered as carry over of the method in [6]. To the best of our knowledge, this is the first result on switched controller reduction in the literature.

The paper is organized as follows. In the next section we review balanced reduction method and the balanced reduction technique based on generalized gramian. Section 2 presents how different gramian based methods can be approximated as generalized gramian based techniques. Balanced reduction within frequency bound based on generalized gramian is also presented in this section. This section ends up with some remarks on numerical implementation of the algorithm and Petrov-Galerkin projection for generalized gramian based reduction methods is suggested instead of balancing and truncation. Section 3, is devoted to develop generalized gramian based reduction method for model reduction of switched systems, followed by a brief discussion on stability, feasibility and error bound. Section 4 shows how the generalized gramian based method can be applied to switched controller reduction followed by some remarks on numerical implementation of the algorithm, stability and error bounds. Section 5 presents our numerical results, and Section 6 concludes the paper.

The notation used in this paper is as follows: $M^*$ denotes the transpose of matrix if $M \in \mathbb{R}^{n \times m}$ and complex conjugate transpose if $M \in \mathbb{C}^{n \times m}$. The norm $\|.\|_\infty$ denotes the $H_\infty$ norm of a rational transfer function. The standard notation $>$, $\geq (<, \leq)$ is used to denote the positive (negative) definite and semidefinite ordering of matrices.

## 2. Balanced Truncation and Generalized Gramians

Balanced truncation is a well-known method for model reduction of dynamical systems [1][2].The basic approach relies on balancing the gramians of the systems. For dynamical systems with minimal realization:

$$G(s) := (A, B, C, D), \tag{1}$$

where $G(s)$ is the transfer matrix with associated state-space representation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(t) \in \mathbb{R}^n, \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{2}$$

gramians are given by the solutions of the Lyapunov equations:

$$\begin{aligned} AP + PA^* + BB^* &= 0, \\ A^*Q + QA + C^*C &= 0. \end{aligned} \tag{3}$$

For stable $A$, they admit unique positive definite solutions $P$ and $Q$, called the

controllability and observability gramians. In balanced reduction, first the system is transformed to the balanced structure in which gramians are equal and diagonal:

$$P = Q = diag(\sigma_1 I_{k_1}, ..., \sigma_q I_{k_q}), \quad \sum_{j=1}^{q} k_j = n. \tag{4}$$

where $\sigma_i > \sigma_{i+1}$ are called Hankel singular values.

The reduced model can be easily obtained by truncating the states which are associated with the set of the least Hankel singular values. Applying the method to stable, minimal $G(s)$, if we keep all the states associated to $\sigma_m (1 \le m \le r)$, and truncate the rest, the reduced model $G_r(s)$ will be minimal and stable and satisfies[1][2]:

$$\left\| G(s) - G_r(s) \right\|_\infty \le 2 \sum_{j=r+1}^{q} \sigma_j . \tag{5}$$

A closely related model reduction method based on the generalized gramian is presented in [20]. In this method, instead of Lyapunov equations (3), the following Lyapunov inequalities are solved:

$$\begin{aligned} AP_g + P_g A^* + BB^* \le 0, \\ A^* Q_g + Q_g A + C^* C \le 0. \end{aligned} \tag{6}$$

For stable $A$, they have positive definite solutions $P_g$ and $Q_g$, called the generalized controllability and observability gramians. Note that these gramians are not unique. The rest of this model reduction method is the same as the aforementioned balanced truncation method, the only difference is that in this algorithm the balancing and truncation are based on the generalized gramian instead of ordinary gramian. In this method, we have generalized Hankel singular values ($\gamma_i$) which are the diagonal elements of balanced generalized gramians instead of Hankel singular values $\sigma_i$. The error bound (5) holds in terms of the generalized Hankel singular values ($\gamma_i$) instead of Hankel singular values ($\sigma_i$). Note that $\gamma_i \ge \sigma_i$ [20]. Therefore the error in balanced reduction based on the generalized gramian is lower bounded by the error of ordinary balanced model reduction.

## 3. Generalized Gramian Framework for Gramian-based Model Reduction Methods

In this section, we first present a general framework to build generalized gramian analogous of gramian based methods. Then we present generalized balanced reduction within frequency bound using this framework, followed by a discussion on numerical implementation of the algorithm based on projection.

**Lemma 1:** Suppose $A$ is stable, $Y$ is symmetric and

$$A^*Y + YA \leq 0,$$
$$A, Y \in \mathbb{R}^{n \times n},$$

(7)

holds. Then $Y \geq 0$, i.e. $Y$ is positive semi-definite.

**Proof:** If $A^*Y + YA \leq 0$, there exists $M \geq 0$ such that:

$$A^*Y + YA + M = 0$$

On the other hand, for any stable $A$, the unique solution to the preceding is:

$$Y = \int_0^\infty e^{A^*\tau} M e^{A\tau} d\tau .$$

In the above structure $M \geq 0$, hence:

$$Y \geq 0$$

$\square$

This lemma leads to the following proposition that makes the relation between Lyapunov equations and Lyapunov inequalities evident.

**Proposition 1[20]:** Suppose $A$ is stable and $X$ is the solution of Lyapunov equation:

$$A^*X + XA + Q = 0 ,$$

(8)

where $Q \geq 0$. If a symmetric $X_g$ satisfies:

$$A^*X_g + X_g A + Q \leq 0$$

(9)

then: $X_g \geq X$ .

**Proof:** It can be proven easily by subtracting (9)-(8) and applying Lemma 1 with $Y = X_g - X$ .

$\square$

Proposition 1 is a direct consequence of Lemma 1, which shows how ordinary gramians can be approximated by the generalized gramians. Balanced reduction based on generalized gramian which we reviewed in the last section is based on Proposition 1. While this method might be less accurate than its gramian based counterpart, the approximation error is still bounded.

By deriving associated Lyapunov equations and relaxing them to inequalities, we can readily generalize other gramian based reduction methods in this framework. In the following, we propose a generalized version of balanced reduction within frequency bound.

*Generalized Balanced Reduction within Frequency Bound*

Over the past two decades, a great deal of attention has been devoted to balanced model reduction and it has been developed and improved from several viewpoints. Frequency weighted balanced reduction method is one of the devised gramian based techniques based on ordinary balanced truncation [1],[2],[22]-[24]. In this method, the model reduction is biased by frequency- dependent input/output weights. In many cases the input and output weights are not given. Instead the problem is to reduce the model over a given frequency range [1][2]. This problem can be attacked directly by balanced reduction within frequency bound, which was first proposed in [25] and then modified in [2] to preserve the stability of the original system and provide an error bound for approximation. In this method, for dynamical system (1) the controllability gramian $P(\omega_1, \omega_2)$ and observability gramians $Q(\omega_1, \omega_2)$ within frequency range $[\omega_1, \omega_2]$ are defined as:

$$P(\omega_1, \omega_2) = P(\omega_2) - P(\omega_1),$$
$$Q(\omega_1, \omega_2) = Q(\omega_2) - Q(\omega_1),$$

$$(10)$$

where:

$$P(\omega) := \frac{1}{2\pi} \int_{-\omega}^{\omega} (Ij\theta - A)^{-1} BB^* (-Ij\theta - A^*)^{-1} d\theta,$$

$$Q(\omega) := \frac{1}{2\pi} \int_{-\omega}^{\omega} (-Ij\theta - A^*)^{-1} C^* C (Ij\theta - A)^{-1} d\theta.$$

$$(11)$$

In order to show the associated Lyapunov equations, we need some more notations:

$$S(\omega) := \frac{1}{2\pi} \int_{-\omega}^{\omega} (Ij\theta - A)^{-1} d\theta,$$

$$(12)$$

$$W_c(\omega) = S(\omega)BB^* + BB^* S^*(\omega),$$
$$W_o(\omega) = C^* C S(\omega) + S^*(\omega) C^* C,$$

$$(13)$$

$$W_c(\omega_1, \omega_2) = W_c(\omega_2) - W_c(\omega_1),$$
$$W_o(\omega_1, \omega_2) = W_o(\omega_2) - W_o(\omega_1).$$

$$(14)$$

The gramians satisfy the following Lyapunov equations [1],[2]:

$$AP(\omega_1, \omega_2) + P(\omega_1, \omega_2)A^* + W_c(\omega_1, \omega_2) = 0,$$
$$A^* Q(\omega_1, \omega_2) + Q(\omega_1, \omega_2)A + W_o(\omega_1, \omega_2) = 0.$$

$$(15)$$

This method is modified in [2] to guarantee stability and to provide a simple error bound.

The modified version starts with the following decomposition of $W_c(\omega_1, \omega_2)$ and $W_o(\omega_1, \omega_2)$ :

$$W_c(\omega_1, \omega_2) := M \Lambda M^* = M \, diag(\lambda_1, ..., \lambda_n)M^*,$$
$$W_o(\omega_1, \omega_2) := N \Delta N^* = N \, diag(\delta_1, ..., \delta_n)N^*, \tag{16}$$

where: $MM^* = NN^* = I_n$, $|\lambda_i| \geq |\lambda_{i+1}| \geq 0$, $|\delta_i| \geq |\delta_{i+1}| \geq 0$.

Note that since $W_c(\omega_1, \omega_2)$ and $W_o(\omega_1, \omega_2)$ are symmetric, decompositions of the form (16) exist. Let:

$$\hat{B} := M \, diag(|\lambda_1|^{1/2}, ..., |\lambda_\xi|^{1/2}, 0, ..., 0),$$
$$\hat{C} := diag(|\delta_1|^{1/2}, ..., |\delta_\rho|^{1/2}, 0, ..., 0) N^*. \tag{17}$$

where:

$$\xi = rank(W_c(\omega_1, \omega_2)),$$
$$\rho = rank(W_o(\omega_1, \omega_2)). \tag{18}$$

The modified gramians satisfy the following Lyapunov equations instead of (15):

$$A\hat{P}(\omega_1, \omega_2) + \hat{P}(\omega_1, \omega_2)A^* + \hat{B}\hat{B}^* = 0,$$
$$A^*\hat{Q}(\omega_1, \omega_2) + \hat{Q}(\omega_1, \omega_2)A + \hat{C}^*\hat{C} = 0. \tag{19}$$

For the generalization, we have the following inequalities:

$$A\hat{P}_g(\omega_1, \omega_2) + \hat{P}_g(\omega_1, \omega_2)A^* + \hat{B}\hat{B}^* \leq 0,$$
$$A^*\hat{Q}_g(\omega_1, \omega_2) + \hat{Q}_g(\omega_1, \omega_2)A + \hat{C}^*\hat{C} \leq 0. \tag{20}$$

Then the generalized modified balanced reduction within frequency bound can be obtained by simultaneously diagonalizing $\hat{P}_g(\omega_1, \omega_2)$ and $\hat{Q}_g(\omega_1, \omega_2)$ and truncating the states associated to the set of the least generalized Hankel singular values.

*Numerical Issues*

Balanced transformation can be numerically ill-conditioned when dealing with systems having some nearly uncontrollable or some nearly unobservable modes. Difficulties associated with computation of the required balanced transformation in [26] drew some attention toward devising alternative numerical methods [27]. Balancing can be badly conditioned even when some states are significantly more controllable than observable or vice versa. In this case, it is advisable to reduce the system in the gramian based framework

without balancing. The Schur method and square root algorithms provide projection matrices to apply balanced reduction without balanced transformation [1],[27]. This method can be easily applied to other Gramian based method. In our generalized method, we can use the same algorithm by plugging generalized gramians into the algorithm instead of ordinary gramians.

## 4. Model Reduction of Switched System

*Model Reduction of Switched Systems Based on Generalized Gramians*

One of the most important subclasses of hybrid systems is the class of linear switched systems. A linear switched system is a dynamical system specified by the following equations:

$$\Sigma : \begin{cases} \dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t), \\ y(t) = C_{\sigma(t)} x(t) + D_{\sigma(t)} u(t), \end{cases} \tag{21}$$

where $x(t) \in \mathbb{R}^n$ is the continuous state, $y(t) \in \mathbb{R}^p$ is the output, $u(t) \in \mathbb{R}^m$ is the continuous input, and $\sigma : \mathbb{R}^{\geq 0} \to \mathrm{K} \subset \mathbb{N}$ is the switching signal that is a piecewise constant map of the time. K is the set of discrete modes, and it is assumed to be a finite set. For each $i \in \mathrm{K}$, $A_i, B_i, C_i, D_i$ are matrices of appropriate dimensions.

In this section we build a framework for model reduction of switched system described by (21). The aim is to find Petrov-Galerkin projectors to project the switched system to a lower dimensional subspace.

Petrov-Galerkin projection for a dynamical system [1]:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) &, \quad x \in \mathbb{R}^n, \\ y(t) = g(x(t), u(t)) &, \end{cases} \tag{22}$$

is defined as a projection $\Pi = VW^*$, where: $W^*V = I_k$, $V, W \in \mathbb{R}^{n \times k}, k < n$.

The reduced order model using this projection is:

$$\begin{cases} \dot{\hat{x}}(t) = W^* f(V\hat{x}(t), u(t)) &, \hat{x} \in \mathbb{R}^k \\ y(t) = g(V\hat{x}(t), u(t)) \end{cases} \tag{23}$$

We can develop generalized gramian framework for model reduction of switched linear

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system by finding common generalized controllability/observability gramian related to subsystems. To do this we need to solve two systems of Lyapunov inequalities, one for finding common generalized controllability gramian and one for common generalized observability gramian. The next step can be simultaneous diagonalization of the common generalized gramians and balancing and reducing all subsystems based on the common generalized Hankel singular values. To improve numerical conditioning and efficiency, we use Schur or square root algorithm (instead of balancing) and directly compute Petrov-Galerkin projection matrices. To clarify the method, we extend generalized balanced reduction within frequency bound presented in previous section for model reduction of switched linear systems.

First we need to find common generalized controllability gramian $\hat{P}_{cg}(\omega_1,\omega_2)$ by solving the system of Lyapunov inequalities :

$$\begin{cases} A_\sigma \hat{P}_{cg}(\omega_1,\omega_2) + \hat{P}_{cg}(\omega_1,\omega_2)A_\sigma^* + \hat{B}_\sigma \hat{B}_\sigma^* < 0, \\ \forall \sigma \in \mathrm{K}. \end{cases} \tag{24}$$

For example in the case of bimodal systems, $\mathrm{K} = \{1,2\}$ , so we have to solve:

$$\begin{cases} A_1 \hat{P}_{cg}(\omega_1,\omega_2) + \hat{P}_{cg}(\omega_1,\omega_2)A_1^* + \hat{B}_1 \hat{B}_1^* < 0, \\ A_2 \hat{P}_{cg}(\omega_1,\omega_2) + \hat{P}_{cg}(\omega_1,\omega_2)A_2^* + \hat{B}_2 \hat{B}_2^* < 0. \end{cases} \tag{25}$$

Common generalized observability gramian $\hat{Q}_{cg}(\omega_1,\omega_2)$ can be obtained similarly by solving the system of Lyapunov inequalities:

$$\begin{cases} A_\sigma^* \hat{Q}_{cg}(\omega_1,\omega_2) + \hat{Q}_{cg}(\omega_1,\omega_2)A_\sigma + \hat{C}_\sigma^* \hat{C}_\sigma < 0, \\ \forall \sigma \in \mathrm{K}. \end{cases} \tag{26}$$

Similarly, in the case of bimodal systems, $\mathrm{K} = \{1,2\}$ , we have:

$$\begin{cases} A_1^* \hat{Q}_{cg}(\omega_1,\omega_2) + \hat{Q}_{cg}(\omega_1,\omega_2)A_1 + \hat{C}_1^* \hat{C}_1 < 0, \\ A_2^* \hat{Q}_{cg}(\omega_1,\omega_2) + \hat{Q}_{cg}(\omega_1,\omega_2)A_2 + \hat{C}_2^* \hat{C}_2 < 0. \end{cases} \tag{27}$$

If we plug in $\hat{P}_{cg}(\omega_1,\omega_2)$ and $\hat{Q}_{cg}(\omega_1,\omega_2)$ to the square root algorithm we can directly obtain projectors for reduction. Note that the results are the same as balancing algorithm. A merit of the square root method is that it relies on the Cholesky factors of the gramians rather than the gramians themselves, which has superior of numerical stability.

One of the important issues in model reduction is preservation of stability. In other words, the question is whether the reduction technique method can preserve the stability of the original model in approximation. In the following proposition, we show that the proposed framework for model reduction of switched systems is stability preserving.

**Proposition 2.** If the switched system described in (21) is stable, the generalized gramian based reduced order model is quadratically stable.

**Proof:**

In the proposed method, we have:

$$W^*V = I_k , \quad V, W \in \mathbb{R}^{n \times k}, k < n,$$

$$\hat{\Sigma} : \begin{cases} \hat{A}_{\sigma(t)} = W^* A_{\sigma(t)} V, \\ \hat{B}_{\sigma(t)} = W^* B_{\sigma(t)}, \\ \hat{C}_{\sigma(t)} = C_{\sigma(t)} V, \\ \hat{D}_{\sigma(t)} = D_{\sigma(t)}, \end{cases} \tag{28}$$

which is a projected switched system (reduced order model). The outcome of square root algorithm for projection [1]: $P_g W = V \Sigma_1$ and $Q_g V = W \Sigma_1$, where $\Sigma_1 \in \mathbb{R}^{k \times k}$ is diagonal and positive definite. Since $P_g$ is common generalized gramian, we have: $A_{\sigma(t)} P_g + P_g A^*_{\sigma(t)} < 0$, which implies:

$$W^* (A_{\sigma(t)} P_g + P_g A^*_{\sigma(t)}) W < 0$$

On the other hand,

$$W^* (A_{\sigma(t)} P_g + P_g A^*_{\sigma(t)}) W = W^* A_{\sigma(t)} P_g W + W^* P_g A^*_{\sigma(t)} W$$

$$= W^* A_{\sigma(t)} V \Sigma_1 + \Sigma_1^* V^* A^*_{\sigma(t)} W = \hat{A}_{\sigma(t)} \Sigma_1 + \Sigma_1 \hat{A}^*_{\sigma(t)}.$$

Hence:

$$\hat{A}_{\sigma(t)} \Sigma_1 + \Sigma_1 \hat{A}^*_{\sigma(t)} < 0 , \tag{29}$$

where $\Sigma_1 \in \mathbb{R}^{k \times k}$ is positive definite. Similarly we can show that:

$$\hat{A}^*_{\sigma(t)} \Sigma_1 + \Sigma_1 \hat{A}_{\sigma(t)} < 0 . \tag{30}$$

In stability theory for switched systems, this is a well-known sufficient condition for quadratic stability [28]. Hence, reduced order model is guaranteed to be quadratically stable.

□

The same results hold, if we use balancing transformation instead of projection. In balanced realization based on generalized gramians, after partitioning we have:

$$M = A_\sigma P_g + P_g A_\sigma^*,$$

$$= \begin{bmatrix} A_{\sigma 11} & A_{\sigma 12} \\ A_{\sigma 21} & A_{\sigma 22} \end{bmatrix} \begin{bmatrix} P_{g11} & 0 \\ 0 & P_{g22} \end{bmatrix} + \begin{bmatrix} P_{g11} & 0 \\ 0 & P_{g22} \end{bmatrix} \begin{bmatrix} A_{\sigma 11} & A_{\sigma 12} \\ A_{\sigma 21} & A_{\sigma 22} \end{bmatrix}^*,$$

$$= \begin{bmatrix} A_{\sigma 11} & A_{\sigma 12} \\ A_{\sigma 21} & A_{\sigma 22} \end{bmatrix} \begin{bmatrix} P_{g11} & 0 \\ 0 & P_{g22} \end{bmatrix} + \begin{bmatrix} P_{g11} & 0 \\ 0 & P_{g22} \end{bmatrix} \begin{bmatrix} A_{\sigma 11}^* & A_{\sigma 21}^* \\ A_{\sigma 12}^* & A_{\sigma 22}^* \end{bmatrix},$$

$$= \begin{bmatrix} A_{\sigma 11}P_{g11} + P_{g11}A_{\sigma 11}^* & A_{\sigma 12}P_{g22} + P_{g11}A_{\sigma 21}^* \\ A_{\sigma 21}P_{g11} + P_{g22}A_{\sigma 12}^* & A_{\sigma 22}P_{g22} + P_{g22}A_{\sigma 22}^* \end{bmatrix},$$

where $P_{g11}, A_{\sigma 11} \in \mathbb{R}^{k \times k}$.

Since $P_g$ is the generalized gramian, $M < 0$. On the other hand, we know If $M \leq 0$, all its leading square partititions are negative semidefinite. Hence:

$$A_{\sigma 11}P_{g11} + P_{g11}A_{\sigma 11}^* < 0, \tag{31}$$

$P_{g11}$ is symmetric and positive definite. Therefore the reduced order switched system $(A_{\sigma 11}, B_{1\sigma}, C_{1\sigma}, D_\sigma)$ associated with the partitions above satisfy the sufficient condition for quadratic stability and is stable.

Clearly, the presented framework for model reduction of switched system is stability preserving. As mentioned, the approximation error for each subsystem is bounded and is given in terms of generalized Hankel singular values.

The system of LMIs in our framework is said to be feasible if common generalized gramian exists. In general, existence of a common Lyapunov function is not guaranteed for switched systems [28], therefore we cannot expect to find common generalized gramian for all linear switched systems. One way to improve the feasibility of the proposed model reduction method is using recently proposed extended notion of generalized gramian called extended gramian [29].


## 5. Switched Controller Reduction Method

In this section we present a method for switched controller reduction followed by a brief discussion on modifications in numerical implementation of the algorithm, studying stability, feasibility and approximation error.

*Generalized Gramian Framework for Switched Controller Reduction*

Consider a general switched system with switched controller in the following closed loop configuration (see Fig. 1.)
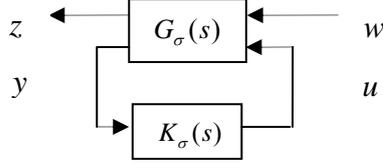
Fig. 1. Closed loop Switched System with Switching Controller

In this configuration $G_\sigma(s) := (A_\sigma, B_\sigma, C_\sigma, D_\sigma)$ is $n$-th order switched system and $K_\sigma(s) := (A_{k\sigma}, B_{k\sigma}, C_{k\sigma}, D_{k\sigma})$ is $m$-th order switched controller. The transfer matrix from $w$ to $z$ is:

$$T_{zw} := (\bar{A}_\sigma, \bar{B}_\sigma, \bar{C}_\sigma, \bar{D}_\sigma), \tag{32}$$

where:

$$\bar{A}_\sigma = \begin{bmatrix} A_\sigma + B_{2\sigma} L_\sigma D_{k\sigma} C_{2\sigma} & B_{2\sigma} L_\sigma C_{k\sigma} \\ B_{k\sigma} F_\sigma C_{2\sigma} & A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma} \end{bmatrix}, \tag{33}$$

$$\bar{B}_\sigma = \begin{bmatrix} B_{1\sigma} + B_{2\sigma} L_\sigma D_{k\sigma} D_{21\sigma} \\ B_{k\sigma} F_\sigma D_{21\sigma} \end{bmatrix}, \tag{34}$$

$$\bar{C}_\sigma = [C_{1\sigma} + D_{12\sigma} D_{k\sigma} F_\sigma C_{2\sigma} \quad D_{12\sigma} L_\sigma C_{k\sigma}], \tag{35}$$

$$\bar{D}_\sigma = D_{11\sigma} + D_{12\sigma} D_{k\sigma} F_\sigma D_{21\sigma}, \tag{36}$$

$$L_\sigma = (I - D_{k\sigma} D_{22\sigma})^{-1}, F_\sigma = (I - D_{22\sigma} D_{k\sigma})^{-1} \tag{37}$$

Note that in the above configuration switched plant $G_\sigma(s)$ is partitioned where inputs to $B_{1\sigma}$ are the disturbances, inputs to $B_{2\sigma}$ are the control inputs, output of $C_{1\sigma}$ are the errors to be kept small, and outputs of $C_{2\sigma}$ are the output measurements provided to the controller.

The goal is to reduce the controller in the way that the closed-loop behavior is preserved as much as possible without sacrificing the stability of the original switched closed loop system. To do this, we develop generalized gramian framework for switched controller reduction which is inspired by the method in [6] which was proposed for controller reduction of linear dynamical systems.

The procedure is similar to generalized gramian framework for model reduction of

switched systems. In the first step, one should find common generalized gramians for closed loop switched system. In other words, we solve:

$$\begin{cases} \bar{A}_\sigma \bar{P}_{cg} + \bar{P}_{cg} \bar{A}_\sigma^* + \bar{B}_\sigma \bar{B}_\sigma^* < 0, \\ \forall \sigma \in \mathrm{K}, \end{cases} \tag{38}$$

and also

$$\begin{cases} \bar{A}_\sigma^* \bar{Q}_{og} + \bar{Q}_{og} \bar{A}_\sigma + \bar{C}_\sigma^* \bar{C}_\sigma < 0, \\ \forall \sigma \in \mathrm{K}, \end{cases} \tag{39}$$

to find the common generalized controllability and observability gramians respectively. Due to the fact that we are interested in the reduction of controller in our framework, we should find generalized gramians that have the following structure:

$$\bar{P}_{cg} = diag(P_1, P_2) > 0, \tag{40}$$

$$\bar{Q}_{og} = diag(Q_1, Q_2) > 0, \tag{41}$$

where $P_2$ and $Q_2$ are simultaneously balanceable and are compatible with the order of the switched controller.
This can be done by solving Lyapunov linear matrix inequalities using common linear matrix inequality (LMI) solvers.
The last step of the framework is to reduce the controller by balanced truncation of each sub-controller based on $P_2$ and $Q_2$.


*Numerical Algorithm*

In our generalized method for controller reduction, we use the same algorithm as the one we have already presented for switched systems with generalized gramians $P_2$ and $Q_2$ instead of ordinary gramians. To improve the numerical properties, we use Petrov-Galerkin projection to reduce the switched controller.


*Stability, Feasibility and Approximation Error*

One of the important issues in model/controller reduction is preservation of stability. In other words, the question is whether the reduction technique method can preserve the stability of the original model in approximation. In the following proposition we show that the proposed framework for switched controller reduction is a stability preserving method. That is, it preserves the stability of the original closed loop system under arbitrary switching.

**Proposition 3.** If the closed loop system described by (32)-(37) is stable, the closed-loop system with reduced switched controller resulting from the proposed algorithm is

quadratically stable.

**Proof:**
In the proposed method for original closed loop system, we have:

$$\begin{cases} \bar{A}_\sigma \bar{P}_{cg} + \bar{P}_{cg} \bar{A}_\sigma^* + \bar{B}_\sigma \bar{B}_\sigma^* < 0, \\ \forall \sigma \in K. \end{cases}$$

where: $\bar{P}_{cg} = diag(P_1, P_2) > 0$ .

Equivalently we have:

$$(A_\sigma + B_{2\sigma} L_\sigma D_{k\sigma} C_{2\sigma}) P_1 + P_1 (A_\sigma + B_{2\sigma} L_\sigma D_{k\sigma} C_{2\sigma})^* +$$
$$(B_{1\sigma} + B_{2\sigma} L_\sigma D_{k\sigma} D_{21\sigma})(B_{1\sigma} + B_{2\sigma} L_\sigma D_{k\sigma} D_{21\sigma})^* < 0, \tag{42}$$

and

$$(A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma}) P_2 + P_2 (A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma})^* +$$
$$(B_{k\sigma} F_\sigma D_{21\sigma})(B_{k\sigma} F_\sigma D_{21\sigma})^* < 0, \tag{43}$$

On the other hand, from our reduction framework for switched controller using Petro-Galerkin projection we have:

$$W^* V = I_k , \quad V, W \in \mathbb{R}^{n \times k}, k < n \tag{44}$$
$$K_r : (\hat{A}_{k\sigma} = W^* A_{k\sigma} V, \hat{B}_{k\sigma} = W^* B_{k\sigma}, \hat{C}_{k\sigma} = C_{k\sigma} V, \hat{D}_{k\sigma} = D_{k\sigma})$$

where $K_r$ is the projected switched controller (reduced order controller). The outcome of square root algorithm for projection[1]:

$$\begin{aligned} P_2 W &= V \Sigma_1, \\ Q_2 V &= W \Sigma_1, \end{aligned} \tag{45}$$

where $\Sigma_1 \in \mathbb{R}^{k \times k}$ is diagonal and positive definite. We know from (43) that:

$$(A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma}) P_2 + P_2 (A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma})^* < 0 ,$$

which implies:

$$W^* ((A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma}) P_2 +$$
$$P_2 (A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma})^*) W < 0. \tag{46}$$

On the other hand using (45) and then (44) we have,

$$W^*((A_{k\sigma} + B_{k\sigma}F_\sigma D_{22\sigma}C_{k\sigma})P_2 + P_2(A_{k\sigma} + B_{k\sigma}F_\sigma D_{22\sigma}C_{k\sigma})^*)W,$$
$$= W^*(A_{k\sigma} + B_{k\sigma}F_\sigma D_{22\sigma}C_{k\sigma})P_2 W + W^*P_2(A_{k\sigma} + B_{k\sigma}F_\sigma D_{22\sigma}C_{k\sigma})^*W,$$
$$= W^*(A_{k\sigma} + B_{k\sigma}F_\sigma D_{22\sigma}C_{k\sigma})V\Sigma_1 + \Sigma_1 V^*(A_{k\sigma} + B_{k\sigma}F_\sigma D_{22\sigma}C_{k\sigma})^*W,$$
$$= (W^*A_{k\sigma}V + W^*B_{k\sigma}F_\sigma D_{22\sigma}C_{k\sigma}V)\Sigma_1 +$$
$$\Sigma_1(W^*A_{k\sigma}V + W^*B_{k\sigma}F_\sigma D_{22\sigma}C_{k\sigma}V)^*,$$
$$= (\hat{A}_{k\sigma} + \hat{B}_{k\sigma}F_\sigma D_{22\sigma}\hat{C}_{k\sigma})\Sigma_1 + \Sigma_1(\hat{A}_{k\sigma} + \hat{B}_{k\sigma}F_\sigma D_{22\sigma}\hat{C}_{k\sigma})^*.$$

Hence:

$$(\hat{A}_{k\sigma} + \hat{B}_{k\sigma}F_\sigma D_{22\sigma}\hat{C}_{k\sigma})\Sigma_1 + \Sigma_1(\hat{A}_{k\sigma} + \hat{B}_{k\sigma}F_\sigma D_{22\sigma}\hat{C}_{k\sigma})^* < 0, \tag{47}$$

where $\Sigma_1 \in \mathbb{R}^{k \times k}$ is positive definite.

Similar to (33) for the closed-loop system with reduced switched controller we have:

$$\bar{A}_{r\sigma} = \begin{bmatrix} A_\sigma + B_{2\sigma}L_\sigma\hat{D}_{k\sigma}C_{2\sigma} & B_{2\sigma}L_\sigma\hat{C}_{k\sigma} \\ \hat{B}_{k\sigma}F_\sigma C_{2\sigma} & \hat{A}_{k\sigma} + \hat{B}_{k\sigma}F_\sigma D_{22\sigma}\hat{C}_{k\sigma} \end{bmatrix}. \tag{48}$$

It is easy to see from (42) and (47) that for $P = diag(P_1, \Sigma_1)$ we have:

$$\bar{A}_{r\sigma}P + \bar{A}_{r\sigma}^*P < 0. \tag{49}$$

Note that $P$ is positive definite. Therefore $x^*Px$ is the common quadratic Lyapunov function for the closed loop switched system with reduced controller.

We know from stability theory for switched systems that this is a well-known sufficient condition for quadratic stability [28]. Hence, reduced order model is guaranteed to be quadratically stable.

□

As can be seen, the presented framework for controller reduction of switched system is stability preserving. The error of approximation for each subsystem of the closed-loop switched system is bounded and is given in terms of generalized Hankel singular values of the controller. This is the direct result of the [6] for linear controller reduction.

In general existence of a common Lyapunov function is not guaranteed for switched systems [28]. Therefore we cannot expect to have common generalized gramians for all linear switched controllers.

## 6. Numerical Examples

In this section, we apply the proposed methods for reduction of two bimodal switched linear systems as well as two switched controllers. The proposed method is not restricted to particular number of discrete modes or particular switching signals. Systems are randomly generated and the results are shown for randomly generated switching signal.

*Fifth Order Switched Linear System:*

Consider a randomly generated single-input-single-output switched linear system of the form (21):

$$A_1 = \begin{bmatrix} -4.23 & 0.4654 & 1.305 & 0.313 & -1.461 \\ 0.4654 & -4.418 & 0.8745 & -0.9324 & -0.7062 \\ 1.305 & 0.8745 & -1.839 & -0.0083 & 0.6652 \\ 0.313 & -0.9324 & -0.0083 & -1.801 & -0.4979 \\ -1.461 & -0.7062 & 0.6652 & -0.4979 & -2.355 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -5.055 & 0.4867 & 0.7761 & -3.765 & -2.702 \\ 0.4867 & -3.034 & 0.0537 & 0.6768 & 0.6030 \\ 0.7761 & 0.0537 & -1.392 & -0.0739 & 0.8858 \\ -3.765 & 0.6768 & -0.0739 & -5.26 & -1.886 \\ -2.702 & 0.603 & 0.8858 & -1.886 & -3.909 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.1721 \\ -0.336 \\ 0.5415 \\ 0 \\ -0.5703 \end{bmatrix}, \; B_2 = \begin{bmatrix} -0.5081 \\ 0.8564 \\ 0.2685 \\ 0.625 \\ -1.047 \end{bmatrix},$$

$$C_1 = [-1.499 \;\; -0.0503 \;\; 0.553 \;\; 0.0835 \;\; 1.578],$$

$$C_2 = [1.536 \;\; 0.4344 \;\; -1.917 \;\; 0 \;\; 0],$$

$$D_1 = D_2 = 0 .$$

This model is reduced to the following third order switched linear model by applying the presented method over $[\omega_1, \omega_2] = [0.1, 100]$:

$$A_{1r} = \begin{bmatrix} -1.031 & 0.0061 & -0.0811 \\ -0.1413 & -1.606 & 0.7891 \\ -0.1708 & 1.028 & -2.723 \end{bmatrix}, A_{2r} = \begin{bmatrix} -0.8714 & 0.0209 & 0.1824 \\ -0.153 & -1.652 & -0.864 \\ 0.0540 & -0.6046 & -2.7 \end{bmatrix},$$

$$B_{1r} = \begin{bmatrix} -0.4154 \\ 0.595 \\ 0.7314 \end{bmatrix}, B_{2r} = \begin{bmatrix} 0.315 \\ 1.136 \\ 2.371 \end{bmatrix},$$

$$C_{1r} = [-0.2443 \;\; -1.076 \;\; 0.1176],$$

$$C_{2r} = [0.5949 \;\; 0.5316 \;\; -0.5847],$$

$$D_{1r} = D_{2r} = 0 .$$

Fig. 2 shows the decay rate of the generalized Hankel singular values. The approximation accuracy for each subsystem is represented in Fig. 3 and Fig. 4. The step response of the

original and reduced order switched systems associated with a randomly generated switching signal of Fig.5 is presented in Fig. 6.
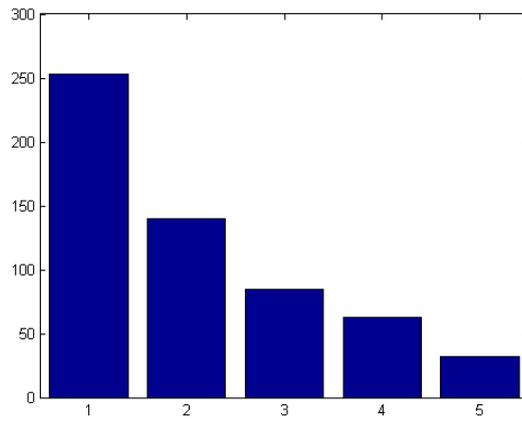


Fig. 2. Generalized Hankel Singular Valuse( $\gamma_i$ ).
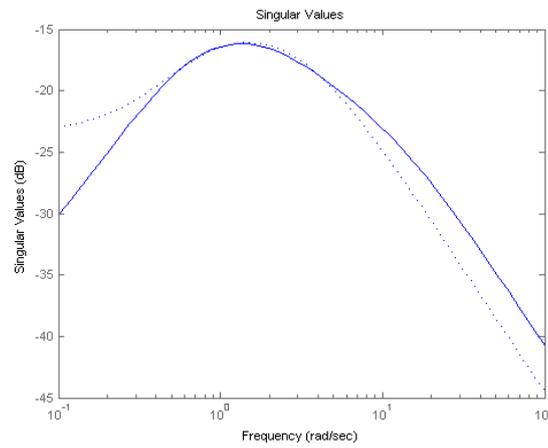


Fig. 3. The infinity norm of original transfer matrix of the first subsystem(solid line) and its third order counterpart (dotted) over frequency domain $[\omega_1, \omega_2] = [0.1, 100]$ .
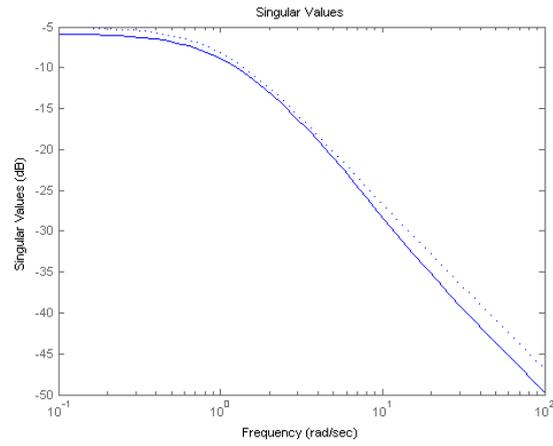
Fig. 4. The infinity norm of original of transfer matrix of second subsystem (solid line) and its third order counterpart (dotted) over frequency domain $[\omega_1, \omega_2] = [0.1, 100]$.
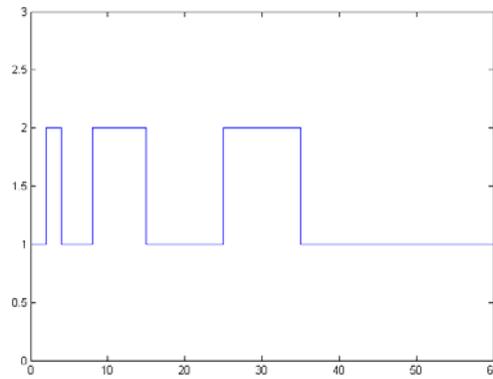


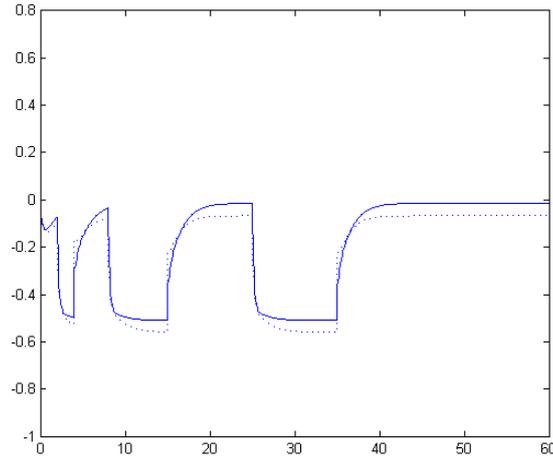Fig. 5. Randomly generated switching signal.

Fig. 6. Step response of original switched linear system(solid line) and the reduced order model (dotted).

Fig. 2. shows that most of the input/output information is in three states of the original systems. The proposed method provides accurate results after reduction of 2 states of the original system (40% of the states) not only locally (see Fig. 3 and Fig. 4) but also globally (see Fig. 6).

*Bimodal Switched linear System of order 100:*

We consider a randomly generated bimodal switched linear system of order 100. This example shows that the presented method can be applied to fairly large systems. The original system is SISO and it is reduced to 87 using the proposed reduction method over $[\omega_1, \omega_2] = [1, 100]$.

The generalized Hankel singular values are shown in Fig. 7. In this example, the infinity norm of the transfer functions of the original subsystems and the reduced counterpart in Fig. 8 and Fig. 9 are compared to show how well the approximation works locally. The approximation accuracy for each subsystem is represented in Fig. 8 and Fig. 9. The step response of the original and reduced order switched systems associated with a randomly generate switching signal of Fig.10 is shown in Fig. 11.

The results after reduction of 13 states of the original system (13% of the states) are accurate locally (see Fig. 8 and Fig. 9) and also globally (see Fig. 11). We already know from Proposition 2 that the reduced order switched system is stable. To better represent how the reduction method performs from a stability viewpoint, we picked randomly generated subsystems that are stable and their poles are close to imaginary axis. Fig. 11 shows that the stability of the original systems is preserved even in such situations for which the step response of the reduced order switched system follows the step response of the original system accurately.
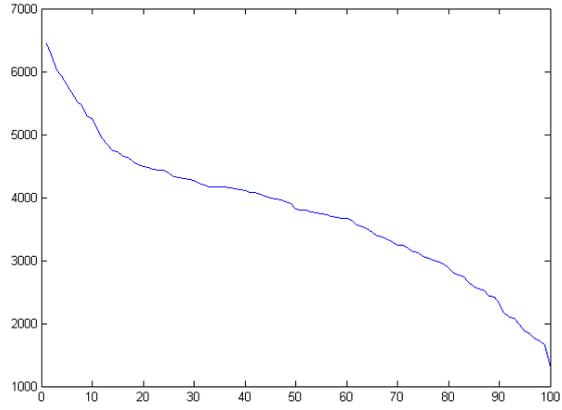
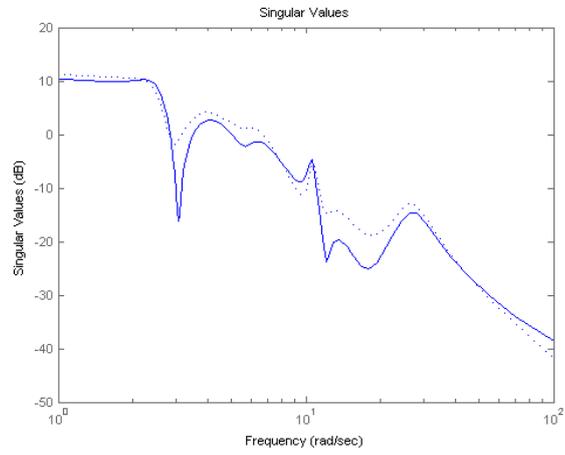Fig. 7. Generalized Hankel Singular Valuse( $\gamma_i$ ).



Fig. 8. The infinity norm of original transfer matrix of first subsystem (solid line) and its reduced order counterpart (dotted) over frequency domain $[\omega_1, \omega_2] = [1, 100]$ .
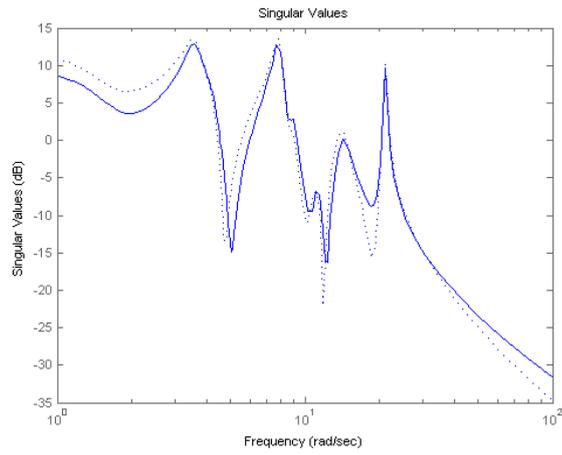
Fig. 9.  The infinity norm of original transfer matrix of second subsystem (solid line) and its reduced order counterpart (dotted) over frequency domain $[\omega_1,\omega_2]=[1,100]$ .
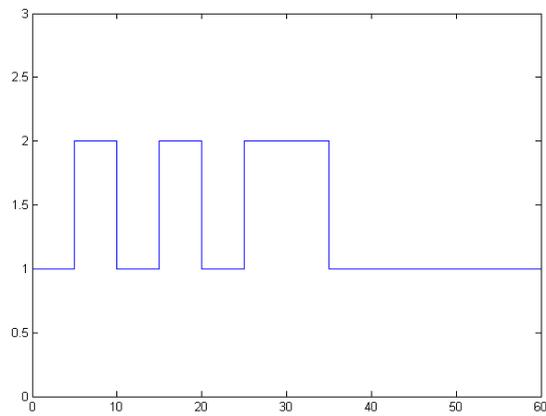


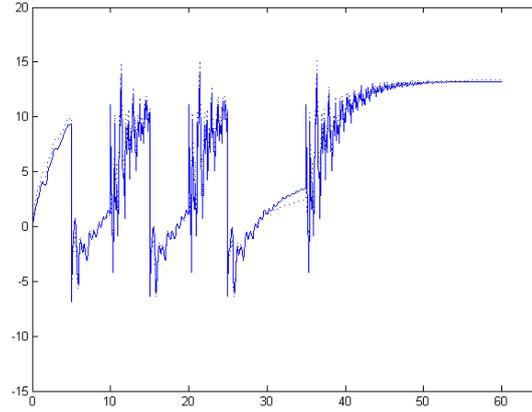Fig. 10.  Randomly generated switching signal.

Fig. 11.   Step response of original switched linear system (solid line) and the reduced order model (dotted).

*Fifth Order Switched controller:*

We consider a switched linear of the form (9) for which we have:

$$A_1 = \begin{bmatrix} -3.3428 & 0.7766 & -0.1894 & 0.5820 & 1.5424 \\ 0.7766 & -1.2319 & 0.3043 & 0.3098 & -0.2189 \\ -0.1894 & 0.3043 & -0.4807 & 0.0478 & 0.3770 \\ 0.5820 & 0.3098 & 0.0478 & -0.7472 & -0.6891 \\ 1.5424 & -0.2189 & 0.3770 & -0.6891 & -1.9965 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.6905 & 0.8334 & -0.8545 & -1.316 & 0.1195 \\ -1.113 & -0.7869 & -1.917 & -0.455 & 1.335 \\ 0.748 & 1.968 & -0.7385 & -0.1609 & -0.0892 \\ -0.8108 & 0.5693 & 0.00355 & -4.924 & -2.485 \\ -0.8452 & -1.418 & 0.2512 & -2.278 & -1.872 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0.5581 \\ 0.1024 & 0 \\ 1.8490 & -1.0816 \\ 1.1762 & 0.0374 \\ 0.2678 & -1.5963 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1.543 & -0.3838 \\ -1.931 & -0.2474 \\ 0 & 0 \\ -0.3468 & -0.05512 \\ -0.1662 & -0.5688 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -0.2914 & 0.0921 & -0.2622 & -0.4689 & -2.0424 \\ 0.8185 & -0.4314 & 0 & -1.2311 & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} -1.78 & 0 & 0 & 0.1578 & -0.04066 \\ 0.9576 & 1.125 & 0 & 1.059 & 0 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0 & -0.2468 \\ 0.2342 & 0.9183 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0.208 \\ 0.3414 & -1.024 \end{bmatrix}.$$

A switched bimodal stabilizing $H_2$ optimal controller $K_\sigma(s) := (A_{k\sigma}, B_{k\sigma}, C_{k\sigma}, D_{k\sigma})$ is synthesized for the above switched system (see Fig. 1) for which we have:

$$A_{k1} = \begin{bmatrix} -0.09566 & 0.1983 & 1.476 & 0.9568 & 9.094 \\ 4.643 & -2.1 & 2.024 & -0.07586 & 7.579 \\ 16.89 & -3.821 & 6.688 & -3.009 & 32.88 \\ 11.72 & -2.181 & 5.018 & -1.812 & 21.85 \\ -0.362 & -0.222 & -1.103 & -2.497 & -8.707 \end{bmatrix},$$

$$A_{k2} = \begin{bmatrix} 13.98 & -1.753 & -0.8545 & -5.245 & 0.5049 \\ 6.806 & -2.233 & -1.917 & -2.626 & 1.544 \\ 12.42 & 0.2864 & -0.7385 & -2.905 & 0.2101 \\ -3.915 & 0.9486 & 0.00355 & -4.263 & -2.563 \\ -9.733 & -0.8385 & 0.2512 & -0.9021 & -2.086 \end{bmatrix},$$

$$B_{k1} = \begin{bmatrix} -0.7196 \\ -1.371 \\ -6.89 \\ -3.92 \\ -0.5589 \end{bmatrix} \quad B_{k2} = \begin{bmatrix} 2.299 \\ 1.285 \\ 1.495 \\ -0.3372 \\ -0.5148 \end{bmatrix}, D_{k1} = D_{k2} = 0,$$

$$C_{k1} = [2.181 \quad -0.2198 \quad 1.367 \quad 1.034 \quad 6.196],$$

$$C_{k2} = [8.599 \quad 0 \quad 0 \quad -0.7589 \quad 0.1955].$$

Fig. 12 shows the decay rate of the generalized Hankel singular values of the switched controller. It is clear from Fig. 12 that reduction of the controller to a fourth order switched controller should provide accurate results.

The step response of the original closed loop system and closed loop system with reduced order controller of order 4 associated with randomly generate switching signal of Fig.13 is presented in Fig. 14.

We reduce the controller as much as possible i. e. to a first order switched controller. The step responses of the original closed loop system and the closed loop system with first order switched controller are shown in Fig. 15. These step responses are also associated to switching signal shown in Fig. 13.

According to Fig. 12 it was expected to be less accurate because too much input/output information are lost by omitting 4 states of the switched controller.

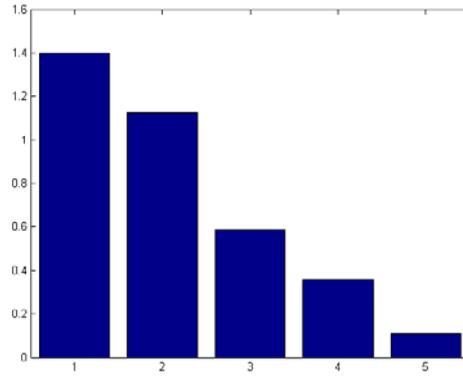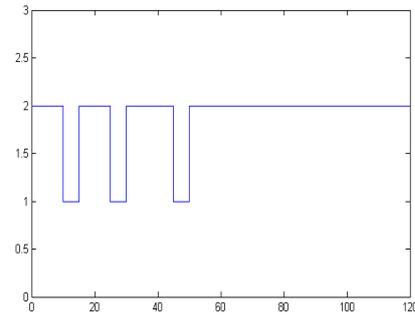Fig. 12.  Generalized Hankel Singular Valuse( $\gamma_i$ ).



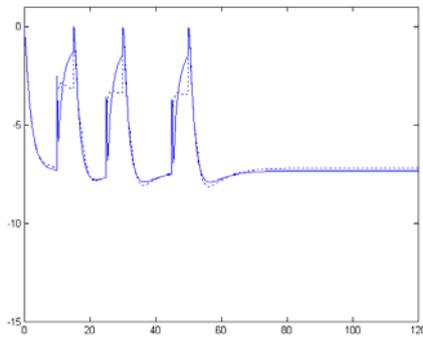Fig. 13.  Randomly generated switching signal.



Fig. 14.  Step response of original closed loop system(solid line) and the closed loop system with the reduced switched controller of order 4(dotted).
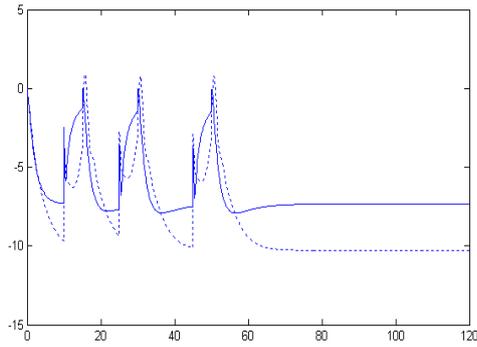
Fig. 15. Step response of original closed loop system(solid line) and the closed loop system with the reduced first order switched controller(dotted).

*Bimodal Switched controller of order 20:*

We consider a randomly generated bimodal switched linear system of order 20. Similar to the previous example, we designed a switched bimodal stabilizing $H_2$ optimal controller for the system. The switched controller is of order 20 with the generalized Hankel singular values which are shown in Fig. 16. It is clear from Fig. 16 that most of the input/output behavior information are embedded in the first two states of the controller. We expect that reduction of the controller to the second order switched controller should provide accurate results. The step response of the original closed loop system and closed loop system with reduced second order switched controller associated with randomly generated switching signal of Fig.17 is shown in Fig. 18.
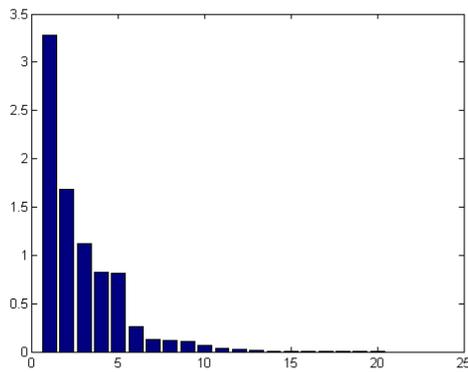


Fig. 16. Generalized Hankel Singular Valuse( $\gamma_i$ ).
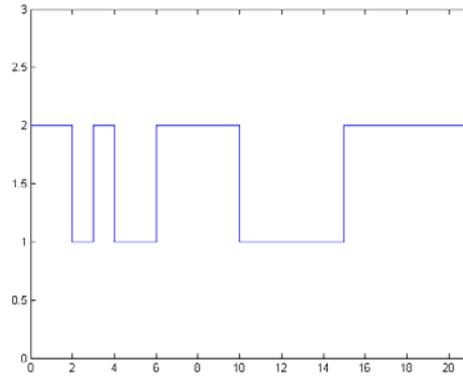
Fig. 17. Randomly generated switching signal.
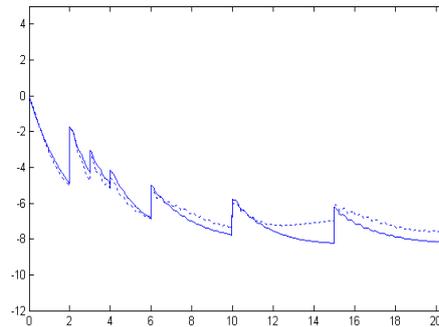


Fig. 18. Step response of original closed loop system (solid line) and the the closed loop system with the reduced second order switched controller(dotted).

## 7. Conclusion

A general framework for model/controller reduction of switched linear dynamical systems has been presented. The proposed method is based on the generalized gramian framework for reduction which requires solving LMI's in the reduction procedure. In this paper, we have reformulated the frequency domain balanced reduction method into this scheme as an example. Other gramian based reduction methods can also be reformulated in the proposed generalized method and can be applied for reduction of switched systems as well as switched controllers. The method preserves the stability under arbitrary switching signals and is applicable to both continuous and discrete time systems. Due to the fact that it uses common generalized gramian, it not only preserves stability but also reduces subsystems/sub-controllers in one step using global projection matrices. One of the

drawbacks of the method is that feasible solutions not always exist because it is not always possible to find a common Lyapunov function for switched systems. Error is bounded, but it is not guaranteed to be always small enough.

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# Paper B

Switched Systems Reduction Framework Based on Convex Combination of
Generalized Gramians

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# 1. Introduction

The highly complicated models are the response to the ever-increasing need for accurate mathematical modeling of physical as well as artificial processes for simulation and control. This problem demands efficient automatic computational tools to replace such complex models by an approximate simpler models, which are capable of capturing dynamical behavior and preserving essential properties of the complex one, either the complexity appears as high order describing dynamical system or complex nonlinear structure. Due to this fact model reduction methods have become increasingly popular over the last two decades [1], [2], [3]. Such methods are designed to extract a reduced order model that adequately describes the behavior of the system in question.

Most of the studies related to model reduction presented so far have been devoted to linear case and just few methods have been proposed for nonlinear cases which are not strong comparing to linear reduction methods.

On the other hand, most of the methods that are proposed so far for control and analysis in hybrid and switched systems theory are suffering from high computational burden when dealing with large-scale dynamical systems. Because of the weakness of standard model reduction techniques in dealing directly with hybrid structure without sacrificing essential features and also pressing needs for efficient analysis and control of large-scale dynamical hybrid and switched systems; it is necessary to study model reduction of hybrid and switched systems in particular. This fact has motivated the researchers in hybrid systems to study model reduction [15]-[27]. Some works have been focused on ordinary model reduction methods that have potential applications in modeling and analysis of hybrid systems [15]-[19] motivated by reachability analysis and safety verification problem. Some researches address the problem of model reduction of switched and hybrid systems directly [20]-[27][34][35].

The model reduction problem for switched systems of Markovian type was studied in [34] and further in [35]. The method that has been presented in [20], deals with abstraction of both continuous and discrete part of hybrid dynamical systems. This framework uses balanced revisualization for reduction of continues part. There is no guarantee for stability preservation for switched system in the framework that has been proposed in [20] and it might happen that guard approximation and reset maps approximation cause non-elegant behavior due to approximation error or possible overlap. In [21] it is presented that the state set can be affinely reduced due to non-observability if and only if a subspace of the classical unobservable subspace, characterized using the normal vectors of the exit facets, is nontrivial. This result does not provide strong tool for reduction of affine systems, as it is an exact reduction which is quite restrictive. Exact reduction is very elegant but the class of systems for which this procedure can be applied is quite small. This method only considers observability for investigating the importance of the states to discard. Although this method has been modified in [25] but lots of problems are still open and should be addressed in this

context. The paper [22] is concerned with the problem of model reduction for discrete switched system. Two different approaches are proposed to solve this problem. The first approach casts the model reduction into a convex optimization problem, which is the first attempt to solve the model reduction problem by using linearization procedure. The second one, based on the cone complementarity linearization, casts the model reduction problem into a sequential minimization problem subject to linear matrix inequality constraints. Both approaches have their own advantages and disadvantages concerning conservatism and computational complexity. These optimization problems will be very hard if not infeasible to solve for a large scale system and also they are not always feasible. This method is not only just applicable to discrete time switched systems furthermore it does not provide us with any hints about the number of states which is suitable to keep prior to the reduction. Similar methods have been developed for more general classes of discrete time switched systems in [23] [24].

In [26] we proposed the generalized gramian framework for model reduction of switched systems based on common generalized gramians of the subsystems. This framework has been developed for controller reduction in [27]. The framework shows to provide satisfactory approximations and it preserves the stability of the original system under arbitrary switching signal but it is over conservative.

 In this paper we propose convex generalized gramian based framework for model reduction of switched system. This general framework can be categorized as gramian based model reduction methods.  Balanced model reduction is one of the most common gramian based model reduction schemes. It was presented in [4] for the first time.

   To apply balanced reduction, first the system is represented in a basis where the states which are difficult to reach are simultaneously difficult to observe. This is achieved by simultaneously diagonalizing the reachability and the observability gramians, which are solutions to the reachability and the observability Lyapunov equations. Then, the reduced model is obtained by truncating the states which have this property. Balanced model reduction method is modified and developed from different viewpoints [1],[2]. One of the methods that are presented based on balanced model reduction is the method based on the generalized gramians instead of gramians[5]. In this method in order to compute the generalized gramians , one should solve Lyapunov inequalities instead of Lyapunov equations. This method is used to devise a technique for structure preserving model reduction methods in [6].

In this paper we first show that the generalized method in [5] can be extended to various gramian based reduction methods. We also modified the original method in [5] to avoid numerical instability and also to achieve more efficiency by building Petrov-Galerkin projection based on generalized gramians. We propose a method based on the balanced model reduction within frequency bound in this framework.  We generalized the framework to model reduction of switched system by constructing Petrov-Galerkin projection based on convex generalized gramian which is a convex combination of generalized gramians. We restrict convex generalized gramian to take stability preservation into account. The feasibility

and also stability preservation of the algorithm is studied. It is shown that the proposed framework is less conservative than its previous counterpart in [26].

The paper is organized as follows: In the next section we review balanced reduction method and balanced reduction technique based on the generalized gramian. Section 2 presents how different gramian based methods can be approximated as generalized gramian based techniques. Balanced reduction within frequency bound based on generalized gramian is also presented in this section. This section ends up with some remarks on numerical implementation of the algorithm and using projection for generalized gramian based reduction methods is suggested instead of balancing and truncation. Section 3 is devoted to develop convex generalized gramian based reduction method for model reduction of switched systems, followed by discussions on stability, feasibility, algorithm parameters and error bound. Section 4 presents our numerical results. Section 5 concludes the paper.

The notation used in this paper is as follows: $M^*$ denotes

transpose of matrix if $M \in \mathbb{R}^{n \times m}$ and complex conjugate transpose if $M \in \mathbb{C}^{n \times m}$. The norm $\|.\|_\infty$ denotes the $H_\infty$, norm of a rational transfer function. The standard notation $>$, $\geq (<, \leq)$ is used to denote the positive (negative) definite and semidefinite ordering of matrices.

## 2. Balanced Truncation and Generalized Gramians

Balanced truncation is a well-known method for model reduction of dynamical systems, see for example [1][2].The basic approach relies on balancing the gramians of the systems. For dynamical systems with minimal realization:

$$G(s) := (A, B, C, D) \tag{1}$$

where $G(s)$ is transfer matrix with associated state-space representation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(t) \in \mathbb{R}^n \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{2}$$

gramians are given by the solutions of the Lyapunov equations:

$$AP + PA^* + BB^* = 0$$
$$A^*Q + QA + C^*C = 0 \tag{3}$$

For stable $A$, they have a unique positive definite solutions $P$ and $Q$, called the controllability and observability gramians. In balanced reduction, first the system is transformed to the balanced structure in which gramians are equal and diagonal:

$$P = Q = diag(\sigma_1 I_{k_1}, \ldots, \sigma_q I_{k_q})$$

$$\sum_{j=1}^{q} k_j = n \tag{4}$$

where $\sigma_i > \sigma_{i+1}$ and they are called Hankel singular values.

The reduced model can be easily obtained by truncating the states which are associated with the set of the least Hankel singular values. Applying the method to stable, minimal $G(s)$, if we keep all the states associated to $\sigma_m (1 \leq m \leq r)$, by truncating the rest, the reduced model $G_r(s)$ will be minimal and stable and satisfies[1][2]:

$$\|G(s) - G_r(s)\|_\infty \leq 2 \sum_{j=r+1}^{q} \sigma_j \tag{5}$$

One of the closely related model reduction methods to the balanced truncation is balanced reduction based on the generalized gramian that is presented in [5]. In this method, instead of Lyapunov equations (3), the following Lyapunov inequalities should be solved:

$$AP_g + P_g A^* + BB^* \leq 0$$

$$A^* Q_g + Q_g A + C^* C \leq 0 \tag{6}$$

For stable $A$, they have positive definite solutions $P_g$ and $Q_g$, called the generalized controllability and observability gramians. Note that these gramians are not unique. The rest of this model reduction method is the same as the aforementioned balanced truncation method, the only difference is that in this algorithm the balancing and truncation are based on generalized gramian instead of ordinary gramian. In this method we have generalized Hankel singular values ($\gamma_i$) which are the diagonal elements of balanced gerealized gramians instead of Hankel singular values $\sigma_i$ which are the diagonal elements of balanced standard gramians. For the error bound also the same result holds but in terms of the generalized Hankel singular values instead of Hankel singular values. It is worth to mention that $\gamma_i \geq \sigma_i$. Therefore the error bound in balanced reduction based on generalized gramian is greater or equal than the error bound in ordinary balanced model reduction.

## 3. Generalized Gramian Framework for Gramian-based Model Reduction Methods

In this section we present a general framework to build generalized gramian version of gramian based methods. Then we present generalized balanced reduction within frequency bound within this framework following by some words about numerical implementation of the algorithm based on projection.

**Lemma 1:** Suppose $A$ is stable, $Y$ is symmetric and

$$A^*Y + YA \leq 0$$

$$A, Y \in \mathbb{R}^{n \times n} \tag{7}$$

is satisfied, then $Y \geq 0$, i.e. $Y$ is positive semi definite.

**Proof:** If $A^*Y + YA \leq 0$, there exists $M \geq 0$ such that:

$$A^*Y + YA + M = 0$$

On the other hand, for any stable $A$, there exists the following unique solution for the equation above:

$$Y = \int_0^\infty e^{A^*\tau} M \, e^{A\tau} d\tau$$

In the above structure $M \geq 0$, hence:

$$Y \geq 0$$

$\square$

This lemma leads to the following proposition that makes the relation between Lyapunov equations and Lyapunov inequalities evident.

**Proposition 1[5]:** Suppose $A$ is stable and $X$ is the solution of Lyapunov equation:

$$A^*X + XA + Q = 0 \tag{8}$$

where $Q \geq 0$. If a symmetric $X_g$ satisfies:

$$A^*X_g + X_g A + Q \leq 0 , \tag{9}$$

then: $X_g \geq X$ .

**Proof:** It can be proven easily by subtracting (9)-(8) and applying Lemma 1 with $Y = X_g - X$ .

$\square$

Proposition 1 shows how the generalized gramian could be an approximation for ordinary gramians. Balanced reduction based on generalized gramian which we reviewed in the last section is based on proposition 1.This method might provide less accurate approximation than its gramian based counterpart but still the approximation error is bounded.

It is possible to propose generalized version of other gramian based reduction methods in this framework. The only step that we need to take is to derive associated Lyapunov equations and their Lyapunov inequalities. In the following we propose generalized version of balanced reduction within frequency bound.

*Generalized Balanced Reduction within Frequency Bound*

Over the past two decades, a great deal of attention has been devoted to balanced model reduction and it has been developed and improved from different viewpoints. Frequency weighted balanced reduction method is one of the devised gramian based techniques based on ordinary balanced truncation [1],[2],[7]-[9]. In this method by using input and output weights and stressing on certain frequency range more accurate results can be achieved. In many cases the input and output weights are not given and the problem is to reduce the model over a given frequency range [1],[2]. This problem can be attacked directly by balanced reduction within frequency bound. This method was first proposed in [10] and then modified in [2] in order to preserve the stability of the original system and to provide an error bound for approximation. In this method, for dynamical system (1) the controllability gramian $P(\omega_1, \omega_2)$ and observability gramians $Q(\omega_1, \omega_2)$ within frequency range $[\omega_1, \omega_2]$ are defined as:

$$P(\omega_1, \omega_2) = P(\omega_1) - P(\omega_2)$$
$$Q(\omega_1, \omega_2) = Q(\omega_1) - Q(\omega_2)$$

(10)

where:

$$P(\omega) =: \frac{1}{2\pi} \int_{-\omega}^{\omega} (Ij\omega - A)^{-1} BB^* (-Ij\omega - A^*)^{-1} d\omega$$

$$Q(\omega) =: \frac{1}{2\pi} \int_{-\omega}^{\omega} (-Ij\omega - A^*)^{-1} C^* C (Ij\omega - A)^{-1} d\omega$$

(11)

In order to show the associated Lyapunov equations, we need some more notations:

$$S(\omega) =: \frac{1}{2\pi} \int_{-\omega}^{\omega} (Ij\omega - A)^{-1} d\omega$$

(12)

$$W_c(\omega) = S(\omega)BB^* + BB^* S^*(-\omega)$$
$$W_o(\omega) = C^* C S(\omega) + S^*(-\omega) C^* C$$

(13)

$$W_c(\omega_1, \omega_2) = W_c(\omega_2) - W_c(\omega_1)$$
$$W_o(\omega_1, \omega_2) = W_o(\omega_2) - W_o(\omega_1)$$

(14)

The gramians satisfy the following Lyapunov equations[1],[2]:

$$AP(\omega_1, \omega_2) + P(\omega_1, \omega_2)A^* + W_c(\omega_1, \omega_2) = 0$$
$$A^* Q(\omega_1, \omega_2) + Q(\omega_1, \omega_2)A + W_o(\omega_1, \omega_2) = 0$$

(15)

This method is modified in [2] to guarantee stability and to provide a simple error bound. The modified version starts with EVD of $W_c(\omega_1,\omega_2)$ and $W_o(\omega_1,\omega_2)$ :

$$W_c(\omega_1,\omega_2) := M\Lambda M^* = M\, diag(\lambda_1,...,\lambda_n)M^*$$
$$W_o(\omega_1,\omega_2) := N\Delta N^* = N\, diag(\delta_1,...,\delta_n)N^*$$

(16)

where: $MM^* = NN^* = I_n$, $|\lambda_i| \geq |\lambda_{i+1}| \geq 0$, $|\delta_i| \geq |\delta_{i+1}| \geq 0$ .

Note that since $W_c(\omega_1,\omega_2)$ and $W_o(\omega_1,\omega_2)$ are symmetric decompositions in the form (16) exist. Let:

$$\hat{B} := M\, diag\left(|\lambda_1|^{1/2},...,|\lambda_\xi|^{1/2},0,...,0\right)$$
$$\hat{C} := diag\left(|\delta_1|^{1/2},...,|\delta_\rho|^{1/2},0,...,0\right)N^*$$

(17)

where:

$$\xi = rank(W_c(\omega_1,\omega_2))$$
$$\rho = rank(W_o(\omega_1,\omega_2))$$

(18)

The modified gramians satisfy the following Lyapunov equations instead of (15):

$$A\hat{P}(\omega_1,\omega_2) + \hat{P}(\omega_1,\omega_2)A^* + \hat{B}\hat{B}^* = 0$$
$$A^*\hat{Q}(\omega_1,\omega_2) + \hat{Q}(\omega_1,\omega_2)A + \hat{C}^*\hat{C} = 0$$

(19)

That is all what we need to present the generalized version of this method:

$$A\hat{P}_g(\omega_1,\omega_2) + \hat{P}_g(\omega_1,\omega_2)A^* + \hat{B}\hat{B}^* \leq 0$$
$$A^*\hat{Q}_g(\omega_1,\omega_2) + \hat{Q}_g(\omega_1,\omega_2)A + \hat{C}^*\hat{C} \leq 0$$

(20)

Then the generalized modified balanced reduction within frequency bound can be obtained by simultaneously diagonalizing $\hat{P}_g(\omega_1,\omega_2)$ and $\hat{Q}_g(\omega_1,\omega_2)$ then by truncating the states associated to the set of the least generalized Hankel singular values.

*Numerical Issues*

Balanced transformation can be ill-conditioned numerically when dealing with the systems with some nearly uncontrollable modes or some nearly unobservable modes. Difficulties associated with computation of the required balanced transformation in [11] draw some attentions to alternative numerical methods [12]. Balancing can be a badly conditioned even

when some states are much more controllable than observable or vice versa. It is advisable then to reduce the system in the gramian based framework without balancing at all. Schur method and Square root algorithm provid projection matrices to apply balanced reduction without balanced transformation [1][12]. This method can be easily applied to other gramian based reduction methods. In our generalized method we use the same algorithm by plugging generalized gramians into the algorithm instead of ordinary gramians.

## 4. Model Reduction of Switched System

*Model Reduction of Switched Systems Based on Convex Generalized Gramians*

One of the most important subclasses of hybrid systems are linear switched systems. Linear switched system is a dynamical system specified by the following equations:

$$\Sigma : \begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{cases} \tag{21}$$

where $x(t) \in \mathbb{R}^n$ is the continuous state, $y(t) \in \mathbb{R}^p$ is the continuous output, $u(t) \in \mathbb{R}^m$ is the continuous input, and $\sigma : \mathbb{R}^{\geq 0} \to \mathrm{K} \subset \mathbb{N}$ is the switching signal that is a piecewise constant map of the time. K is the set of discrete modes, and it is assumed to be finite. For each $i \in \mathrm{K}$, $A_i$, $B_i$, $C_i$, $D_i$ are matrices of appropriate dimensions.

In this section we build a framework for model reduction of switched system described by (21). The aim is to find projection that maps the state-space of a switched system to lower dimensional subspace. Definition 1, describes the general definition of Petrov-Galerkin projection.

**Definition1.** Petrov-Galerkin projection for a dynamical system:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \quad , x \in \mathbb{R}^n \\ y(t) = g(x(t), u(t)) \end{cases} \tag{22}$$

is defined as a projection $\Pi = VW^*$, where: $W^*V = I_k$, $V, W \in \mathbb{R}^{n \times k}, k < n$ [1].

The reduced order model using this projection is:

$$\begin{cases} \dot{\hat{x}}(t) = W^* f(V\hat{x}(t), u(t)) \quad , \hat{x} \in \mathbb{R}^k \\ y(t) = g(V\hat{x}(t), u(t)) \end{cases} \tag{23}$$

In our framework we construct the aforementioned projection based on the convex generalized gramian which is defined as following:

**Definition2.** Convex controllability (observability) generalized gramian for the dynamical system (21) is defined as:

$$\Psi^{\gamma}_g = \sum_{i=1}^{|\mathcal{K}|} \gamma_i P_{gi}$$

(24)

where

$$\sum_{i=1}^{|\mathcal{K}|} \gamma_i = 1 \, , \gamma_i \in \mathbb{R}^{\geq 0}$$ 

(25)

$P_{gi}$ is generalized controllability (observability) generalized gramian associated to the $i^{th}$ subsystem of (21).

One easy way to develop generalized gramian framework to model reduction of switched linear system is to apply the method locally on each subsystem independently, in other words, to reduce each subsystem by generalized gramian reduction method independently. Independent reduction of subsystems poses an extra computational burden for construction of the independent projection matrices for each subsystem. Therefore it is preferable to construct single projection which is capable of reduction of all subsystems in one shot. Due to this fact, we introduce convex generalized gramian. Building the projection based on the convex generalized gramian enables us to reduce all subsystem in one shot and reduces the extra computational burden which the methods based on independent reduction of subsystems like the one in [20] suffer from. On the other hand, the elegant structure of convex gramian gives us more flexibility to play with the parameters and also to deploy some stability results.

At this point it is possible to develop different gramian based reduction methods into this framework for reduction of switched system finding generalized controllability/observability gramian for each subsystem, constructing convex controllability/observability generalized gramian. The next step can be simultaneous diagonalization  of the convex generalized gramian and balancing and reduction of all subsystems based on Hankel singular values of the convex generalized gramian. In order to avoid numerical bad conditioning and also to increase the efficiency we use Schur or square root algorithm instead of balancing and directly Petrov-Galerkin projection matrices can be computed. This procedure is less conservative and provides more accurate results.

In the method that we proposed in [26] the stability of the original switched systems under arbitrary switching signal is guaranteed to be preserved which was the main reason for conservatism. We can also modify the convex generalized gramian based framework to

preserve the stability. We modify the method based on the stability results which are less conservative than their counterpart which we used in [26]. This is a compromise between stability preservation and feasibility. The matrix pencil and the convex hull of matrices which will be used in the algorithm for this purpose need to be defined.

**Definition3.** The matrix pencil $\xi_\alpha(A_1, A_2)$ is defined as the one-parameter family of matrices $\alpha A_1 + (1-\alpha)A_2$, $\alpha \in [0,1]$ [28].

In general this is the convex hull of the family of matrices which is defined as:

$$Co(A_1, A_2,...,A_n) = \left\{ A : A = \sum_{i=1}^n \alpha_i A_i , \sum_{i=1}^n \alpha_i = 1, \ \alpha_i \in \mathbb{R}^{\geq 0} \right\} \tag{26}$$

The procedure is almost the same as what we mentioned before. The only deference is that we restrict one of the convex gramians to satisfy:

$$A(\alpha)^* \Psi_g + \Psi_g A(\alpha) < 0 \tag{27}$$

where $A(\alpha) \in \xi_\alpha(A_1, A_2)$ for bimodal systems. In the case of multimodal switched systems a stable $A(\alpha)$ is picked from $Co(A_1, A_2,...,A_n)$.

In order to clarify the method we extend generalized balanced reduction within frequency bound that is presented in previous section, for model reduction of switched linear system.

First we need to find, the generalized controllability gramian $\hat{P}_{g,\sigma}(\omega_1, \omega_2)$ of each subsystem within frequency domain by solving the system of Lyapunov inequalities:

$$\begin{cases} A_\sigma \hat{P}_{g,\sigma}(\omega_1, \omega_2) + \hat{P}_{g,\sigma}(\omega_1, \omega_2) A_\sigma^* + \hat{B}_\sigma \hat{B}_\sigma^* < 0 \\ \forall \sigma \in \mathrm{K} \end{cases} \tag{28}$$

For example in the case of bimodal systems, $\mathrm{K} = \{1,2\}$, we have to solve:

$$\begin{cases} A_1 \hat{P}_{g,1}(\omega_1, \omega_2) + \hat{P}_{g,1}(\omega_1, \omega_2) A_1^* + \hat{B}_1 \hat{B}_1^* < 0 \\ A_2 \hat{P}_{g,2}(\omega_1, \omega_2) + \hat{P}_{g,2}(\omega_1, \omega_2) A_2^* + \hat{B}_2 \hat{B}_2^* < 0 \end{cases} \tag{29}$$

The convex controllability gramian within $(\omega_1, \omega_2)$ frequency bound is computed according to Definition.2:

$$\Psi_{cg}^\gamma(\omega_1, \omega_2) = \sum_{i=1}^{|K|} \gamma_i \hat{P}_{g,i}(\omega_1, \omega_2) \tag{30}$$

In (30), we are free to tune $\gamma_i \in [0,1]$. We can do the same to compute the convex observability gramian within $(\omega_1, \omega_2)$ frequency bound $\Psi_{og}^{\gamma'}(\omega_1, \omega_2)$ :

$$\Psi_{og}^{\gamma'}(\omega_1, \omega_2) = \sum_{i=1}^{|K|} \gamma_i' \hat{Q}_{g,i}(\omega_1, \omega_2) \tag{31}$$

where $\hat{Q}_{g,i}(\omega_1, \omega_2)$ is the generalized observability gramian of $i^{th}$ subsystem within frequency domain $(\omega_1, \omega_2)$, i.e. we have:

$$\begin{cases} A_\sigma^* \hat{Q}_{g,\sigma}(\omega_1, \omega_2) + \hat{Q}_{g,\sigma}(\omega_1, \omega_2) A_\sigma + \hat{C}_\sigma^* \hat{C}_\sigma < 0 \\ \forall \sigma \in K \end{cases} \tag{32}$$

If stability preservation is of concern we have to choose $\gamma_i' \in [0,1]$ such that:

$$A(\alpha)^* \Psi_{og}^{\gamma'}(\omega_1, \omega_2) + \Psi_{og}^{\gamma'}(\omega_1, \omega_2) A(\alpha) < 0 \tag{33}$$

to be satisfied.

If we plug $\Psi_{cg}^{\gamma}(\omega_1, \omega_2)$ and $\Psi_{og}^{\gamma'}(\omega_1, \omega_2)$ into the square root algorithm, we directly obtain projectors for reduction. Note that the results are same as balancing algorithm. A merit of the Square Root method is that it relies on the Cholesky factors of the gramians rather than the gramians themselves, which has advantages in terms of numerical stability.

*Stability, Parameters and Feasibility,*

One of issues in model reduction is preservation of the stability which needs to be studied. We need to recall two stability results Theorem.1 and Theorem.2 which is the generalization of the first one.

**Theorem 1.** Switched bimodal dynamical system (21) (i.e. $K = \{1, 2\}$ ) for some switching signal is stable iff there exists $A(\alpha) \in \xi_\alpha(A_1, A_2)$ which is stable[29].

**Theorem 2.** Switched dynamical system (21) for some switching signal is stable if there exists $A(\alpha) \in Co(A_1, A_2, ..., A_{|K|})$ which is stable [29].

The proofs for these theorems are by construction, in other words in the proofs the switching signal for which the switched system is stable are constructed based on $\alpha$ and dynamics of the systems [29].

**Proposition2.** Consider $A(\alpha) \in Co(A_1, A_2, ..., A_{|K|})$ associated to the coefficients $\alpha = (\alpha_1, \alpha_2, ..., \alpha_{|K|})$ and $\hat{A}(\alpha)$ is its reduced order counterpart using convex

generalized gramian.

If $A(\alpha)$ is stable then $\hat{A}(\alpha)$ is also stable.

**Proof:**

In the proposed method, we have:

$$W^*V = I_k, \quad V,W \in \mathbb{R}^{n \times k}, k < n$$

$$\hat{\Sigma}: \begin{cases} \hat{A}_{\sigma(t)} = W^* A_{\sigma(t)} V \\ \hat{B}_{\sigma(t)} = W^* B_{\sigma(t)} \\ \hat{C}_{\sigma(t)} = C_{\sigma(t)} V \\ \hat{D}_{\sigma(t)} = D_{\sigma(t)} \end{cases} \tag{34}$$

which is projected switched system (reduced order model). The outcome of Square root algorithm for projection[1]: $\Psi_{cg}^{\gamma} W = V \Sigma_1$ and $\Psi_{og}^{\gamma} V = W \Sigma_1$, where $\Sigma_1 \in \mathbb{R}^{k \times k}$ is diagonal and positive definite. From (33): $A(\alpha)^* \Psi_{og}^{\gamma'} + \Psi_{og}^{\gamma'} A(\alpha) < 0$, which implies:

$$V^* (A(\alpha)^* \Psi_{og}^{\gamma'} + \Psi_{og}^{\gamma'} A(\alpha)) V < 0$$

On the other hand,

$$V^* (A(\alpha)^* \Psi_{og}^{\gamma'} + \Psi_{og}^{\gamma'} A(\alpha)) V$$

$$= V^* A(\alpha)^* \Psi_{og}^{\gamma'} W + V^* \Psi_{og}^{\gamma'} A(\alpha) V$$

$$= V^* A(\alpha)^* W \Sigma_1 + \Sigma_1^* W^* A(\alpha) V$$

$$= V^* (\sum_{i=1}^{|K|} \alpha_i A_i)^* W \Sigma_1 + \Sigma_1^* W^* (\sum_{i=1}^{|K|} \alpha_i A_i) V$$

$$= \sum_{i=1}^{|K|} \alpha_i V^* A_i^* W \Sigma_1 + \Sigma_1^* \sum_{i=1}^{|K|} \alpha_i W^* A_i V$$

$$= \sum_{i=1}^{|K|} \alpha_i \hat{A}_i^* \Sigma_1 + \Sigma_1^* \sum_{i=1}^{|K|} \alpha_i \hat{A}_i$$

$$= \hat{A}(\alpha) \Sigma_1 + \Sigma_1 \hat{A}^*(\alpha)$$

Hence:

$$\hat{A}(\alpha) \Sigma_1 + \Sigma_1 \hat{A}^*(\alpha) < 0 \tag{35}$$

where $\Sigma_1 \in \mathbb{R}^{k \times k}$ is positive definite. Hence $\hat{A}(\alpha)$ is stable.

This proposition along with the Theorem.1 and Theorem.2 shows that at least for stabilizing switching signals which have been used in the proofs of the Theorem.1 and Theorem.2 the reduced order dynamical system is stable.

In the particular scenarios the stability of the original switched system is guaranteed to be preserved under arbitrary switching signal. This is shown in Proposition 3.

**Proposition3.** The Convex Generalized Grammian framework is stability preserving under arbitrary switching signal if:

$$P_{gi} = P_g \tag{36}$$
$$\text{or}$$
$$Q_{gi} = Q_g \tag{37}$$

**Proof:**
We have:

$$\Psi_{cg}^{\gamma} = \sum_{i=1}^{|K|} \gamma_i P_{g,i} = \sum_{i=1}^{|K|} \gamma_i P_g = (\sum_{i=1}^{|K|} \gamma_i) P_g = P_g$$

Similarly:

$$\Psi_{og}^{\gamma'} = \sum_{i=1}^{|K|} \gamma_i' Q_{g,i} = \sum_{i=1}^{|K|} \gamma_i' Q_g = (\sum_{i=1}^{|K|} \gamma_i') Q_g = Q_g$$

Assume that (36) is satisfied, the outcome of Square root algorithm for projection[1]: $P_g W = V \Sigma_1$ and $Q_g V = W \Sigma_1$, where $\Sigma_1 \in \mathbb{R}^{k \times k}$ is diagonal and positive definite. Since $P_g$ is common controllability generalized gramian:

$$A_{\sigma(t)} P_g + P_g A^*_{\sigma(t)} < 0 \,,$$

which implies:

$$W^* (A_{\sigma(t)} P_g + P_g A^*_{\sigma(t)}) W < 0$$

On the other hand,

$$W^* (A_{\sigma(t)} P_g + P_g A^*_{\sigma(t)}) W = W^* A_{\sigma(t)} P_g W + W^* P_g A^*_{\sigma(t)} W$$
$$= W^* A_{\sigma(t)} V \Sigma_1 + \Sigma_1^* V^* A^*_{\sigma(t)} W = \hat{A}_{\sigma(t)} \Sigma_1 + \Sigma_1 \hat{A}^*_{\sigma(t)}$$

Hence:

$$\hat{A}_{\sigma(t)} \Sigma_1 + \Sigma_1 \hat{A}^*_{\sigma(t)} < 0 \,,$$

where $\Sigma_1 \in \mathbb{R}^{k \times k}$ is positive definite.

In stability theory for switched system it is well-known sufficient condition for quadratic

stability [13]. Hence, reduced order model is guaranteed to be quadratic stable.

In the case that (37) is satisfied we can prove in a same way starting with $V^*(A^*_{\sigma(t)}Q_g + Q_g A_{\sigma(t)})V < 0$ and using $Q_g V = W\Sigma_1$ which again proves the existence of the common Lyapunov function. We show that (27) is also satisfied for all $A(\alpha) \in Co(A_1, A_2, ..., A_{|K|})$ in this case.

We have: $A = \sum_{i=1}^{n} \alpha_i A_i$, $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i \in \mathbb{R}^{\geq 0}$, on the other hand, $A^*_i Q_g + Q_g A_i < 0$ and consequently $\alpha_i (A^*_i Q_g + Q_g A_i)\alpha_i \leq 0$, knowing that at least one of $\alpha_i$'s must be nonzero we have:

$$\sum_{i=1}^{|K|} \alpha_i (A^*_i Q_g + Q_g A_i)\alpha_i < 0$$

which implies:

$$A(\alpha)^* Q_g + Q_g A(\alpha)$$
$$= A(\alpha)^* \Psi_g + \Psi_g A(\alpha) < 0$$

$\square$

Some research has been focused on conditions for finding $\alpha$ which leads to stable $A(\alpha)$ which is in general an NP-hard problem [30]-[32].

Let $\| . \|$ be the induced matrix norm, $I$ identity matrix and $\mu(A_i)$ the matrix measure of $A_i$ defined as:

$$\mu(A_i) = \lim_{\delta \to 0^+} \frac{\|I + \delta A_i\| - I}{\delta} \tag{38}$$

In Proposition 4 we give a general condition which provides us freedom of choosing any $\alpha$ in our framework.

**Proposition 4.** For all $\alpha$ associated to $A(\alpha) \in Co(A_1, A_2, ..., A_{|K|})$ the original switched system described by (21) and its reduced order counterpart using convex gramian is stable for stabilizing switching signal if there exists a norm such that:

$$\mu(A_i) < 0, \text{ for all } i = 1, ..., |K| \tag{39}$$

**Proof:**

$A(\alpha) = \sum_{i=1}^{|K|} \alpha_i A_i$, $\sum_{i=1}^{|K|} \alpha_i = 1$, $\alpha_i \in \mathbb{R}^{\geq 0}$ for all $\alpha$. Moreover $\mu$ is convex and $\text{Re}[\lambda_i(A(\alpha))] \leq \mu(A(\alpha))$ [33]. Hence we have:

$$\text{Re}[\lambda_i(A(\alpha))] \le \mu(A(\alpha)) \le \sum_{i=1}^{|K|}(\alpha_i \cdot \mu(A_i)) < 0 \qquad (40)$$

Therefore the sufficient condition for the stability of $A(\alpha)$ is (39), hence this is also sufficient condition for the stability of $\hat{A}(\alpha)$ according to Proposition2. On the other hand Theorem 1 and Theorem 2 ensure us about the stability of the original and reduced order switched system under stabilizing switching sequence when $A(\alpha)$ and $\hat{A}(\alpha)$ are stable.

$\square$

Our framework is said to be feasible if (27) is satisfied. This can not be always satisfied; One way to improve the feasibility of the proposed model reduction method is using recently proposed extended notion of generalized grammian which is called extended grammian [14].

## 5. Numerical Examples

In this section we have applied the proposed method for reduction of two bimodal switched linear systems. The first example is of order 5 and the second one is of order 25.

*Fifth Order Switched linear System:*

Consider a single-input-single output switched linear of the form (21):

$$A_1 = \begin{bmatrix} -0.9569 & -0.1636 & 0.1179 & -0.00943 & 0.00425 \\ -0.1636 & -0.9735 & 0.255 & -0.1064 & 0.1422 \\ 0.1179 & 0.255 & -1.284 & 0.1509 & -0.2352 \\ -0.00943 & -0.1064 & 0.1509 & -0.9284 & 0.1775 \\ 0.00425 & 0.1422 & -0.2352 & 0.1775 & -0.8085 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.9347 & 2.752 & 0.1713 & 0.5116 & -0.3569 \\ -2.514 & -1.746 & 0.6784 & -2.997 & -3.009 \\ 0.047 & -0.8559 & -0.6181 & -0.1723 & -0.2124 \\ -1.225 & 2.703 & 0.3607 & -0.9974 & -0.6158 \\ -0.4173 & 3.033 & 0.4358 & -0.2138 & -1.01 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.1345 \\ 0 \\ 0.9017 \\ 0.07619 \\ 0.3617 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1.422 \\ 0 \\ 0.1575 \\ 0.3783 \\ 0.1787 \end{bmatrix}$$

$$C_1 = [\text{-2.059} \quad \text{-2.332} \quad \text{-0.3709} \quad 1.286 \quad 0.557]$$

$$C_2 = [1.536 \quad 0.4344 \quad -1.917 \quad 0 \quad 0]$$
$$D_1 = -0.1802$$
$$D_2 = 2.301$$

In order to reduce the switched system first we construct convex gramians over the frequency domain $[\omega_1, \omega_2] = [0.1, 100]$ associated with $\alpha = 0.4$ , $\gamma_i = \gamma'_i = 0.49$ :

$$\Psi^\gamma_{cg}(\omega_1, \omega_2) =$$

$$\begin{bmatrix}
105.6176 & 3.4651 & 7.4806 & -20.7915 & -21.6220 \\
3.4651 & 60.1041 & -1.2246 & -9.4240 & 2.9827 \\
7.4806 & -1.2246 & 116.6340 & 6.9075 & 2.8129 \\
-20.7915 & -9.4240 & 6.9075 & 101.9774 & -18.3694 \\
-21.6220 & 2.9827 & 2.8129 & -18.3694 & 103.3487
\end{bmatrix}$$

$$\Psi^{\gamma'}_{og}(\omega_1, \omega_2) =$$

$$\begin{bmatrix}
484.5217 & -19.7167 & 33.3448 & -46.3701 & -58.1727 \\
-19.7167 & 394.1394 & 38.4309 & -40.8157 & 31.0604 \\
33.3448 & 38.4309 & 455.2840 & 34.0563 & -35.5232 \\
-46.3701 & -40.8157 & 34.0563 & 485.5720 & 3.1139 \\
-58.1727 & 31.0604 & -35.5232 & 3.1139 & 526.3895
\end{bmatrix}$$

The resulting third order switched linear model by applying the presented method is:

$$A_{1r} = \begin{bmatrix}
-0.7431 & -0.051 & 0.07166 \\
0.1496 & -0.935 & 0.03146 \\
-0.09937 & -0.04228 & -1.262
\end{bmatrix} \quad A_{2r} = \begin{bmatrix}
-0.7214 & 0.2683 & -0.1391 \\
-0.2549 & -0.6095 & 0.1671 \\
0.3458 & -0.04548 & -0.7505
\end{bmatrix}$$

$$B_{1r} = \begin{bmatrix}
-0.1704 \\
-0.262 \\
-1.317
\end{bmatrix}, \quad B_{2r} = \begin{bmatrix}
1.594 \\
-1.169 \\
-0.1603
\end{bmatrix}$$

$$C_{1r} = [1.464 \quad -1.157 \quad 0.4734]$$
$$C_{2r} = [-0.1603 \quad -0.8186 \quad -0.5214]$$
$$D_{1r} = -0.1802, D_{2r} = 2.301$$

Fig. 1 shows the decay rate of the genentized Hankel singular values. The step response of the original and reduced order switched systems associated to the switching signal of Fig.2 is presented in Fig. 3.
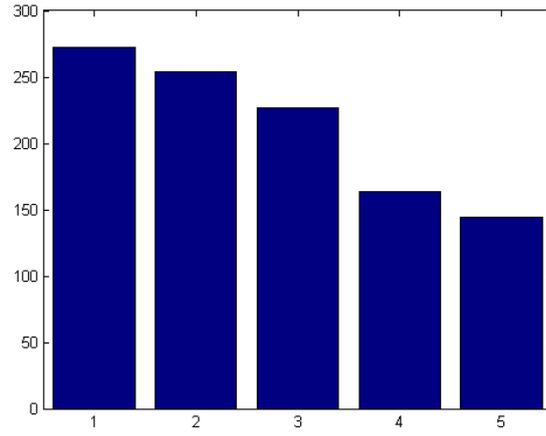
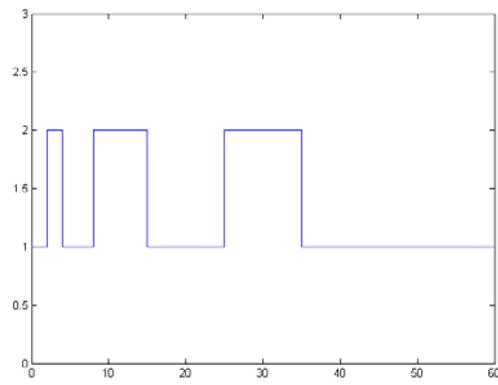Fig. 1.  Generalized Hankel Singular Values($\gamma_i$)



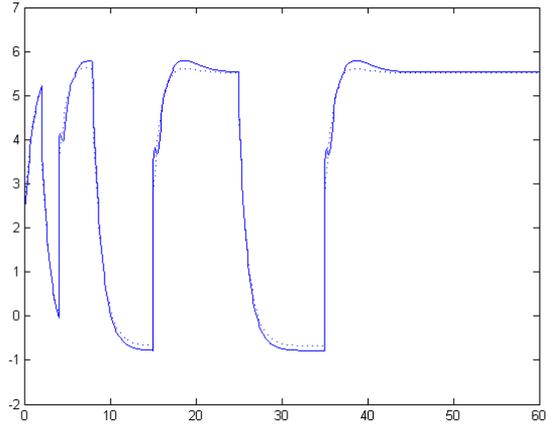Fig. 2.  Randomly generated switching signal

Fig. 3. Step response of original switched linear system(solid line) and the reduced order model (dotted).

Fig. 1. shows that most of the input/output information is in three states of the original systems. The proposed method provides accurate results after reduction of 2 states of the original system (40% of the states).

*Bimodal Switched linear System of order 25:*

Consider bimodal switched linear system of order 25. The original system is SISO and it is reduced to 14, 17, 18 and 19 using the proposed reduction method over $[\omega_1, \omega_2] = [0.001, 1000]$.

The generlized Hankel singular values are shown in Fig. 4.

The step responses of the original and reduced order switched systems associated to the switching signal of Fig.5 is shown in Fig. 6 – Fig. 9 .

In this example also we represent the infinity norm of transfer function of the original subsystems and the reduced counterpart which is of order 19 in Fig. 10 and Fig. 11 to show how accurate the approximation works locally. As we expected by reduction of more states we loose more input-output information in reduced order dynamical system and it leads to less accurate approximation. The quality of the approximation is highly dependent of the decay rate of singular values.
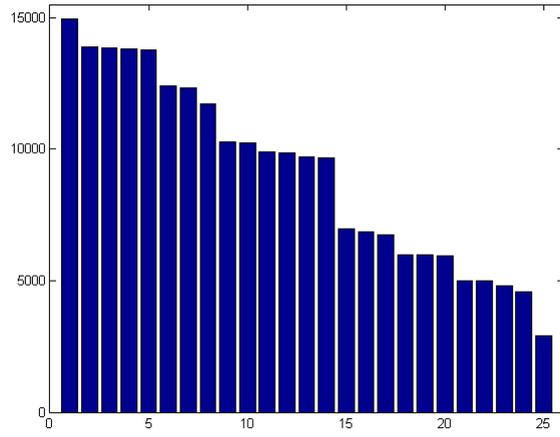
Fig. 4. Generalized Hankel Singular Valuse($\gamma_i$)



Fig. 5. switching signal

Fig. 6.   Step response of original switched linear system (solid line) and the reduced order model which is of order 19(dotted).
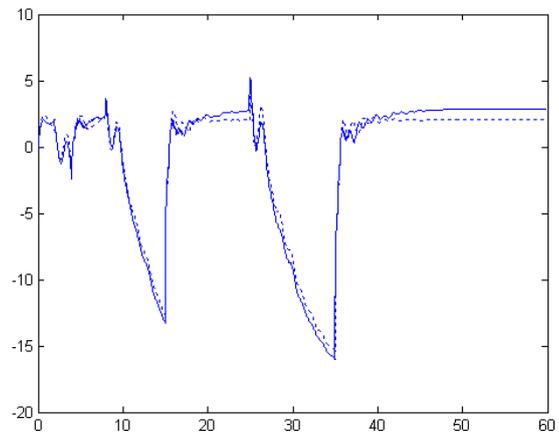


Fig. 7.   Step response of original switched linear system (solid line) and the reduced order model which is of order 18(dotted).
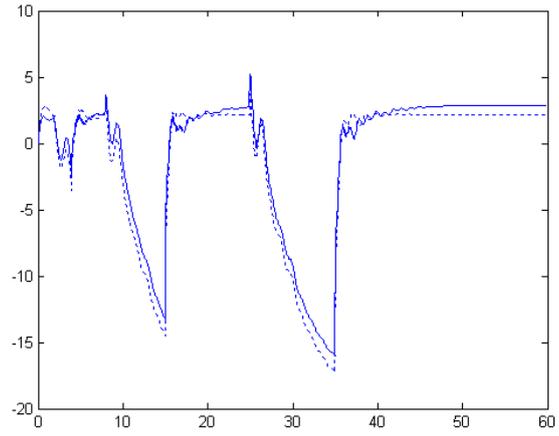
Fig. 8. Step response of original switched linear system (solid line) and the reduced order model which is of order 17(dotted).
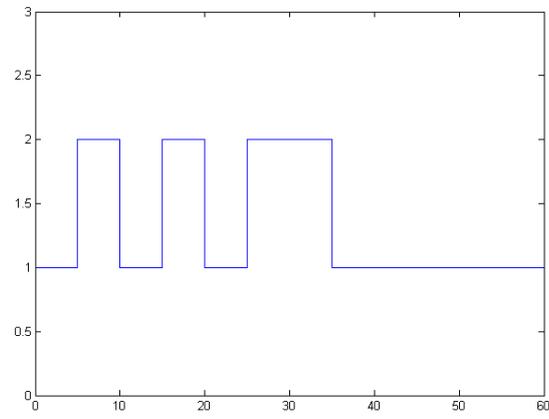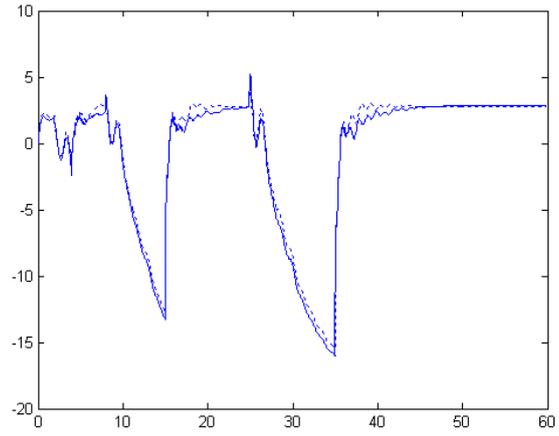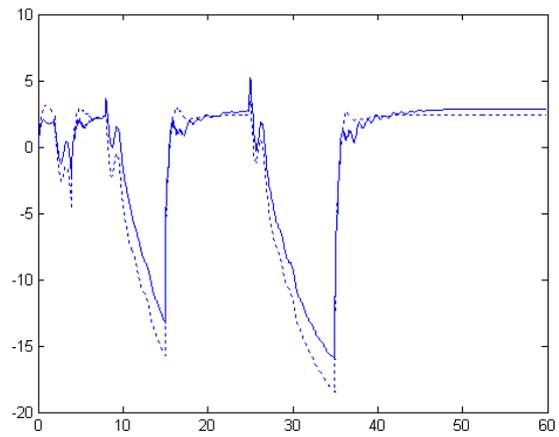


Fig.9 . Step response of original switched linear system (solid line) and the reduced order model which is of order 14(dotted).
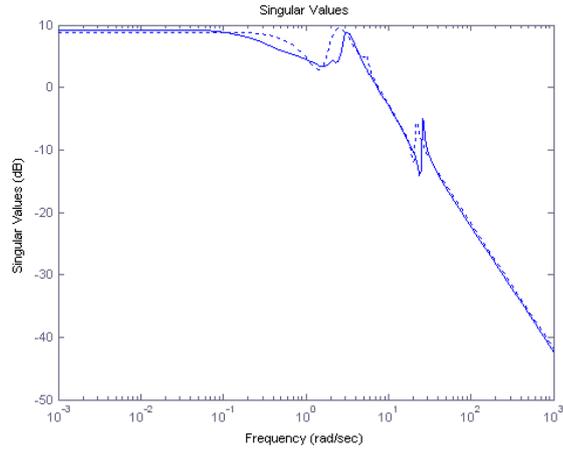
Fig. 10. The infinity nom of original of transfer matrix of first subsystem (solid line) and its reduced order counterpart of order 19(dotted) over frequency domain $[\omega_1, \omega_2] = [0.001, 1000]$.
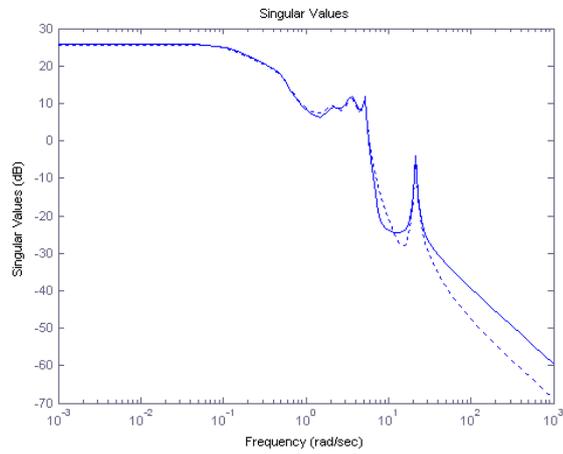


Fig. 11. The infinity nom of original of transfer matrix of second subsystem (solid line) and its reduced order counterpart of order 19(dotted) over frequency domain $[\omega_1, \omega_2] = [0.001, 1000]$.

## 6. Conclusion

A general framework for model order reduction of switched linear dynamical systems has

been presented. In this paper we have reformulated the frequency domain balanced reduction method into this scheme but generally various gramian based reduction methods can be reformulated in the proposed generalized method easily and can be applied for reduction of switched system. The stability issue has been studied in the paper. The method provides single projectors for all subsystems which enable us to reduce all of the subsystems in one step. It is less conservative then previous method based on common generalized gramian. The method is dependent to selection of parameters. This opens a window toward further modifications in optimization framework.

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# Paper C

Model Reduction of Switched Systems Based on Switching Generalized Gramians

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This paper is submitted to:

# 1. Introduction

Complexity of models are increasing as the response to the ever-increasing need for accurate mathematical modeling of physical as well as artificial processes for simulation and control. This problem demands efficient automatic computational tools to replace such complex models by an approximate simpler models, which are capable of capturing dynamical behavior and preserving essential properties of the complex one, either the complexity appears as high order describing dynamical system or complex nonlinear structure. Due to this fact model reduction methods have become increasingly popular over the last two decades [1],[2],[3]. Such methods are designed to extract a reduced order model that adequately describes the behavior of the system in question.

Most of the methods that are proposed so far for control and analysis in hybrid and switched systems theory are suffering from high computational burden when dealing with large-scale dynamical systems. This fact has motivated the researchers in hybrid systems to study model reduction [15]-[28]. Some works have been focused on ordinary model reduction methods that have potential applications in modeling and analysis of hybrid systems [15]-[19] motivated by reachability analysis and safety verification problem. Some researches address the problem of model reduction of switched and hybrid systems directly [20]-[28].

The method that has been presented in [20], deals with abstraction of both continuous and discrete part of hybrid dynamical systems. This framework uses balanced residualization for reduction of continues part. Stability preservation for switched system in the framework is not guaranteed and it also might happen that guard approximation and reset maps approximation cause overlap. In [21] it is shown that the state set can be affinely reduced due to non-observability if and only if a subspace of the classical unobservable subspace, characterized using the normal vectors of the exit facets, is nontrivial. This result does not provide strong tool for reduction of affine systems, as it is an exact reduction which is quite restrictive. Exact reduction is very elegant but the class of systems for which this procedure can be applied is quite small. This method only considers observability for investigating the importance of the states to discard. Although this method has been modified in [25] but lots of problems are still remained open and should be addressed in this context. The paper [22] is concerned with the problem model reduction for discrete switched system. Two different approaches are proposed to solve this problem. The first approach casts the model reduction into a convex optimization problem, which is the first attempt to solve the model reduction problem by using linearization procedure. The second one, based on the cone complementarity linearization, casts the model reduction problem into a sequential

minimization problem subject to linear matrix inequality constraints. Both approaches have their own advantages and disadvantages concerning conservatism and computational complexity. These optimization problems will be very hard if not infeasible to solve for a large scale system. This method is just applicable to discrete time switched systems and it does not provide information about the number of states which is suitable to keep prior to the reduction. Similar methods have been developed for more general classes of discrete time switched systems in [23] [24].

In [26] we proposed the generalized gramian framework for model reduction of switched systems based on common generalized gramians of the subsystems. This framework has been developed for controller reduction in [27]. The framework is shown to provide satisfactory approximations and it preserves the stability of the original system under arbitrary switching signal but it is over conservative. The method which is reported in [28] is based on the convex generalized gramian concept. Although this method is less conservative than its counterpart in [27] and by choosing suitable tuning parameters in the algorithm can be more accurate but the stability preservation is not guaranteed for all switching sequences in this method in general.

In this paper we propose framework for model reduction of switched system based on switching generalized gramian. This general framework can be categorized as gramian based model reduction methods. Balanced model reduction is one of the most common gramian based model reduction schemes. It was presented in [4] for the first time.

To apply balanced reduction, first the system is represented in a basis where the states which are difficult to reach are simultaneously difficult to observe. This is achieved by simultaneously diagonalizing the reachability and the observability gramians, which are solutions to the reachability and the observability Lyapunov equations. Then, the reduced model is obtained by truncating the states which have this property. Balanced model reduction method is modified and developed from different viewpoints [1],[2]. One of the methods that are presented based on balanced model reduction is the method based on the generalized gramians instead of gramians[5]. In this method in order to compute the generalized gramians , one should solve Lyapunov inequalities instead of Lyapunov equations. This method is used to devise a technique for structure preserving model reduction methods in [6].

In this paper we first show that the generalized method in [5] can be extended to various gramian based reduction methods. We also modified the original method in [5] to avoid numerical instability and also to achieve more efficiency by building Petrov-Galerkin projection based on generalized gramians. We propose a method based on the balanced model reduction within frequency bound in this framework. We generalized the framework

to model reduction of switched system by constructing switching Petrov-Galerkin projection based on switching generalized gramian. This framework is developed in a way that preserves stability under arbitrary switching signal and furthermore it is a general method in the sense that different classical gramian based method can be developed for reduction of switched system within this framework. The feasibility and also stability preservation of the algorithm is studied. It is shown that the proposed framework is less conservative than its previous counterparts.

The paper is organized as follows: In the next section we review balanced reduction method and balanced reduction technique based on generalized gramian. Section 2 presents how different gramian based methods can be approximated as generalized gramian based techniques. Balanced reduction within frequency interval based on generalized gramian is also presented in this section. This section ends up with some remarks on numerical implementation of the algorithm and using projection for generalized gramian based reduction methods is suggested instead of balancing and truncation. Section 3 is devoted to develop switching generalized gramian based reduction method for model reduction of switched systems, followed by discussions on stability, feasibility, algorithm parameters and error bound. Section 4 presents our numerical results. Section 5 concludes the paper.

The notation used in this paper is as follows: $M^*$ denotes

transpose of matrix if $M \in \mathbb{R}^{n \times m}$ and complex conjugate transpose if $M \in \mathbb{C}^{n \times m}$. The norm $\|.\|_\infty$ denotes the $H_\infty$, norm of a rational transfer function. The standard notation $>, \geq (<, \leq)$ is used to denote the positive (negative) definite and semidefinite ordering of matrices.

## 2. Balanced Truncation and Generalized Gramians

Balanced truncation is a well-known method for model reduction of dynamical systems, see for example [1][2].The basic approach relies on balancing the gramians of the systems. For dynamical systems with minimal realization:

$$G := (A, B, C, D) \tag{1}$$

where $G$ is transfer matrix with associated state-space representation:

$$\begin{cases} \eta x(t) = Ax(t) + Bu(t), & x(t) \in \mathbb{R}^n \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{2}$$

where $\eta$ is either the derivative operator $\eta f(t) = \dfrac{df(t)}{dt}$ , $t \in \mathbb{R}$ or the shift $\eta f(t) = f(t+1)$ , $t \in \mathbb{Z}$ .

Gramians for continuous time systems are given by the solutions of the Lyapunov equations:

$$AP + PA^* + BB^* = 0$$
$$A^*Q + QA + C^*C = 0$$

(3)

and for discrete time systems by:

$$APA^* - P + BB^* = 0$$
$$A^*QA - Q + C^*C = 0$$

(4)

For stable $A$ , they have a unique positive definite solutions $P$ and $Q$ , called the controllability and observability gramians. In balanced reduction, first the system is transformed to the balanced structure in which gramians are equal and diagonal:

$$P = Q = diag(\sigma_1 I_{k_1}, ..., \sigma_q I_{k_q})$$
$$\sum_{j=1}^{q} k_j = n$$

(5)

where $\sigma_i > \sigma_{i+1}$ and they are called Hankel singular values.

The reduced model can be easily obtained by truncating the states which are associated with the set of the least Hankel singular values. Applying the method to stable, minimal $G$ , if we keep all the states associated to $\sigma_m (1 \le m \le r)$ , by truncating the rest, the reduced model $G_r$ will be minimal and stable and satisfies[1][2]:

$$\|G - G_r\|_\infty \le 2 \sum_{j=r+1}^{q} \sigma_j$$

(6)

One of the closely related model reduction methods to the balanced truncation is balanced reduction based on generalized gramian that is presented in [5]. In this method, instead of Lyapunov equations (3), the following Lyapunov inequalities should be solved:

$$AP_g + P_g A^* + BB^* \le 0$$
$$A^*Q_g + Q_g A + C^*C \le 0$$

(7)

For stable $A$ , they have positive definite solutions $P_g$ and $Q_g$ , called the generalized

controllability and observability gramians. Note that these gramians are not unique. The rest of this model reduction method is the same as the aforementioned balanced truncation method, the only difference is that in this algorithm the balancing and truncation are based on generalized gramian instead of ordinary gramian. In this method we have generalized Hankel singular values ( $\gamma_i$ ) which are the diagonal elements of balanced gerealized gramians instead of Hankel singular values $\sigma_i$ which are the diagonal elements of balanced standard gramians. For the error bound also the same result holds but in terms of the generalized Hankel singular values instead of Hankel singular values. It is worth to mention that $\gamma_i \geq \sigma_i$. Therefore the error bound in balanced reduction based on generalized gramian is greater or equal than the error bound in ordinary balanced model reduction. In order to achieve more accurate results we can find $P_g$ and $Q_g$ in (7), such that, they minimize $tr(Q_g)$ and $tr(P_g)$ respectively.

## 3. Generalized Gramian Framework for Gramian-based Model Reduction Methods

In this section we present a general framework to build generalized gramian version of gramian based methods. Then we present generalized balanced reduction within frequency bound within this framework following by some words about numerical implementation of the algorithm based on projection.

*Lyapunov Equations, Lyapunov Inequalities and Reduction*

**Lemma 1:** Suppose $A$ is stable, $Y$ is symmetric and

$$A^*YA - Y \leq 0$$
$$A, Y \in \mathbb{R}^{n \times n}$$

(8)

is satisfied, then $Y \geq 0$, i.e. $Y$ is positive semi definite.

**Proof:** If $A^*YA - Y \leq 0$, there exists $M \geq 0$ such that:

$$A^*YA - Y + M = 0$$

On the other hand, for any stable $A$, there exists the following solution for the equation above:

$$Y = \sum_{k=0}^{\infty} (A^*)^k M A^k$$

In the above structure $M \geq 0$, hence:

$$Y \geq 0$$

□

This lemma leads to the following proposition that makes the relation between Lyapunov equations and Lyapunov inequalities evident. Let $\Gamma : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ be the operator which is defined as:

$$\Gamma_{A,B}(X) := A^* XA - X + B \qquad (9)$$

**Proposition 1:** Suppose $A$ is stable and $X$ is the solution of Lyapunov equation:

$$\Gamma_{A,Q}(X) = A^* XA - X + Q = 0 \qquad (10)$$

where $Q \geq 0$. If a symmetric $X_g$ satisfies:

$$\Gamma_{A,Q}(X) = A^* X_g A - X_g + Q \leq 0 \qquad (11)$$

Then: $X_g \geq X$.

**Proof:** Subtract (11)-(10) and apply Lemma 1 with $Y = X_g - X$.

□

Proposition 1 shows how the generalized gramian could be an approximation for ordinary gramians. Similar to the method in [5], based on Proposition 1, we can propose generalized version of different gramian based reduction methods in this framework. This method might provide less accurate approximation than its gramian based counterpart but still the approximation error is bounded and the reduced model is stable. The only step that we need to take is to derive associated Lyapunov equations (10) and the associated Lyapunov inequalities (11). In the following we propose generalized version of discrete time balanced reduction within frequency interval.

*Generalized Balanced Reduction within Frequency Bound*

Over the past two decades, a great deal of attention has been devoted to balanced model reduction and it has been developed and improved from different viewpoints. Frequency weighted balanced reduction method is one of the devised gramian based techniques based on ordinary balanced truncation [1],[2],[7]-[9]. In this method by using input and output weights and stressing on certain frequency range more accurate results can be achieved. In many cases the input and output weights are not given and instead the problem is to reduce the model over a given frequency range [1],[2]. This is problem can be attacked directly by

balanced reduction within frequency bound. This method first proposed in [10] and then modified in [2] for continuous time systems in order to preserve the stability of the original system and to provide an error bound for approximation. In [29], similar method have been proposed for discrete-time systems and further improved in [30] to preserve stability and to provide computable error bound. In this method, for discrete time dynamical system (1) the controllability gramian $P(\omega_1, \omega_2)$ and observability gramians $Q(\omega_1, \omega_2)$ within frequency range of operation $[\omega_1, \omega_2]$ are defined as:

$$P(\omega_1, \omega_2) := \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} (I - Ae^{-j\omega})^{-1} BB^* (I - A^* e^{j\omega})^{-1} d\omega$$
$$Q(\omega_1, \omega_2) := \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} (I - A^* e^{j\omega})^{-1} C^* C (I - Ae^{-j\omega})^{-1} d\omega$$

(12)

where: $0 \le \omega_1 < \omega_2 \le \pi$.

Due to the symmetry of the Furier transform the integration is carried out over $[\omega_1, \omega_2]$ and $[-\omega_2, -\omega_1]$, therefore the gramians are always real.

In order to show the associated Lyapunov equations, we need some more notations:

$$F(\omega_1, \omega_2) := -\frac{\omega_2 - \omega_1}{4\pi} I + \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} (I - Ae^{-j\omega})^{-1} d\omega$$

(13)

$$X(\omega_1, \omega_2) = F(\omega_1, \omega_2) BB^* + BB^* F(\omega_1, \omega_2)^*$$

(14)

$$Y(\omega_1, \omega_2) = C^* C F(\omega_1, \omega_2) + F(\omega_1, \omega_2)^* C^* C$$

(15)

The gramians satisfy the following Lyapunov equations [29],[30]:

$$AP(\omega_1, \omega_2) A^* - P(\omega_1, \omega_2) + X(\omega_1, \omega_2) = 0$$
$$A^* Q(\omega_1, \omega_2) A - Q(\omega_1, \omega_2) + Y(\omega_1, \omega_2) = 0$$

(16)

This method is modified in [30] to guarantee stability and to provide a simple error bound. The modified version starts with Schur decomposition of $X$ and $Y$:

$$X(\omega_1, \omega_2) = U \Lambda U^* = U \, diag(\lambda_1, ..., \lambda_n) U^*$$
$$Y(\omega_1, \omega_2) = V \Delta V^* = V \, diag(\delta_1, ..., \delta_n) V^*$$

(17)

where: $UU^* = VV^* = I_n, |\lambda_i| \geq |\lambda_{i+1}| \geq 0, |\delta_i| \geq |\delta_{i+1}| \geq 0$.

Note that since $X(\omega_1, \omega_2)$ and $Y(\omega_1, \omega_2)$ are real and symmetric decompositions in the form (16) exist. Let:

$$\hat{B} := U \, diag(|\lambda_1|^{1/2}, ..., |\lambda_n|^{1/2})$$
$$\hat{C} := diag(|\delta_1|^{1/2}, ..., |\delta_n|^{1/2}) V^*$$

(18)

The modified gramians satisfy the following Lyapunov equations instead of (16):

$$A\hat{P}(\omega_1, \omega_2)A^* - \hat{P}(\omega_1, \omega_2) + \hat{B}\hat{B}^* = 0$$
$$A^*\hat{Q}(\omega_1, \omega_2)A - \hat{Q}(\omega_1, \omega_2) + \hat{C}^*\hat{C} = 0$$

(19)

That is all what we need to present the generalized version of this method:

$$A\hat{P}(\omega_1, \omega_2)A^* - \hat{P}(\omega_1, \omega_2) + \hat{B}\hat{B}^* \leq 0$$
$$A^*\hat{Q}(\omega_1, \omega_2)A - \hat{Q}(\omega_1, \omega_2) + \hat{C}^*\hat{C} \leq 0$$

(20)

Then the generalized modified balanced reduction within frequency bound can be obtained by simultaneously diagonalizing $\hat{P}_g(\omega_1, \omega_2)$ and $\hat{Q}_g(\omega_1, \omega_2)$ then by truncating the states associated to the set of the least generalized Hankel singular values.

*Numerical Issues*

Balanced transformation can be ill-conditioned numerically when dealing with the systems with some nearly uncontrollable modes or some nearly unobservable modes. Difficulties associated with computation of the required balanced transformation in [11] draw some attentions to alternative numerical methods [12]. Balancing can be a badly conditioned even when some states are much more controllable than observable or vice versa. It is advisable then to reduce the system in the gramian based framework without balancing at all. Schur method and square root algorithms provides projection matrices to apply balanced reduction without balanced transformation [1][12]. This method can be easily applied to other gramian based method. In our generalized method we can use the same algorithm by plugging generalized gramians into the algorithm instead of ordinary gramians.

## 4. Model Reduction of Switched System

*Model Reduction of Switched Systems Based on Convex Generalized Gramians*

One of the most important subclasses of hybrid systems are Linear switched systems[13]. Linear switched system is a dynamical system specified by the following equations:

$$\Sigma : \begin{cases} \eta x(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t) \\ y(t) = C_{\sigma(t)} x(t) + D_{\sigma(t)} u(t) \end{cases} \tag{21}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^p$ is the output, $u(t) \in \mathbb{R}^m$ is the input, and $\sigma : \mathbb{R}^{\geq 0} \to K \subset \mathbb{N}$ is the switching signal that is a piecewise constant map of the time. K is the set of discrete modes, and it is assumed to be finite. For each $i \in K$, $A_i, B_i, C_i, D_i$ are matrices of appropriate dimensions. The indicator function is defined as:

$$\zeta_i(t) = \begin{cases} 1, & \text{when the switched system is described} \\ & \text{by the i}^{th} \text{ mode matrices } (A_i, B_i, C_i, D_i) \\ 0, & \text{otherwise} \end{cases} \tag{22}$$

The switched system (21) can also be written as following using indicator function:

$$\Sigma : \begin{cases} \eta x(t) = \sum_{i=1}^{|K|} \zeta_i (A_i x(t) + B_i u(t)) \\ y(t) = \sum_{i=1}^{|K|} \zeta_i (C_i x(t) + D_i u(t)) \end{cases} \tag{23}$$

In this section we build a framework for model reduction of switched system described by (21). The aim is to find projection that maps the state-space of a switched system to lower dimensional subspace. Definition 1, describes the general definition of Petrov-Galerkin projection.

**Definition1.** Petrov-Galerkin projection for a dynamical system:

$$\begin{cases} \eta x(t) = f(x(t), u(t)) \quad , x \in \mathbb{R}^n \\ y(t) = g(x(t), u(t)) \end{cases} \tag{24}$$

is defined as a projection $\Pi = VW^*$, where: $W^*V = I_k$, $V, W \in \mathbb{R}^{n \times k}$, $k < n$ [1].

The reduced order model using this projection is:

$$\begin{cases} \eta \hat{x}(t) = W^* f(V\hat{x}(t), u(t)) & , \quad \hat{x} \in \mathbb{R}^k \\ y(t) = g(V\hat{x}(t), u(t)) \end{cases} \tag{25}$$

In our framework we construct the aforementioned projection based on the switching generalized gramian which is defined as following:

**Definition2.** Switching controllability (observability) generalized gramian for the dynamical system (21) is defined as:

$$\Psi_g(t) = \sum_{i=1}^{|K|} \zeta_i(t) P_{g,i} , \tag{26}$$

where $P_{g,i}$ is controllability (observability) generalized gramian associated to the $i^{th}$ mode of (21).

In order to develop generalized gramian framework to model reduction of switched linear system the generalized gramian reduction framework can be applied locally on each subsystem to reduce each subsystem independently. As opposed to ordinary gramians, the generalized gramians are not unique, therefore we can choose the generalized gramians for subsystems such that the reduction framework preserves important properties of the original system such as stability.

At this point it is possible to develop different gramian based reduction methods into this framework for reduction of switched system finding generalized controllability/observability gramian for each subsystem, constructing switching controllability/observability generalized gramian. The next step can be simultaneous diagonalization of the switching generalized gramian and balancing and reduction of all subsystems based on Hankel singular values of the switching generalized gramian in each mode. In order to avoid numerical bad conditioning and also to increase the efficiency we use Schur or square root algorithm instead of balancing and directly Petrov-Galerkin projection matrices can be computed. This procedure is less conservative and provides more accurate results.

In the method that we proposed in [26] the stability of the original switched systems under arbitrary switching signal is guaranteed to be preserved due to existence of common - function. This was the main reason for conservatism. In our new framework the generalized

gramians are computed such that the existence of piecewise quadratic Lyapunov function for the switched system is guaranteed and the stability of the reduced switched system is guaranteed consequently. In following we first propose our general framework for model reduction of switched system.

Let the observability gramian $Q_i$ and the controllability gramian $P_i$ corresponding a general gramian based method for each subsystem is derived as the solution to the following Lyapunov equation:

$$\Gamma_{A_i, M_i}(Q_i) = 0 \tag{27}$$

$$\Gamma_{A_i^*, N_i}(P_i) = 0 \tag{28}$$

where $M_i$, $N_i$ are positive semi-definite.

In order to develop the gramian based reduction method for switched system which preserves the stability of the original system, the switching controllability generalized gramian $\Psi_{cg}(t)$ and switching controllability generalized gramian $\Psi_{og}(t)$ are obtained:

$$\Psi_{cg}(t) = \sum_{i=1}^{|K|} \zeta_i(t) P_{g,i} \tag{29}$$

$$\Psi_{og}(t) = \sum_{i=1}^{|K|} \zeta_i(t) Q_{g,i} \tag{30}$$

where the generalized observability gramian $Q_{g,i}$ and the generalzed controllability gramian $P_{g,i}$ are the solutions to:

$$\Gamma_{A_i^*, N_i}(P_{g,i}) < 0$$

$$\Gamma_{A_i, M_i}(Q_{g,i}) < 0 \tag{31}$$

$$A_i^* Q_{g,j} A_i - Q_{g,i} < 0$$

For all $i \in K$.

The next step is to simply construct Petrov-Galerkin projection for each subsystem based on the switching gramians in each mode.

In the following in order to clarify the proposed general framework we extend generalized balanced reduction within frequency interval that is presented in previous section, for model reduction of switched linear system.

First we need to find, the generalized controllability gramian $\hat{P}_{g,i}(\omega_1, \omega_2)$ and the generalized observability gramian $\hat{Q}_{g,i}(\omega_1, \omega_2)$ for each subsystem within frequency domain

satisfying (31) for all $i, j \in K$. In other words, the following LMI's need to be solved:

$$A_i \hat{P}_{g,i}(\omega_1, \omega_2) A_i^* - \hat{P}_{g,i}(\omega_1, \omega_2) + \hat{B}_i \hat{B}_i^* < 0 \tag{32}$$

$$A_i^* \hat{Q}_{g,i}(\omega_1, \omega_2) A_i - \hat{Q}_{g,i}(\omega_1, \omega_2) + \hat{C}_i^* \hat{C}_i < 0 \tag{33}$$

$$A_i^* \hat{Q}_{g,j}(\omega_1, \omega_2) A_i - \hat{Q}_{g,i}(\omega_1, \omega_2) < 0 \tag{34}$$

The switching generalized gramians are:

$$\Psi_{cg}(t) = \sum_{i=1}^{|K|} \zeta_i(t)(P_{g,i}(\omega_1, \omega_2)) \tag{35}$$

$$\Psi_{og}(t) = \sum_{i=1}^{|K|} \zeta_i(t)(Q_{g,i}(\omega_1, \omega_2)) \tag{36}$$

If we plug $\Psi_{cg}(t)$ and $\Psi_{og}(t)$ into the square root algorithm we can directly obtain projectors associated to all subsystems for reduction. Note that the results are the same as balancing algorithm. A merit of the square root method is that it relies on the Cholesky factors of the gramians rather than the gramians themselves, which has advantages in terms of numerical stability.

*Stability and Feasibility,*

One of the important issues in model reduction is preservation of the stability which needs to be studied. In our proposed framework the stability of the original switched system is guaranteed to be preserved. In order to prove the stability preservation first we need to recall a theorem on stability of discrete time switched system from [31][32].

**Theorem 1.** If there exist $|K|$ symmetric matrices $S_1, S_2, ..., S_{|K|}$ for discrete time dynamical system (21), satisfying:

$$\begin{bmatrix} S_i & A_i^* S_j \\ S_j A_i & S_j \end{bmatrix} > 0 \qquad \forall (i, j) \in K \times K \tag{37}$$

then the switched system is asymptotic stable and the Lyapunov function is give by:

$$V(t, x(t)) = x(t)^* (\sum_{i=1}^{|K|} \zeta_i(t) S_i) x(t) . \tag{38}$$

This theorem propose a sufficient condition for stability of switched system based on

existence of the piecewise quadratic Lyapunov function for switched system which is less conservative than the condition for stability based on common Lyapunov function (See [31] and [32] for more details and proofs).

In the following proposition we show that our reduction framework for reduction of switched system is stability preserving.

**Proposition2.** If the discrete-time switched system described in (21) is stable, the generalized gramian based reduced order model is guaranteed to be asymptotic stable.

**Proof:**

In the proposed method, we have:

$$W_i^* V_i = I_k , \quad V_i, W_i \in \mathbb{R}^{n \times k}, k < n$$

$$\begin{cases} \hat{A}_i = W_i^* A_i V_i \\ \hat{B}_i = W_i^* B_i \\ \hat{C}_i = C_i V \\ \hat{D}_i = D_i \end{cases} \tag{34}$$

which is projected system matrices associated to the reduced order switched model.

$$\hat{\Sigma} : \begin{cases} \eta \hat{x}(t) = \sum_{i=1}^{|K|} \zeta_i (\hat{A}_i \hat{x}(t) + \hat{B}_i u(t)) \\ y(t) = \sum_{i=1}^{|K|} \zeta_i (\hat{C}_i \hat{x}(t) + \hat{D}_i u(t)) \end{cases} \tag{35}$$

We know $Q_{g,i}$ is the generalized observability gramian and the original switched system satisfy (31), therefore:

$$A_i^* Q_{g,i} A - Q_{g,i} + M_i < 0, \quad Q_{g,i} > 0 \tag{36}$$

$$A_i^* Q_{g,j} A_i - Q_{g,i} < 0 \tag{37}$$

which lead to:

$$\begin{bmatrix} Q_{g,i} & A_i^* Q_{g,j} \\ Q_{g,j} A_i & Q_{g,j} \end{bmatrix} > 0 \qquad \forall (i,j) \in K \times K \tag{38}$$

based on the Schur complement inequality. The original system is asymptotic stable according to theorem 1 and if we find $|K|$ symmetric matrices which satisfy (37) for the reduced order switched system, the reduced order switched model will be stable as well. From (36) and (37) we have:

$$V_i^* (A_i^* Q_{g,i} A - Q_{g,i} + M_i) V_i < 0 \tag{39}$$

$$V_i^* (A_i^* Q_{g,j} A_i - Q_{g,i}) V_i < 0 \tag{40}$$

On the other hand, the outcome of square root algorithm for projection is[1]: $P_{g,i} W_i = V_i \Sigma_i$ and $Q_{g,i} V_i = W_i \Sigma_i$, where $\Sigma_i \in \mathbb{R}^{k \times k}$ is diagonal and positive definite and we have:

$$V_i^* (A_i^* Q_{g,i} A - Q_{g,i} + M_i) V_i$$
$$= V_i^* A_i^* Q_{g,i} A V_i - V_i^* Q_{g,i} V_i + V_i^* M_i V_i$$
$$= V_i^* A_i^* W_i \Sigma_i W_i^* A V_i - V_i^* W_i \Sigma_i + V_i^* M_i V_i$$
$$= (W_i^* A V_i)^* \Sigma_i (W_i^* A V) - \Sigma_i + V_i^* M_i V_i$$

Hence:

$$\hat{A}_i^* \Sigma_i \hat{A}_i - \Sigma_i + V_i^* M_i V_i < 0$$

and consequently:

$$\hat{A}_i^* \Sigma_i \hat{A}_i - \Sigma_i < 0 \tag{41}$$

Let $\Phi_{ij} := V_i^* Q_{g,j} V_i$, therefore $Q_{g,j} = W_i \Phi_{ij} W_i^*$. We have from (40):

$$V_i^* (A_i^* Q_{g,j} A_i - Q_{g,i}) V_i =$$
$$= V_i^* A_i^* Q_{g,j} A_i V_i - V_i^* Q_{g,i} V_i$$
$$= V_i^* A_i^* W_i \Phi_{ij} W_i^* A_i V_i - V_i^* Q_{g,i} V_i$$
$$= \hat{A}_i^* \Phi_{ij} \hat{A}_i - \Sigma_i$$

Hence:

$$\hat{A}_i^* \Phi_{ij} \hat{A}_i - \Sigma_i < 0 \tag{42}$$

Note that: $\Phi_{ij} = \Phi_{ij}^* > 0$ and $\Phi_{ii} = \Sigma_i$.

Let $S_j = \Phi_{ij}$ and $S_i = \Sigma_i$, according to Theorem 1. The reduced order switched model (35) is stable under arbitrary switching sequence.

$$\square$$

Our framework is said to be feasible if (36)-(37) are satisfied. These can not be always satisfied; but the framework is much less conservative comparing to its counterparts in [26] and [27]. One way to improve the feasibility of the proposed model reduction method is using recently proposed extended notion of generalized grammian which is called extended grammian [14].

## 5. Numerical Examples

In this section we have applied the proposed method for reduction of two bimodal switched linear systems. The first example is of order 7 and the second one is of order 25.

$7^{th}$ *Order Switched linear System:*

Consider a single-input-single output switched linear of the form (21):

$$A_1 = \begin{bmatrix} -0.334 & 0.3046 & -0.03543 & -0.07088 & 0.1474 & -0.2414 & -0.07635 \\ 0.1292 & -0.05956 & -0.03945 & 0.2164 & -0.3475 & -0.1074 & -0.2008 \\ -0.1205 & -0.02622 & -0.115 & -0.1031 & -0.05692 & -0.1377 & 0.02162 \\ -0.1308 & 0.01855 & -0.1999 & -0.6649 & -0.1376 & -0.0985 & -0.072 \\ -0.09125 & -0.3183 & -0.04991 & 0.1481 & -0.2894 & -0.1928 & 0.02208 \\ -0.3358 & 0.08599 & -0.05365 & 0.08062 & 0.07906 & -0.3054 & 0.01544 \\ -0.1247 & -0.1874 & 0.0197 & -0.01706 & 0.02899 & -0.01897 & 0.1089 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.02764 & 0.2331 & -0.3819 & 0.1918 & 0.1083 & -0.0531 & 0.412 \\ 0.2406 & -0.5743 & 0.06595 & 0.275 & -0.1156 & 0.3873 & 0.3771 \\ -0.3711 & 0.07406 & -0.3554 & 0.09365 & 0.2317 & 0.02326 & 0.3513 \\ 0.129 & 0.2794 & 0.1674 & 0.3015 & 0.1313 & 0.09701 & -0.05687 \\ 0.1283 & -0.1153 & 0.2107 & 0.1169 & 0.2967 & 0.3146 & -0.2963 \\ -0.01531 & 0.385 & -0.02076 & 0.09491 & 0.3066 & 0.2628 & -0.2449 \\ 0.4385 & 0.3797 & 0.3264 & -0.09338 & -0.2908 & -0.2239 & 0.3117 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.07866 \\ -0.6817 \\ -1.025 \\ -1.234 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.1497 \\ 0 \\ 0 \\ 1.535 \\ 0 \\ -1.347 \\ 0.4694 \end{bmatrix}$$

$$C_1 = [0 \quad 0 \quad 0.0558 \quad 0 \quad -0.465 \quad 0.371 \quad 0.7283]$$

$$C_2 = [-0.9036 \quad 0 \quad -0.6275 \quad 0.5354 \quad 0.5529 \quad -0.2037 \quad -2.054]$$

$$D_1 = 0$$
$$D_2 = 0.1326$$

In order to reduce the switched system we solve LMI's (32)-(34) to compute the switching gramians over the frequency domain $[\omega_1, \omega_2] = [0.0001, 1]$. The switching observability generalized gramian is computed by solving (33) and (34):

$$\Psi_{og}(t) = \sum_{i=1}^{2} \zeta_i(t)(Q_{g,i}(\omega_1, \omega_2))$$

where

$\hat{Q}_{g,1}(\omega_1, \omega_2) =$

$$\begin{bmatrix}
432.3740 & -2.0884 & 36.5183 & 22.1324 & -26.9003 & 27.7670 & 5.6355 \\
-2.0884 & 474.9520 & -15.9972 & 0.2649 & 32.5495 & -23.7194 & 2.1711 \\
36.5183 & -15.9972 & 426.6650 & 3.4264 & -9.5463 & 7.0706 & -19.7154 \\
22.1324 & 0.2649 & 3.4264 & 462.4184 & 8.0981 & 20.2084 & 9.2145 \\
-26.9003 & 32.5495 & -9.5463 & 8.0981 & 425.7597 & 29.0492 & -0.5272 \\
27.7670 & -23.7194 & 7.0706 & 20.2084 & 29.0492 & 424.6442 & -4.5798 \\
5.6355 & 2.1711 & -19.7154 & 9.2145 & -0.5272 & -4.5798 & 437.7793
\end{bmatrix}$$

$\hat{Q}_{g,2}(\omega_1, \omega_2) =$

$$\begin{bmatrix}
438.1188 & -1.8337 & 37.8168 & 18.3223 & -32.8578 & 25.2266 & 14.1855 \\
-1.8337 & 474.0864 & -15.9894 & 2.5844 & 29.8504 & -25.6838 & 7.0294 \\
37.8168 & -15.9894 & 427.4084 & 5.2123 & -10.4245 & 6.8873 & -20.0638 \\
18.3223 & 2.5844 & 5.2123 & 457.5497 & 15.1847 & 24.8077 & 9.0909 \\
-32.8578 & 29.8504 & -10.4245 & 15.1847 & 434.9824 & 34.4873 & -14.7420 \\
25.2266 & -25.6838 & 6.8873 & 24.8077 & 34.4873 & 431.6313 & -16.5326 \\
14.1855 & 7.0294 & -20.0638 & 9.0909 & -14.7420 & -16.5326 & 456.5890
\end{bmatrix}$$

The switching controlability generalized gramian $\Psi_{cg}(t)$ is computed similarly by solving (32). The resulting fourth order switched linear model by applying the presented method is:

$$A_{1r} = \begin{bmatrix}
-0.7885 & 0.03459 & -0.1212 & -0.1066 \\
0.1418 & -0.5086 & -0.3072 & -0.1373 \\
-0.008041 & 0.2888 & -0.5294 & -0.04207 \\
-0.05953 & -0.07607 & -0.1193 & 0.2978
\end{bmatrix}$$

$$A_{2r} = \begin{bmatrix}
-0.9483 & 0.01865 & -0.01575 & -0.02349 \\
0.01436 & 0.9308 & 0.04869 & -0.005533 \\
0.01945 & 0.04652 & -0.9376 & 0.04768 \\
-0.03012 & 0.03389 & -0.03229 & 0.7319
\end{bmatrix}$$

$$B_{1r} = \begin{bmatrix} 0.4763 \\ -1.272 \\ -0.6668 \\ -2.061 \end{bmatrix}, B_{2r} = \begin{bmatrix} 0.2252 \\ 0.3751 \\ 0.26 \\ 0.5943 \end{bmatrix}$$

$$C_{2r} = \begin{bmatrix} 0.9824 & -2.738 & -0.5533 & -0.9583 \end{bmatrix}$$

$$C_{1r} = \begin{bmatrix} -0.155 & -0.072 & 0.01531 & 0.1706 \end{bmatrix}$$

$$D_{1r} = 0, D_{2r} = 0.1326$$

Fig. 1 shows the generlized Hankel singular values of the first subsystem and Fig. 2 shows the generalized Hankel singular values of the second subsystem. The step response of the original and reduced order switched systems associated to the switching signal of Fig.3 is presented in Fig. 4.
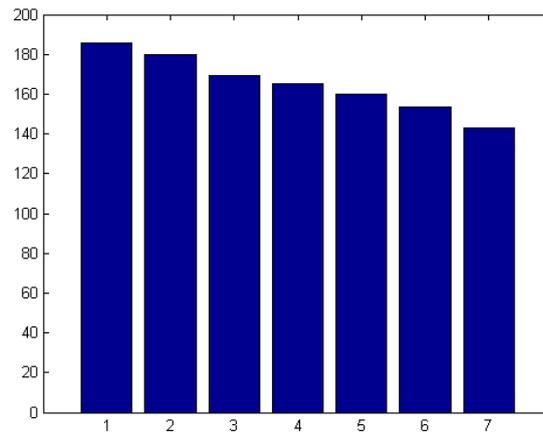


Fig. 1. Generalized Hankel Singular Valuse( $\gamma_i$ ) of the first subsystem
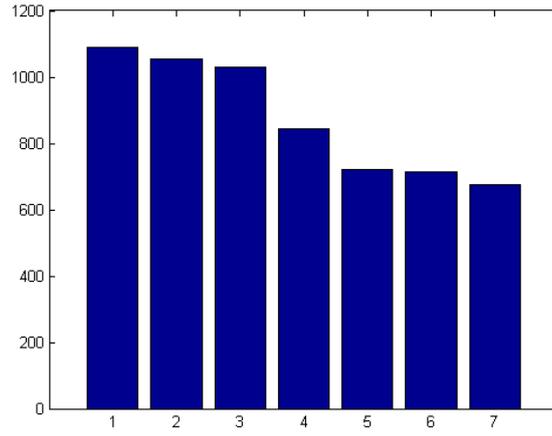
Fig. 2. Generalized Hankel Singular Valuse( $\gamma_i$ ) of the second subsystem



Fig. 3. Randomly generated switching signal
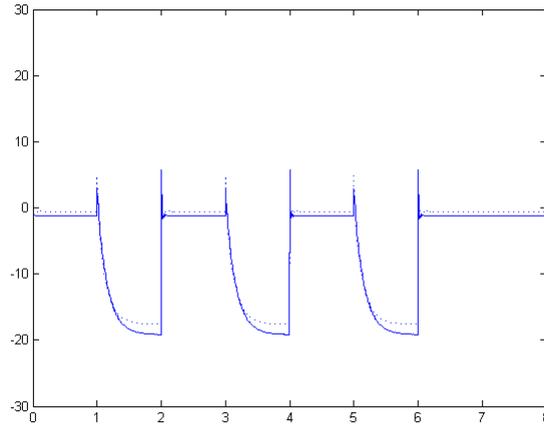
Fig. 3.  Step response of original switched linear system(solid line) and the reduced order model (dotted).

Fig. 1. and Fig. 2 show that most of the input/output information is in four states of the original systems. The proposed method provides accurate results after reduction of 3 states of the original system (42.8 % of the states).


*Bimodal Switched linear System of order 25:*

Consider bimodal switched linear system of order 25. The original system is SISO and it is reduced to 17 using the proposed reduction method over $[\omega_1, \omega_2] = [0.001, 1000]$.

The generlized Hankel singular values are shown in Fig. 4 and Fig. 5.
The step responses of the original and reduced order switched systems associated to the switching signal of Fig.6 is shown in Fig. 7.

Fig. 4. Generalized Hankel Singular Valuse( $\gamma_i$ ) of the first subsystem


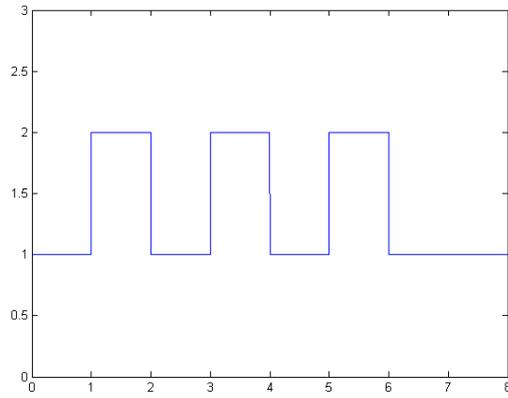
Fig. 5. Generalized Hankel Singular Valuse( $\gamma_i$ ) of the first subsystem
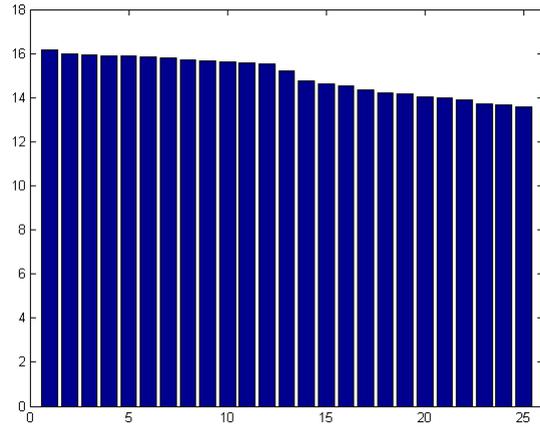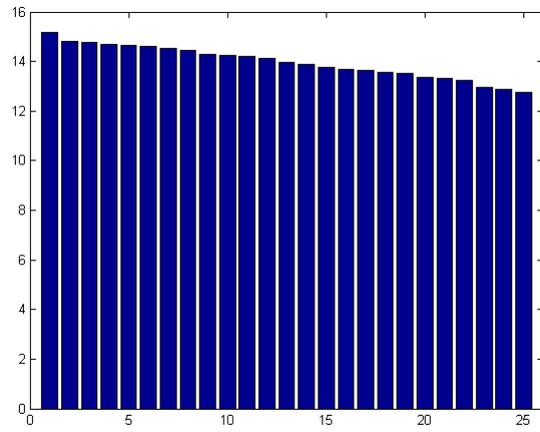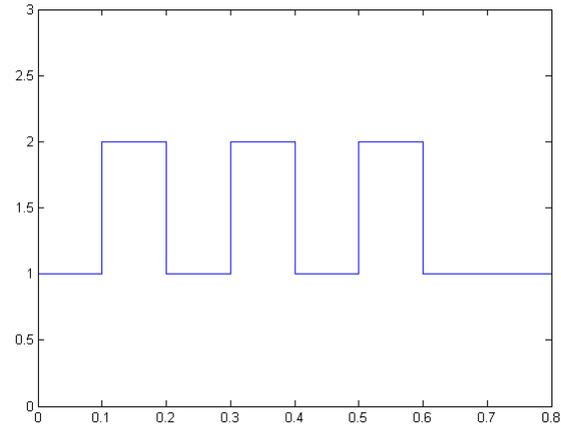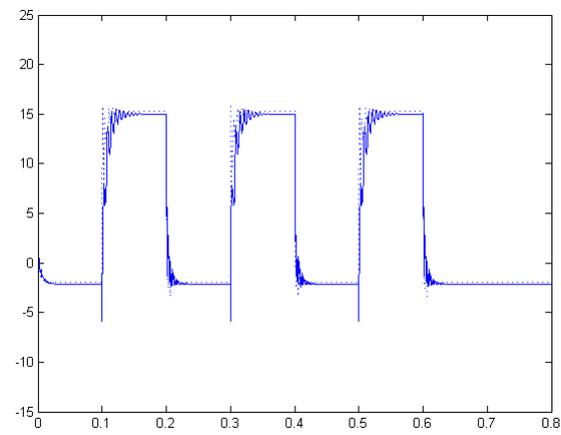
Fig. 5. switching signal



Fig. 6. Step response of original switched linear system (solid line) and the reduced order model which is of order 17(dotted).

# 6. Conclusion

A general framework for model order reduction of switched linear dynamical systems has been presented. In this paper we have reformulated the frequency domain balanced reduction method into the generalized gramian framework but generally various gramian based reduction methods can be reformulated in the proposed framework easily and can be applied for reduction of switched system. It is proved that the proposed framework preserves the stability of the original system. The method is much less conservative then previous method based on common generalized gramian. The method can be further extended for reduction of switching controllers and   for closed loop model reduction with embedded switching which will be addressed in the future works.

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# Paper D

On Exact/Approximate Reduction of Dynamical Systems Living on
Piecewise linear Partition

Hamid Reza Shaker and Rafael Wisniewski

# 1 Introduction

Over the past two decades model reduction has become an ubiquitous tool in a variety of application areas and, accordingly, a research focus for many mathematicians and engineers [1]. Most of the methods that are proposed so far for control and analysis of hybrid and switched systems are suffering from high computational burden when dealing with large-scale dynamical systems. This fact has motivated the researchers in hybrid systems to study model reduction. Because of the weakness of nonlinear model reduction techniques and also pronounced needs for efficient analysis and control of large-scale dynamical hybrid and switched systems; it is essential to study model reduction of hybrid and switched systems in particular. One of the most important classes of hybrid systems which has been studied extensively in the literature is a class of piecewise affine systems. This class is equivalent to many other hybrid system classes such as mixed logical dynamical systems, linear complementary systems, and maxmin-plus-scaling systems and thus form a very general class of linear hybrid systems. To our knowledge the only available study in the context of reduction of affine systems in the literature is the work done by Habets and Schuppen [2] which has considered the problem of the exact reduction due to non-observability. Model reduction problem for dynamical systems which are defined on piecewise linear partitioning is addressed in this paper. Our presented work is generalization and modification of the method in [2]. It is easy to show that in our method if we restrict our attention just to reduction due to non-observability the method also provides the same results as [2]. The technique presented is based on the transformation of affine dynamical systems inside the cells to a new structure and it can be applied to both exact reduction and also approximate model reduction. In this framework both controllability and observability of the affine system inside the polytopes are considered for reduction purpose. The paper is organized as follows: In the next section we review some definitions and notions which clarify our problem formulation. Section 3 presents the main contribution of this paper. In this section we show the technique to transform affine dynamical systems inside the cells to a new structure in which switching information and input/output relation information are embedded. This section ends up with some remarks on reduction which is the step after transformation.

Section 4 presents our numerical results followed by a brief discussion. Section 5 concludes the paper.

## 2  Linear Partitions, Affine Systems and Reduction

Let $J$ be a finite index set and cardinality of $J$ is $\#J$ . A polyhedral set $P$ in $\mathbb{R}^n$ is the intersection of a family of closed half spaces $H_j^- = \{x \in \mathbb{R}^n \mid \langle x, N_j \rangle \leq a_j\}$ for $N_j \in \mathbb{R}^n$ and $a_j \in \mathbb{R}$ , where $j \in J$ and $\langle .,. \rangle$ is scalar product in $\mathbb{R}^n$ , i.e. $P := \bigcap_{j \in J} H_j^-$ . The polyhedral set $P$ can be expressed by the inequality (1) to be understood components wise:

$$P = \{x \in \mathbb{R}^n \mid Nx \leq a\} \tag{1}$$

where $N = [N_1^T \ldots N_{\#J}^T]^T$ , $a = [a_1 \ldots a_{\#J}]^T$ .

Let $K = \{P_j \mid j \in I\}$ be a polyhedral Complex with the index set $I$ .

$$|K| := \bigcup_{j \in I} P_j \subseteq \mathbb{R}^n$$

Let $E$ be any polyhedral set ( $\mathbb{R}^n$ inclusively). A piecewise linear partition of $E$ is a polyhedral complex $K$ such that $E = |K|$ .The elements of $K$ will be called cells.

We define $K_i := \{P \in K \mid \dim(P) = i\}$ . The class of dynamical systems that we deal with in this paper is the class of affine dynamical systems living on full dimensional cells $K_n$ of linear partition associated to a quadruple $(E, K, U, S)$ , where $E$ is a polyhedral set (a polytope) in $\mathbb{R}^n$ , $K$ is a piecewise affine partition of $E$ , $U$ is a polyhedral set (of admissible inputs) in $\mathbb{R}^m$ , and $S = \{s_p : P \in K_n\}$ is a family of piecewise affine systems:

$$s_p : \begin{cases} \dot{x} = A_p x + B_p u + a_p \\ y = C_p x + D_p u \end{cases} \tag{2}$$

The problem that we address is the reduction of this class of dynamical systems. In model reduction the goal is to reduce the order of dynamical systems, input/output behaviour must be preserved when the reduction is in the exact sense. Approximate reduction keeps the input/output behaviour close to the original system while we reduce the order of dynamical system.

# 3 Reduction Framework for Affine Systems on Linear Partitions

Our framework has two main steps. First, the system should be transformed to a new structure which contains the switching information and is suitable for reduction. Second main step is the reduction part. In this step we can check if the system is reducible in exact sense and if it is we can reduce it. We can also apply linear model reduction techniques at this point for approximate reduction. The system can be retransformed to the structure (2) at the end.

## 3.1 Transformation

In the following we first transform $s_p$ into a new structure. In this structure input/output information and also switching information is embedded. We can apply linear reduction methods easily to the new structure and it can be retransformed to the original structure after reduction. If we introduce the new input vector:

$$W_p := \begin{bmatrix} u \\ a_p \end{bmatrix} \tag{3}$$

the transformed system will be:

$$s_p : \begin{cases} \dot{x} = A_p x + \bar{B}_p W_p \\ y = C_p x + \bar{D}_p W_p \end{cases} \tag{4}$$

where :

$$\begin{aligned} \bar{B}_p &= [B_p \quad I] \\ \bar{D}_p &= \begin{bmatrix} D_p & O \end{bmatrix} \end{aligned} \tag{5}$$

Transformation to this structure makes sense because the reduction procedure has nothing to do with the vector of inputs and it is obvious that based on the dimension of $a_p$ we can recover the new constant vector in the reduced system.

The next step is to find a way to embed the switching information to the structure; in other words information of the cell in which our affine system is defined (1), this will help us to pay attention to the importance of the states which are probably not important from local input/output maps but they are actively involved the switching conditions. The idea is to define new output and using the advantage of exact/approximate preservation of input-output behaviour in model reduction.

In other words, for the structure (4) we define a new output vector:

$$Y_{new} := \begin{bmatrix} y \\ Nx \end{bmatrix} \qquad (6)$$

Hence we have:

$$Y_{new} = \begin{bmatrix} C_p \\ N \end{bmatrix} x + \begin{bmatrix} \bar{D}_p \\ O \end{bmatrix} W_p , \qquad (7)$$

which gives us new $C, D$ matrices. This new structure can be retransformed to the original structure after reduction by partitioning based on the length of the vector $Nx$ and the original output.

The transformed LTI systems contain state contribution in local input/output behaviour and their contribution to the switching actions.

### 3.2    Order Reduction

At this point, we are in position to use several results from linear system theory regarding conditions for exact reduction and also methods to find appropriate projection for exact/ approximate reduction. In the case of exact reduction, applying ordinary controllability/observability tests for LTI systems on the aforementioned transformed system provides us with conditions for exact reducibility. In these propositions for exact reduction we have conditions on the rank of controllability/observability matrices of the transformed system and consequently conditions on affine system matrices and $N$. One can also approach the problem using Grammians which leads the same results. It is also straightforward to find appropriate projection to remove the states due to non-observability or non-cotrollability [3]. In the case of approximate reduction after transformation of the affine system to the aforementioned structure one can use different reduction techniques such as balanced reduction techniques and then it is possible to recover the original structure of the system by partitioning the system based on original output and input. Although this method provides satisfactory approximate results but in approximate reduction a lot of other issues arise which needs more investigations and further research in this context.

## 4    Illustrative Example

In this section we illustrate the proposed framework with a numerical example. Consider a randomly generated dynamical system:

$$\begin{cases} \dot{x} = \begin{bmatrix} -1.119 & 0.2557 & -0.01542 \\ 0.2557 & -1.892 & 0.1438 \\ -0.01542 & 0.1438 & -1.889 \end{bmatrix} x + \begin{bmatrix} 1.444 \\ 0 \\ 0.6232 \end{bmatrix} u \\ y = \begin{bmatrix} 0.799 & 0.9409 & -0.9921 \end{bmatrix} x \end{cases}$$

which is defined on the cell: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} x \le \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

This dynamical system is linear i. e. $a_p = 0$ therefore defining new input (3) is not needed in this case and we can skip this step. The transformed system will be:

$$\dot{x} = \begin{bmatrix} -1.119 & 0.2557 & -0.01542 \\ 0.2557 & -1.892 & 0.1438 \\ -0.01542 & 0.1438 & -1.889 \end{bmatrix} x + \begin{bmatrix} 1.444 \\ 0 \\ 0.6232 \end{bmatrix} u,$$

$$y_{new} = \begin{bmatrix} 0.799 & 0.9409 & -0.9921 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \qquad (8)$$

If we calculate the observability matrix, we can see that it is a full column rank matrix therefore the system is not reducible in the exact sense due to non-observability. The rank of controllability matrix is 3 which shows, the associated system is not reducible in the exact sense due to non-controllability. In order to apply balanced truncation for approximate reduction we first should transform (8) to the balanced realization.

The associated singular values are: $\{1.2594, 0.0920, 0.0014\}$. If we reduce the system to the second order system and retransform the original structure we have:

$$\begin{cases} \dot{x} = \begin{bmatrix} -1.034 & 0.01587 \\ -0.7875 & -2.046 \end{bmatrix} x + \begin{bmatrix} 1.614 \\ 0.6135 \end{bmatrix} u \\ y = \begin{bmatrix} 0.5064 & -0.46 \end{bmatrix} x \end{cases} \qquad (9)$$

with the switching inequality:

$$\begin{bmatrix} 0.8679 & 0.0655 \\ 0.0892 & -0.2289 \\ 0.2731 & 0.2993 \\ -1.2302 & -0.1359 \end{bmatrix} x \le \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{10}$$

In Fig.1 the states for both reduced and original systems are shown in terms of time. The solid line shows the switching signal which is 1 when the dynamical system hits the facet and switching occurs. As we can, both systems hit the facet and switch almost simultaneously at 0.0495. This Figure confirms that approximation provides us with accurate results regarding the switching time. Fig.2 shows that approximate reduction also preserve input/output behaviour quite well. In general for exact reduction this framework works very well but in the case of approximate reduction some other issues should be taken into account such as stability preservation. It might happen that the framework can not keep the stability of original hybrid system. Although the accuracy of the method inside the cell is quite well depending on the dynamics outside of the cell it might also happen that the approximation in the neighbourhood and outside of the cell is not satisfactory. These problems need further investigation and research to be done.
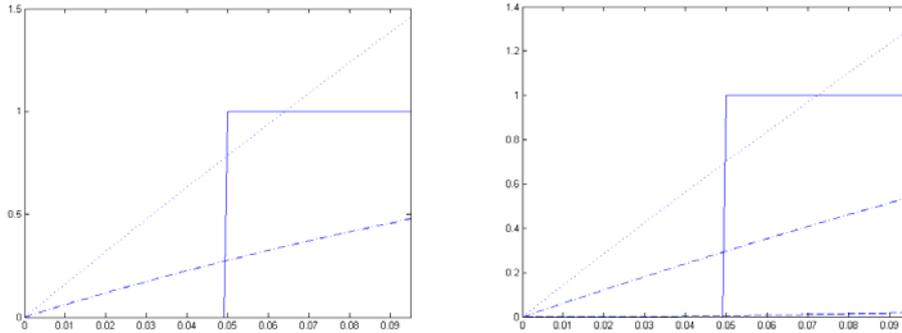


**Figure 1**. Left: reduced system( $x_1$ :dotted, $x_2$ :dash dotted, switch: solid ) Right: original system( $x_1$ :dotted, $x_2$ :dash dotted, , $x_3$ :dashed ,switch: solid)
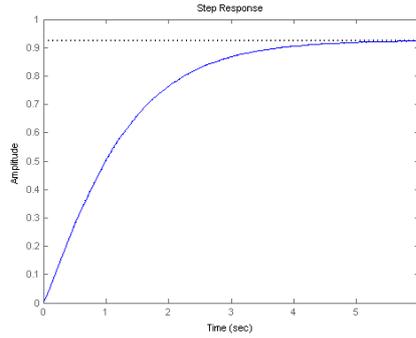
**Figure 2**. Step response of reduced system (dotted) and original system(solid)

## 5 Conclusion

Model reduction problem for dynamical systems which are defined on piecewise linear partitioning was addressed in this paper. The method compromises generalization and modification of [2]. The technique presented is based on the transformation of affine dynamical systems inside the cells to a new structure and it can be applied for both exact reduction and also approximate model reduction. In the case of exact reduction the method works very well. Although in the approximate reduction numerical results are satisfactory but still several issues like stability preservation, guard overlaps need to be studied.

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# Paper E

Stability Analysis for a Class of Switched Nonlinear Systems

Hamid Reza Shaker and Jonathan P. How

This paper will be published in:

## 1. Introduction

Hybrid and switched dynamical systems received a lot of attention over the last decade due to their capability for mathematical modeling of physical systems as well as man-made systems [1]. There is a growing demand in industry for methods to model, analyze, and control hybrid and switched dynamical systems.

Among different problems in this field, a lot of research has been devoted to the study of the stability of switched and hybrid systems [1][3]. Most of the methods which have been proposed so far for stability analysis of hybrid systems are devoted to switched linear systems. The stability of switched systems under arbitrary switching signal is guaranteed by the existence of a common Lyapunov function. In the linear case, many approaches have been presented to construct common Lyapunov functions. The problem is more complicated for switched nonlinear systems and relatively fewer results have been reported in this context. The existence of a common Lyapunov function is only sufficient for the stability of switched systems and can be rather conservative. There are examples of systems that do not posses a common Lyapunov function, but are stable under arbitrary switching signals. Due to the conservatism of the methods for stability analysis which are based on the common Lyapunov function, some attention has been paid to a less conservative class of Lyapunov functions, called switched quadratic Lyapunov functions [2],[12]. In this paper, we propose sufficient stability conditions based on switched quadratic Lyapunov functions for a class of switched nonlinear systems. The conditions are similar to their counterpart in [2] for switched linear systems. These linear matrix inequalities (LMI) conditions are easy to check and suitable for controller synthesis.

We study the class of discrete-time switched nonlinear systems of the form:

$$\Sigma : \begin{cases} x(k+1) = A_{\sigma(k)}\Phi_{\sigma(k)}(x(k)) \\ y(k) = C_{\sigma(k)}\Phi_{\sigma(k)}(x(k)) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state, $y(k) \in \mathbb{R}^p$ is the output and $\sigma : \mathbb{Z}^{\geq 0} \to \mathrm{K} = \{1, 2, ..., |\mathrm{K}|\}$, is the switching signal that is a piecewise constant map of the time index. K is the set of discrete modes, which is assumed to be finite. For each $i \in \mathrm{K}$, $A_i$, $C_i$ are matrices of appropriate dimensions. Furthermore:

$$\Phi_i(x(k)) \coloneqq \begin{bmatrix} \bar{\Phi}_i(x_1(k)) \\ \bar{\Phi}_i(x_2(k)) \\ \vdots \\ \bar{\Phi}_i(x_n(k)) \end{bmatrix} \qquad (2)$$

where:

$$\bar{\Phi}_i \in OL \coloneqq \left\{ \phi : \mathbb{R} \to \mathbb{R} \middle| \; \forall s, t \in \mathbb{R}, \; \left| \phi(s) + \phi(t) \right| \le \left| s + t \right| \right\}.$$

It is worth mentioning that subsystems with the above description are from the class of so-called $\Phi$-systems [4]. In some literature, they have been called $\sigma$-systems[7][8]. It is clear from the description that the nonlinearity of this class of systems is odd and 1-Lipschitz. The standard saturation and the hyperbolic tangent (popular activation function in neural network) are examples of this type of nonlinear systems[4][7]-[10]. The discrete-time recurrent artificial neural network is a special case of $\Phi$-systems [7]-[9]. Furthermore, results related to this class of nonlinear systems have potential applications in the classical problems related to uncertain nonlinearities such as Lur'e systems[11].

The notation used in this paper is as follows: $M^*$ denotes transpose of matrix if $M \in \mathbb{R}^{n \times m}$ and complex conjugate transpose if $M \in \mathbb{C}^{n \times m}$. The standard notation $>$, $\ge (<, \le)$ is used to denote the positive (negative) definite and semidefinite ordering of matrices.

## 2. Positive Diagonal Dominant Matrices

In this section we recall a definition and results which we will use in the sequel.

*Definition 1:* A matrix $P$ is said to be positive diagonally dominant (pdd) if:

$$\begin{cases} P > 0, \\ \left| p_{ii} \right| > \sum_{j \ne i} \left| p_{ij} \right|, \qquad \forall i. \end{cases} \qquad (3)$$

This definition simply says that a matrix is pdd if it is positive definite and row diagonally dominant.

**Lemma 1** [4]: $P$ is pdd if and only if $P > 0$ and there exists a symmetric $R = [r_{ij}]$ such that:

$$\begin{cases} r_{ij} \ge 0, \quad p_{ij} + r_{ij} \ge 0, & \forall i \ne j, \\ p_{ii} \ge \sum_{j \ne i} (p_{ij} + 2r_{ij}) & \forall i. \end{cases} \tag{4}$$

**Lemma 2** [4]: $P$ is pdd if and only if:

$$P > 0$$
$$\forall \Phi \in OL, \qquad \forall \gamma \in \mathbb{R}^n, \qquad \Phi(\gamma)^* P \Phi(\gamma) \le \gamma^* P \gamma. \tag{5}$$

Lemma 2 shows the elegant property of the pdd matrices which is useful for finding conditions for quadratic stability of $\Phi$-systems.

## 3. Stability of Switched $\Phi$ - Systems

Consider the family of the switched $\Phi$-systems described in (1). This class of systems can also be represented as:

$$\Sigma : \begin{cases} x(k+1) = \sum_{i=1}^{|K|} \zeta_i(k) A_i \Phi_i(x(k)) \\ y(k) = \sum_{i=1}^{|K|} \zeta_i(k) C_i \Phi_i(x(k)) \end{cases} \tag{6}$$

where:

$\zeta_i(k)$ is the indicator function which is defined as:

$$\zeta_i(k) = \begin{cases} 1, & \text{when the switched system is described} \\ & \text{by the i}^{th} \text{ mode matrices } (A_i, C_i) \text{ and } \Phi_i \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

A sufficient condition for stability is the following:

**Proposition 1.** The switched system (6) is asymptotically stable under an arbitrary switching signal if there exist $|\kappa|$ symmetric pdd matrices, $P_1, P_2, ..., P_{|K|}$ satisfying:

$$\begin{bmatrix} P_i & A_i^* P_j \\ P_j A_i & P_j \end{bmatrix} > 0 \qquad \forall (i, j) \in K \times K \tag{8}$$

**Proof:**

Let:

$$V(k, \gamma(k)) := \gamma(k)^* \left( \sum_{i=1}^{|K|} \zeta_i(k) P_i \right) \gamma(k), \qquad \gamma(k) \in \mathbb{R}^n \tag{9}$$

$$\zeta(k) = [\zeta_1(k) \quad \zeta_2(k) \quad \cdots \quad \zeta_{|K|}(k)],$$

$$A(\zeta(k)) = \sum_{i=1}^{|K|} \zeta_i(k) A_i, \quad P(\zeta(k)) = \sum_{i=1}^{|K|} \zeta_i(k) P_i$$

$$\Phi_\sigma(x(k)) = \sum_{i=1}^{|K|} \zeta_i(k) \Phi_i(x(k))$$

then:

$$\Delta V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k))$$
$$= x(k+1)^* P(\zeta(k+1)) x(k+1) - x(k)^* P(\zeta(k)) x(k)$$
$$= \Phi_\sigma^*(x(k)) A^*(\zeta(k)) P(\zeta(k+1)) A(\zeta(k)) \Phi_\sigma(x(k))$$
$$\quad - x(k)^* P(\zeta(k)) x(k)$$
$$= \Phi_\sigma^*(x(k))[A^*(\zeta(k)) P(\zeta(k+1)) A(\zeta(k)) - P(\zeta(k))] \Phi_\sigma(x(k))$$
$$\quad + \Phi_\sigma^*(x(k)) P(\zeta(k)) \Phi_\sigma(x(k)) - x(k)^* P(\zeta(k)) x(k)$$

On the other hand, since:

$P_1, P_2, ..., P_{|K|}$ are all pdd and $\Phi_\sigma \in OL$ we have from Lemma 2 that:

$$\Phi_\sigma^*(x(k)) P(\zeta(k)) \Phi_\sigma(x(k)) - x(k)^* P(\zeta(k)) x(k) \le 0$$

The Schur complement of (8), shows that:

$$A^*(\zeta(k)) P(\zeta(k+1)) A(\zeta(k)) - P(\zeta(k)) < 0$$

Therefore:

$$\Delta V(k, x(k)) < 0,$$

which proves the stability of the switched system(6).

$\square$

Note that the switched quadratic Lyapunov function is a common Lyapunov function when $P_1 = P_2 = ... = P_{|K|}$. Therefore, the stability condition based on the switched quadratic Lyapunov function generalizes the approaches based on the common Lyapunov function and is usually less conservative.

The next proposition is similar to Proposition 1. In the stability condition of Proposition 2,

we have slack variables which makes the proposition more suitable for design problems.

**Proposition2.** The switched system (6) is asymptotically stable under an arbitrary switching signal if there exist $|K|$ symmetric positive diagonal matrices, $S_1, S_2, ..., S_{|K|}$ and $|K|$ matrices, $G_1, G_2, ..., G_{|K|}$, satisfying:

$$\begin{bmatrix} G_i + G_i^* - S_i & G_i^* A_i^* \\ A_i G_i & S_j \end{bmatrix} > 0 \qquad \forall (i, j) \in K \times K \qquad (10)$$

**Proof:**

From (10), we have: $G_i + G_i^* - S_i > 0$. Since $S_i$ is a positive diagonal matrix, $S_i^{-1}$ is also the positive diagonal, which implies that: $(G_i - S_i)^* S_i^{-1} (G_i - S_i) \geq 0$. Moreover:

$$(G_i - S_i)^* S_i^{-1} (G_i - S_i) = G_i^* S_i^{-1} G_i - G_i^* - G_i + S_i.$$

Hence:

$$\begin{bmatrix} G_i^* S_i^{-1} G_i & G_i^* A_i^* \\ A_i G_i & S_j \end{bmatrix} > 0 \qquad \forall (i, j) \in K \times K$$

On the other hand:

$$\begin{bmatrix} G_i^* S_i^{-1} G_i & G_i^* A_i^* \\ A_i G_i & S_j \end{bmatrix} = \begin{bmatrix} G_i^* & 0 \\ 0 & S_j \end{bmatrix} \begin{bmatrix} S_i^{-1} & A_i^* S_j^{-1} \\ S_j^{-1} A_i & S_j^{-1} \end{bmatrix} \begin{bmatrix} G_i & 0 \\ 0 & S_j \end{bmatrix}$$

Therefore:

$$\begin{bmatrix} S_i^{-1} & A_i^* S_j^{-1} \\ S_j^{-1} A_i & S_j^{-1} \end{bmatrix} > 0$$

where $S_i^{-1}$ and $S_j^{-1}$ are positive diagonal. $S_i^{-1}$ and $S_j^{-1}$ are obviously pdd matrices. From Proposition 1, we conclude that the switched system (6) is stable.

$\square$

In Proposition 2, $S_1, S_2, ..., S_{|K|}$ matrices in general do not have to be diagonal. The only restriction is that the inverse of these matrices need to be pdd.

Note that specifying a matrix to be pdd is LMI (Lemma 1) and therefore to check the proposed conditions, we need to solve an LMI.

## 4. Numerical Examples

In this section the proposed method is applied to two numerical examples: one is a second order switched $\Phi$-system and the other one is a third order switched system.

*Second Order Switched $\Phi$-system:*

Consider a bimodal switched $\Phi$-system with the system matrices:

$$A_1 = \begin{bmatrix} -0.3099 & -0.6063 \\ -0.6063 & 0.3684 \end{bmatrix}, A_2 = \begin{bmatrix} -0.04904 & -0.02485 \\ 0.02485 & -0.04904 \end{bmatrix}.$$

The LMI condition (8) is feasible with the solution:

$$P_1 = \begin{bmatrix} 34.5791 & -0.5189 \\ -0.5189 & 35.0735 \end{bmatrix}, P_2 = \begin{bmatrix} 26.3119 & 0.0760 \\ 0.0760 & 26.2510 \end{bmatrix},$$

with

$$R_1 = \begin{bmatrix} 0 & 9.4905 \\ 9.4905 & 0 \end{bmatrix}, R_2 = \begin{bmatrix} 0 & 5.7751 \\ 5.7751 & 0 \end{bmatrix}.$$

Therefore, the switched $\Phi$-system is stable. The inverse of a two-dimensional pdd matrix is always a pdd matrix. This is not always true for higher dimensions. Because of the fact that the inverse of a two-dimensional pdd matrix is always a pdd, the following $S_1$ and $S_2$ satisfy (10).

$$S_1 = \begin{bmatrix} 0.0289 & 0.0004 \\ 0.0004 & 0.0285 \end{bmatrix}, S_2 = \begin{bmatrix} 0.0380 & -0.0001 \\ -0.0001 & 0.0381 \end{bmatrix}.$$

Proposition 2 for second order systems is less conservative than higher order systems and $S_1, S_2, ..., S_{|K|}$ do not have to be diagonal matrices.

*Third Order Switched $\Phi$-system:*

Consider a third order $\Phi$-system with the system matrices:

$$A_1 = \begin{bmatrix} -0.06515 & -0.4744 & 0.3041 \\ -0.4744 & 0.4872 & 0.3732 \\ 0.3041 & 0.3732 & -0.1271 \end{bmatrix}, A_2 = \begin{bmatrix} 0.04419 & 0.3155 & -0.04247 \\ 0.1451 & -0.04931 & -0.2805 \\ 0.2833 & -0.01418 & 0.1554 \end{bmatrix}.$$

The LMI condition (8) is feasible for this example with the solution:

$$
P_1 = \begin{bmatrix} 16.5100 & -0.4733 & -1.7526 \\ -0.4733 & 16.1032 & -0.4243 \\ -1.7526 & -0.4243 & 13.3561 \end{bmatrix}, P_2 = \begin{bmatrix} 14.6794 & 0.1369 & 0.1619 \\ 0.1369 & 11.8281 & -0.1509 \\ 0.1619 & -0.1509 & 12.0436 \end{bmatrix}, R_1 = \begin{bmatrix} 0 & 2.5952 & 3.2349 \\ 2.5952 & 0 & 2.5707 \\ 3.2349 & 2.5707 & 0 \end{bmatrix},
$$

$$
R_2 = \begin{bmatrix} 0 & 2.0286 & 2.0161 \\ 2.0286 & 0 & 2.1725 \\ 2.0161 & 2.1725 & 0 \end{bmatrix}.
$$

The switched system is stable under arbitrary switching signals.

LMI condition (10) is also feasible with the solution:

$$
S_1 = \begin{bmatrix} 15.4911 & 0 & 0 \\ 0 & 13.4929 & 0 \\ 0 & 0 & 16.0713 \end{bmatrix}, S_2 = \begin{bmatrix} 17.2202 & 0 & 0 \\ 0 & 17.2319 & 0 \\ 0 & 0 & 17.2063 \end{bmatrix}
$$

$$
G_1 = \begin{bmatrix} 14.7675 & 0.4846 & 1.5626 \\ 0.4846 & 12.2697 & 0.1082 \\ 1.5626 & 0.1082 & 15.4927 \end{bmatrix}, G_2 = \begin{bmatrix} 16.9289 & -0.0223 & -0.0117 \\ -0.0223 & 16.9435 & 0.0143 \\ -0.0117 & 0.0143 & 16.9115 \end{bmatrix}
$$

Therefore Proposition 2 confirms the stability of the switched system under the arbitrary switching signals.

## 5. Conclusion

Two LMI-based sufficient conditions for stability analysis of a class of switched nonlinear systems are proposed. These conditions are extensions of the LMI-based stability conditions for switched linear systems to switched $\Phi$-systems. The proposed stability results are based on the switched quadratic Lyapunov functions which are usually less conservative than their counterparts which are based on common Lyapunov functions. These results can be used for controller design problems as well as model reduction of switched nonlinear systems.

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# Paper F

Output feedback control for a class of switched nonlinear systems

Hamid Reza Shaker

# 1. Introduction

Hybrid and switched dynamical systems received a lot of attention over the last decade due to their capability for mathematical modeling of physical systems as well as man-made systems [1]. There is a growing demand in industry for methods to model, analyze, and control hybrid and switched dynamical systems. Among different problems in this field, a lot of research has been devoted to the study of the stability and control of switched and hybrid systems [1]-[4]. Most of the methods which have been proposed so far for stability analysis of hybrid systems are devoted to switched linear systems. The stability of switched systems under arbitrary switching signal is guaranteed by the existence of a common Lyapunov function. In the linear case, many approaches have been presented to construct common Lyapunov functions. The problem is more complicated for switched nonlinear systems and relatively fewer results have been reported in this context. The existence of a common Lyapunov function is only sufficient for the stability of switched systems and can be rather conservative. There are examples of systems that do not posses a common Lyapunov function, but are stable under arbitrary switching signals[1][3][4]. Due to the conservatism of the methods for stability analysis which are based on the common Lyapunov function, some attention has been paid to a less conservative class of Lyapunov functions, called switched quadratic Lyapunov functions [2],[5]. In this paper, we propose sufficient stability conditions based on switched quadratic Lyapunov functions for a class of switched nonlinear systems. The conditions are similar to their counterparts in [2] for switched linear systems. These linear matrix inequality (LMI) conditions are used for controller synthesis. We study the class of discrete-time switched nonlinear systems of the form:

$$\Sigma : \begin{cases} x(k+1) = A_{\sigma(k)} \Phi_{\sigma(k)}(x(k)) & , \\ y(k) = C_{\sigma(k)} \Phi_{\sigma(k)}(x(k)) & , \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state, $y(k) \in \mathbb{R}^p$ is the output and $\sigma : \mathbb{Z}^{\geq 0} \to \mathrm{K} = \{1, 2, ..., |\mathrm{K}|\}$, is the switching signal that is a piecewise constant map of the time index. K is the set of discrete modes, which is assumed to be finite. For each $i \in \mathrm{K}$, $A_i$, $C_i$ are matrices of appropriate dimensions. Furthermore:

$$\Phi_i(x(k)) := \begin{bmatrix} \bar{\Phi}_i(x_1(k)) \\ \bar{\Phi}_i(x_2(k)) \\ \vdots \\ \bar{\Phi}_i(x_n(k)) \end{bmatrix}, \tag{2}$$

where: $\bar{\Phi}_i \in OL := \left\{ \phi : \mathbb{R} \to \mathbb{R} \mid \forall s,t \in \mathbb{R}, \; |\phi(s) + \phi(t)| \le |s + t| \right\}$.

It is worth mentioning that subsystems with the above description are from the class of so-called $\Phi$-systems [6],[7]. In some literature, they have been called $\sigma$-systems [8]-[10]. The standard saturation and the hyperbolic tangent (popular activation function in neural networks) are examples of this type of nonlinearities [6]-[11],[13]. The discrete-time recurrent artificial neural network is a special case of $\Phi$-systems [10]-[12]. Furthermore, results related to this class of nonlinear systems have potential applications in the classical problems related to uncertain nonlinearities such as Lur'e systems [14].

The notation used in this paper is as follows: $M^*$ denotes transpose of matrix if $M \in \mathbb{R}^{n \times m}$ and complex conjugate transpose if $M \in \mathbb{C}^{n \times m}$. The standard notation $>$, $\ge (<, \le)$ is used to denote the positive (negative) definite and semidefinite ordering of matrices.

## 2. Positive diagonal dominant matrices

In this section we recall a definition and results which we will use in the sequel.

***Definition 1:*** A matrix $P$ is said to be positive diagonally dominant (PDD) if:

$$\begin{cases} P = P^* > 0, \\ |p_{ii}| > \sum_{j \ne i} |p_{ij}|, \qquad \forall i. \end{cases} \tag{3}$$

This definition simply says that a matrix is PDD if it is positive definite and row diagonally dominant. A matrix $P$ is said to be inverse positive diagonal dominant (IPDD), if $P^{-1}$ is PDD.

**Lemma 1** [6]: $P$ is PDD if and only if $P > 0$ and there exists a symmetric $R = [r_{ij}]$ such that:

$$\begin{cases} r_{ij} \geq 0, \quad p_{ij} + r_{ij} \geq 0, \qquad \forall i \neq j, \\ p_{ii} \geq \sum_{j \neq i} (p_{ij} + 2r_{ij}), \qquad \forall i. \end{cases} \tag{4}$$

**Lemma 2** [6]: $P$ is PDD if and only if:

$$P > 0,$$
$$\forall \Phi \in OL, \qquad \forall \gamma \in \mathbb{R}^n, \qquad \Phi(\gamma)^* P \Phi(\gamma) \leq \gamma^* P \gamma. \tag{5}$$

Lemma 2 shows the elegant property of the PDD matrices which is useful for finding conditions for quadratic stability of $\Phi$-systems.

**Lemma 3** [7]: Let $P$ be $n \times n$ matrix, $n \geq 2$. $P$ is IPDD if :

$$P > 0,$$
$$\forall i \neq j, \qquad |p_{ij}| \leq \frac{1}{2n-3}|p_{ii}|. \tag{6}$$

Note that Lemma 3 is LMI sufficient condition. Further results on IPDD can be found in [7].

## 3. Stability of switched nonlinear systems

Consider the family of the switched $\Phi$-systems described in (1). This class of systems can also be represented as:

$$\Sigma : \begin{cases} x(k+1) = \sum_{i=1}^{|K|} \zeta_i(k) A_i \Phi_i(x(k)), \\ y(k) = \sum_{i=1}^{|K|} \zeta_i(k) C_i \Phi_i(x(k)), \end{cases} \tag{7}$$

where:

$\zeta_i(k)$ is the indicator function which is defined as:

$$\zeta_i(k) = \begin{cases} 1, & \text{when the switched system is described} \\ & \text{by the } i^{th} \text{ mode matrices } (A_i, C_i) \text{ and } \Phi_i, \\ 0, & \text{otherwise.} \end{cases} \qquad (8)$$

A sufficient condition for stability is the following:

**Proposition 1.** The switched system (7) is asymptotically stable under an arbitrary switching signal if there exist $|\kappa|$ symmetric pdd matrices, $P_1, P_2, ..., P_{|K|}$ satisfying:

$$\begin{bmatrix} P_i & A_i^* P_j \\ P_j A_i & P_j \end{bmatrix} > 0 \qquad \forall (i,j) \in K \times K . \qquad (9)$$

**Proof:**

Let:

$$V(k, \gamma(k)) := \gamma(k)^* \left( \sum_{i=1}^{|K|} \zeta_i(k) P_i \right) \gamma(k), \qquad \gamma(k) \in \mathbb{R}^n \qquad (10)$$

$$\zeta(k) = [\zeta_1(k) \quad \zeta_2(k) \quad \cdots \quad \zeta_{|K|}(k)] ,$$

$$A(\zeta(k)) := \sum_{i=1}^{|K|} \zeta_i(k) A_i , \quad P(\zeta(k)) := \sum_{i=1}^{|K|} \zeta_i(k) P_i ,$$

$$\Phi_\sigma(x(k)) = \sum_{i=1}^{|K|} \zeta_i(k) \Phi_i(x(k)) .$$

Then:

$$\Delta V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k)),$$
$$= x(k+1)^* P(\zeta(k+1)) x(k+1) - x(k)^* P(\zeta(k)) x(k),$$
$$= \Phi_\sigma^*(x(k)) A^*(\zeta(k)) P(\zeta(k+1)) A(\zeta(k)) \Phi_\sigma(x(k)) - x(k)^* P(\zeta(k)) x(k),$$
$$= \Phi_\sigma^*(x(k)) [A^*(\zeta(k)) P(\zeta(k+1)) A(\zeta(k)) - P(\zeta(k))] \Phi_\sigma(x(k))$$
$$+ \Phi_\sigma^*(x(k)) P(\zeta(k)) \Phi_\sigma(x(k)) - x(k)^* P(\zeta(k)) x(k).$$

On the other hand, since:

$P_1, P_2, ..., P_{|K|}$ are all pdd and $\Phi_\sigma \in OL$ ,we have from Lemma 2 that:

$$\Phi_\sigma^*(x(k)) P(\zeta(k)) \Phi_\sigma(x(k)) - x(k)^* P(\zeta(k)) x(k) \leq 0 .$$

The Schur complement of (9), shows that:

$$A^*(\zeta(k))P(\zeta(k+1))A(\zeta(k)) - P(\zeta(k)) < 0 .$$

Therefore:

$$\Delta V(k, x(k)) < 0 ,$$

which proves the stability of the switched system(7).

□

Note that the switched quadratic Lyapunov function is a common Lyapunov function when $P_1 = P_2 = ... = P_{|K|}$. Therefore, the stability condition based on the switched quadratic Lyapunov function generalizes the approaches based on the common Lyapunov function and is usually less conservative.

The next proposition is similar to Proposition 1. In the stability condition of Proposition 2, we have slack variables which makes the proposition more suitable for design problems.

**Proposition 2.** The switched system (7) is asymptotically stable under an arbitrary switching signal if there exist $|K|$ matrices $S_1, S_2,..., S_{|K|}$ which are IPDD and $|K|$ matrices, $G_1, G_2,..., G_{|K|}$ ,satisfying:

$$\begin{bmatrix} G_i + G_i^* - S_i & G_i^* A_i^* \\ A_i G_i & S_j \end{bmatrix} > 0, \qquad \forall (i, j) \in K \times K. \tag{11}$$

**Proof:**

From (11), we have: $G_i + G_i^* - S_i > 0$ . Since $S_i$ is IPDD, $S_i^{-1}$ is PDD, which implies that: $(G_i - S_i)^* S_i^{-1}(G_i - S_i) \geq 0$ . Moreover:

$$(G_i - S_i)^* S_i^{-1}(G_i - S_i) = G_i^* S_i^{-1} G_i - G_i^* - G_i + S_i .$$

Hence:

$$\begin{bmatrix} G_i^* S_i^{-1} G_i & G_i^* A_i^* \\ A_i G_i & S_j \end{bmatrix} > 0, \qquad \forall (i, j) \in K \times K.$$

On the other hand:

$$\begin{bmatrix} G_i^* S_i^{-1} G_i & G_i^* A_i^* \\ A_i G_i & S_j \end{bmatrix} = \begin{bmatrix} G_i^* & 0 \\ 0 & S_j \end{bmatrix} \begin{bmatrix} S_i^{-1} & A_i^* S_j^{-1} \\ S_j^{-1} A_i & S_j^{-1} \end{bmatrix} \begin{bmatrix} G_i & 0 \\ 0 & S_j \end{bmatrix}.$$

Therefore:

$$\begin{bmatrix} S_i^{-1} & A_i^* S_j^{-1} \\ S_j^{-1} A_i & S_j^{-1} \end{bmatrix} > 0,$$

where $S_i$ and $S_j$ are IPDD. $S_i^{-1}$ and $S_j^{-1}$ are obviously PDD matrices. From Proposition 1, we conclude that the switched system (7) is stable.

$\square$

## 4. Output feedback control for switched nonlinear system

Consider the problem of the output feedback design:

$$u(k) = (\sum_{i=1}^{|K|} \zeta_i(k) F_i) \ y(k) , \tag{12}$$

for the following switched nonlinear dynamical system:

$$\Sigma : \begin{cases} x(k+1) = \sum_{i=1}^{|K|} \zeta_i(k)(A_i \Phi_i(x(k)) + B_i u(k)), \\ y(k) = \sum_{i=1}^{|K|} \zeta_i(k) C_i \Phi_i(x(k)), \end{cases} \tag{13}$$

where $\Phi_i(x(k))$ is defined by (2). The closed loop system is in the form (1) with the system matrix:

$$A_{cl}(\zeta(k)) = \sum_{i=1}^{|K|} \zeta_i(k)(A_i + B_i F_i C_i) . \tag{14}$$

If we apply the stability results from the previous section directly to find output feedback, the problem is non-convex. The following propositions show how to determine stabilizing output feedback control for the aforementioned switched systems based on the stability results of the last section. These conditions are convex and numerically tractable.

**Proposition 3.** If there exist IPDD matrices $S_i$, matrices $U_i$ and $V_i$, such that:

$$\begin{bmatrix} S_i & (A_i S_i + B_i U_i C_i)^* \\ A_i S_i + B_i U_i C_i & S_j \end{bmatrix} > 0, \qquad \forall (i, j) \in K \times K, \tag{15}$$

and

$$V_i C_i = C_i S_i, \qquad \forall i \in K, \tag{16}$$

then the output feedback (12) with

$$F_i = U_i V_i^{-1}, \qquad \forall i \in K, \tag{17}$$

stabilizes (13).

**Proof:**

From (16) and (17) we have:

$U_i C_i = F_i C_i S_i \qquad \forall i \in K$.

Replacing $U_i C_i$ by $F_i C_i S_i$ in (15), Schur complement leads to:

$$S_i^{-1} - (A_i + B_i F_i C_i)^* S_j^{-1} (A_i + B_i F_i C_i) > 0 \tag{18}$$

$S_i$ and $S_j$ are IPDD, therefore $S_i^{-1}$ and $S_j^{-1}$ PDD which implies that (9) is satisfied with $P_i = S_i^{-1}$ and $P_j = S_j^{-1}$ for the closed loop system. Therefore the closed loop system is stable.

$\square$

The following proposition is based on Proposition 2. Using the slack variables in this proposition provides less conservative sufficient conditions for output feedback control design.

**Proposition 4.** If there exist IPDD matrices $S_i$, matrices $G_i$, $U_i$ and $V_i$, such that:

$$\begin{bmatrix} G_i + G_i^* - S_i & (A_i G_i + B_i U_i C_i)^* \\ A_i G_i + B_i U_i C_i & S_j \end{bmatrix} > 0, \qquad \forall (i, j) \in K \times K, \tag{19}$$

and

$$V_i C_i = C_i G_i, \qquad \forall i \in K, \tag{20}$$

then the output feedback (12) with

$$F_i = U_i V_i^{-1}, \qquad \forall i \in K, \tag{21}$$

stabilizes (13).

**Proof:**

From (20) and (21) we have:

$$U_i C_i = F_i C_i G_i, \qquad \forall i \in K.$$

Replacing $U_i C_i$ by $F_i C_i G_i$ in (19):

$$
\begin{bmatrix} G_i + G_i^* - S_i & (A_i G_i + B_i F_i C_i G_i)^* \\ A_i G_i + B_i F_i C_i G_i & S_j \end{bmatrix}
$$
$$
= \begin{bmatrix} G_i + G_i^* - S_i & G_i^* (A_i + B_i F_i C_i)^* \\ (A_i + B_i F_i C_i) G_i & S_j \end{bmatrix} > 0, \qquad \forall (i, j) \in K \times K. \tag{22}
$$

Proposition 2 applies and the closed loop system is stable.

$\square$

# 5. Numerical examples

In this section we design output feedback controllers for switched nonlinear systems based on Proposition 3 and Proposition 4. The first example is a bimodal system and the second one is a switched system with three discrete modes.

*5.1. Bimodal Switched System:*

Consider a switched nonlinear system in the form (13) with the matrices:

$$A_1 = \begin{bmatrix} -0.5835 & 0.5445 & -0.2751 \\ 0.5445 & 0.02276 & -0.6126 \\ -0.2751 & -0.6126 & -0.4681 \end{bmatrix}, B_1 = \begin{bmatrix} 2.768 & -0.6573 \\ -0.4953 & -1.717 \\ 0.4688 & 1.471 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.6941 & 0.1134 & -1.462 \\ -0.5107 & -0.2298 & -2.882 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.3118 & -0.00226 & 0.1697 \\ -0.00226 & -0.2147 & -0.06306 \\ 0.1697 & -0.06306 & -0.4528 \end{bmatrix}, B_2 = \begin{bmatrix} 1.258 & 0 \\ 1.469 & 0.4757 \\ 0.03294 & 0.3685 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 2.192 & 0.9073 & -0.5384 \\ 0 & 0.4023 & -0.1152 \end{bmatrix}.$$

LMI sufficient conditions of Proposition 3 are feasible. The corresponding output feedback is:

$$u(k) = (\sum_{i=1}^{2} \zeta_i(k)F_i) \, y(k),$$

where:

$$F_1 = \begin{bmatrix} 0.1042 & -0.0616 \\ -0.0616 & 0.0426 \end{bmatrix} \text{ and } F_2 = \begin{bmatrix} 0.1103 & -0.2744 \\ -0.2764 & 1.1758 \end{bmatrix}.$$

LMI sufficient conditions of Proposition 4 are also feasible. The designed output feedback is with:

$$F_1 = \begin{bmatrix} 0.0979 & -0.0641 \\ -0.0643 & 0.0449 \end{bmatrix} \text{ and } F_2 = \begin{bmatrix} 0.1086 & -0.2640 \\ -0.2681 & 1.1331 \end{bmatrix}.$$

### 5.2. A switched nonlinear system with three modes

Consider switched system (13) defined by matrices:

$$A_1 = \begin{bmatrix} 0.02867 & -0.1549 & -0.1331 \\ 0.1983 & 0.1208 & -0.3969 \\ 0.0488 & 0.4157 & 0.1073 \end{bmatrix}, B_1 = \begin{bmatrix} -0.8407 & 1.952 \\ -0.3495 & 0.459 \\ 0 & -0.3993 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 1.307 & -0.7917 \\ -0.2803 & -0.2844 & 0.2406 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.2602 & -0.1601 & -0.04529 \\ -0.1661 & 0.3563 & 0.004146 \\ 0.008821 & 0.04462 & 0.141 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0.4682 \\ 0.3259 & 0.3259 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.1 & 0.02382 & -0.008758 \\ 0.3014 & -0.02205 & 0.9295 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -0.08873 & -0.7651 & -0.3403 \\ -0.7651 & 0.08916 & -0.2076 \\ -0.3403 & -0.2076 & 0.8621 \end{bmatrix}, B_3 = \begin{bmatrix} -0.1734 & 2.072 \\ -0.01403 & 0.637 \\ -0.6127 & 0.07489 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 1.123 & -0.09775 & -0.6155 \\ -0.03313 & -0.5566 & 1.605 \end{bmatrix}.$$

LMI sufficient conditions of Proposition 3 are feasible for this system. The corresponding output feedback is:

$$u(k) = (\sum_{i=1}^{3} \zeta_i(k)F_i)\, y(k)\,,$$

where:

$$F_1 = \begin{bmatrix} 1.3178 & 0.5831 \\ 0.6320 & 0.4464 \end{bmatrix}, F_2 = \begin{bmatrix} 1.1333 & -0.4973 \\ -0.5177 & 0.0998 \end{bmatrix} \text{and } F_3 = \begin{bmatrix} -0.6917 & 0.0950 \\ 0.0823 & 0.0888 \end{bmatrix}.$$

LMI sufficient conditions of Proposition 4 are also feasible. The designed output feedback is with:

$$F_1 = \begin{bmatrix} 1.3070 & 0.4498 \\ 0.6341 & 0.4152 \end{bmatrix}, F_2 = \begin{bmatrix} 1.2082 & -0.4903 \\ -0.4962 & 0.1125 \end{bmatrix} \text{and } F_3 = \begin{bmatrix} -0.5856 & 0.1427 \\ 0.0545 & 0.0761 \end{bmatrix}.$$

*5.3. Feasibility*

Numerical experiment suggests that feasible solutions to the output feedback control problem using proposition 3 and 4 are more readily found than using the methods based on the common Lyapunov function. The sufficient conditions for stability in Proposition 1 and 2 are less conservative than stability conditions based on the existence of common Lyapunov function. However, when dealing with switched systems with a large number of discrete modes, finding feasible solution remains a challenge. The LMI conditions in Proposition 4

are less conservative than conditions in Proposition 3 because the equality constraint in Proposition 3 restricts $S_i$ while in Proposition 4, the slack variables are restricted instead.

## 6. Conclusion

Two LMI-based sufficient conditions for stabilizing output feedback design are proposed. These conditions are based on LMI conditions for stability analysis of a class of switched nonlinear systems which are extensions of analogous conditions for switched linear systems, to switched $\Phi$-systems. The proposed results are based on the switched quadratic Lyapunov functions which are usually less conservative than their counterparts based on common Lyapunov functions.

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