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Abstract— We consider control of a cooling system with several consumers that require cooling from a common source. The flow feeding coolant to the consumers can be controlled, but due to significant physical distances between the common source and the consumers, the coolant flow takes a non-negligible amount of time to travel to the consumers, giving rise to input-dependent state delays. We first present a simple bilinear model of the system, followed by a state feedback control design that is able to stabilize the system at a chosen equilibrium in spite of the delays. We also present a heuristic, performance-oriented improvement to the design. The strategy is illustrated with some simulation examples.

I. INTRODUCTION

Transport delays are a common annoyance and source of instability in many real-life systems, such as flow systems, chemical processes, rolling mill systems, traffic systems, communication networks, etc. As a consequence, systems with delays in states and/or inputs have received considerable attention in the literature; see for instance [1] and the references therein.

In this paper, we consider a simplified model of a single-phase cooling system with non-negligible cooling media transport between a central cooler and a number of consumers that require heat to be removed in real time. The system is bilinear, with delays in the states that depend on the input signal. A real-world example of such a system is the cooling system for main and auxiliary machinery onboard ocean-going ships, documented in [2].

The main challenge of this particular system is that the delay is dependent on the control input; in particular, the delay is inversely proportional to the flow rate in the system, which is one of the controlled inputs. Computing an input signal to compensate for a delay based on feedback of a delayed state, where the delay depends on the same input in the first place, is naturally a difficult problem, and results on input-dependent delays in literature are consequently relatively few. Somewhat related problems have been treated in [3], [4], [5] and [6], which considered time-varying input delays in various settings. More recently, [7] presented a predictor-based methodology for compensating state-dependent input delays for both linear and nonlinear systems. A class of nonlinear systems with input-dependent parameters and delays was considered in [8], in which an open-loop motion planning problem was solved using an explicit parametrization of

trajectories. Building on results developed in [3], [9] and [10] tackled a system that is superficially similar to the cooling system considered in this paper (mixing hot and cold water in a shower), but in their case the transport delay was treated as an input delay and some of the aforementioned complications could thus be avoided (although the design still turned out to be quite complex).

In the present paper, we present a comparatively simple design that overcomes the input-dependent delay difficulties by fixing the flow rate and deriving a Lyapunov-stable feedback law for the remaining inputs. In addition, we present a preliminary, heuristic design which approaches an optimal nonlinear feedback design in some regions of state space and converges to our Lyapunov-stable design as the state approaches the origin. No proof of stability for this design has been established yet, although simulation examples indicate that the heuristic design may work well in practice.

The structure of the remainder of the paper is as follows. Section II first provides an overview of the system under consideration and presents a nominal design for the delay-free case. Section III then provides a stabilizing Lyapunov-Krasovskii-based control design that exploits the structure of the system, along with a performance-oriented, heuristic improvement to the design. Section IV presents some simulation examples, and finally Section V sums up the contributions of the work.

II. SINGLE-PHASE COOLING SYSTEM

A. Physical model

We consider a simple thermodynamical model of a cooling system; see Figure 1.
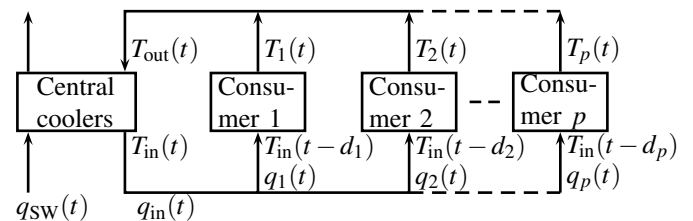
Fig. 1. Cooling system structure. q_i are flows of coolant to the consumers, indexed by $i = 1, \ldots, p$. T_i are the corresponding temperatures of coolant exiting the consumers' heat exchangers. d_i are flow-dependent transport delays.

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The system consists of a cooling source (the surrounding sea, in the case of a ship system), a set of central coolers, and a number of consumers that draw coolant (water) from the central coolers. Each consumer is equipped with a heat exchanger that enables excess heat to be transferred from the consumer to the coolant. Each consumer is situated in parallel, but at different distances from the central coolers, which implies that the coolant has to travel different distances to reach each consumer. The transport of coolant is facilitated by variable-speed pumps; these pumps are equipped with local controllers, which ensure fast flow control compared to the temperature dynamics that we shall consider in the following. Thus, we can assume that the volumetric flows $q_{SW}(t)$, $q_1(t)$, $q_2(t),\ldots,q_p(t)$ are control inputs ($q_{in}(t)$ is controlled to enforce $q_{in}(t) = \sum_{i=1}^{p} q_i(t)$ at all times).

The thermodynamics of the system cover heat transfer between seawater, coolant and the consumers. For simplicity, it is assumed that heat transfer only takes place within the heat exchangers; consequently, a single model with consumer-specific parameters can be used to describe the thermodynamics of each consumer. Indeed, the heat balance for the heat exchanger in each consumer is modelled as a simple first-order ordinary differential equation:

$$\dot{T}_i(t) = \frac{1}{V_i} q_i(t)(T_{in}(t - d_i) - T_i(t)) + w_i(t) \qquad (1)$$

where $T_i(t)$ is the temperature of the cooling medium when it leaves the i'th consumer, V_i is the effective volume of the consumer's heat exchanger and $w_i(t) \in [\underline{w}_i; \overline{w}_i]$ is a slowly varying disturbance (the consumer heat load). $T_{in}(t - d_i)$ is the temperature of the cooling medium as it enters the i'th consumer (ignoring any heat loss along the way).

d_i are *input-dependent transport delays* caused by the fact that the coolant has to travel from the central coolers to each individual consumer; simply put, the slower the coolant flows, the longer the delay becomes. Mass conservation and some simplification provide expressions of the form

$$d_i = d_i(q_i(t)) = \frac{\alpha_i}{q_i(t)} \qquad (2)$$

where α_i are consumer-specific constants that primarily depends on pipe length and diameter of the piping between the central coolers and the consumer in question [2].

The central coolers are modeled in the same way as the consumers, except that in this case the seawater side is receiving heat from the return flow of coolant, and we control the flow of sea water into the central coolers $q_{SW}(t)$:

$$\dot{T}_{in}(t) = \frac{1}{V_{CC}} q_{in}(t)(T_{out}(t) - T_{in}(t))$$
$$+ q_{SW}(t) \frac{\rho_{SW} c_{p,SW}}{\rho c_p}(T_{SW,in}(t) - T_{SW,out}(t)) \quad (3)$$

In (3), V_{CC} is the effective cooler volume while ρ_{SW}, ρ, $c_{p,SW}$ and c_p are densities and specific heat capacity of seawater and cooling media, respectively. $T_{SW,in}(t)$ and $T_{SW,out}(t)$ are the seawater in- and outlet temperatures, respectively; they are not really relevant for the model, except to justify that

for given $T_{SW,in}(t)$, $q_{SW}(t)$ can be adjusted to achieve a desired coolant temperature $T_{in}(t)$. The return flow from the consumers is modeled as

$$\begin{aligned} T_{out}(t) &= \frac{1}{\sum_{i=1}^{p} q_i(t)}(q_1(t)T_1(t) + \cdots + q_p(t)T_p(t)) \\ &= \frac{1}{q_{in}(t)}(q_1(t)T_1(t) + \cdots + q_p(t)T_p(t)) \end{aligned}$$

i.e., simple mixing of the return flow from each consumer.

The goal is to stabilize the temperatures $T_i(t)$ at some desired values $\overline{T}_i$ in the face of strictly positive loads w_i. The reference and load values will be considered constant; in practice, they will vary with the ship's overall operating conditions, i.e., whether it is in a harbor, at sea, close to the Equator, etc., but such variations will be very slow compared to the system dynamics.

B. Bilinear model with delays

We will now rewrite the 'physical' model above in a form more amenable to control design.

With p consumers, we have the set of model equations

$$\dot{T}_1(t) = \frac{q_1(t)}{V_1}(T_{in}(t - d_1) - T_1(t)) + w_1 \qquad (4)$$

$$\vdots$$

$$\dot{T}_p(t) = \frac{q_p(t)}{V_p}(T_{in}(t - d_p) - T_p(t)) + w_p \qquad (5)$$

$$\dot{T}_{in}(t) = \frac{1}{V_{CC}}(q_1(t)T_1(t) + \cdots + q_p(t)T_p(t))$$
$$- \frac{1}{V_{CC}} \sum_{i=1}^{p} q_i(t)T_{in}(t) + Q(t) \qquad (6)$$

where $Q(t) = q_{SW}(t)\rho_{SW}c_{p,SW}(T_{SW,in}(t) - T_{SW,out}(t))/(\rho c_p)$ is considered a control input.

In steady-state operation, for given fixed w_i and with the consumer outlet temperatures equal to their respective reference temperatures $\overline{T}_i$, we have the static relations

$$w_1 = -\frac{\overline{q}_1}{V_1}(\overline{T}_{in} - \overline{T}_1) \qquad (7)$$

$$\vdots$$

$$w_p = -\frac{\overline{q}_p}{V_p}(\overline{T}_{in} - \overline{T}_p) \qquad (8)$$

$$\overline{Q} = -\sum_{i=1}^{p} w_i \qquad (9)$$

where $\overline{T}_{in}$ is the steady-state temperature of the coolant leaving the central coolers and

$$\overline{q}_i = \frac{V_i w_i}{\overline{T}_i - \overline{T}_{in}}, \quad i = 1,\ldots,p \qquad (10)$$

are the corresponding steady-state flows.

Let $n = p + 1$ and define the new coordinates

$$x_i(t) = T_i(t) - \overline{T}_i, u_i(t) = q_i(t) - \overline{q}_i, \quad i = 1,\ldots,n-1$$
$$x_n(t) = T_{in}(t) - \overline{T}_{in}, u_n(t) = Q(t) - \overline{Q}$$

We can then write (4)–(6) on the bilinear form

$$\begin{aligned}
\dot{x}_1 &= a_1(-x_1(t)+x_n(t-d_1)) \\
&\quad +(-b_1-x_1(t)+x_n(t-d_1))u_1(t) \quad (11)
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
\dot{x}_{n-1} &= a_{n-1}(-x_{n-1}(t)+x_n(t-d_{n-1})) \\
&\quad +(-b_{n-1}-x_{n-1}(t)+x_n(t-d_{n-1}))u_{n-1}(t) \quad (12)
\end{aligned}$$

$$\begin{aligned}
\dot{x}_n &= \sum_{i=1}^{n-1} a_i(x_i(t)-x_n(t)) \\
&\quad +\sum_{i=1}^{n-1}(b_i+x_i(t)-x_n(t))u_i(t)+u_n(t) \quad (13)
\end{aligned}$$

where $a_i = \bar{q}_i/V_i$ and $b_i = \bar{T}_i - \bar{T}_{\text{in}}$ are positive constants. Furthermore, initial conditions on the states are given by

$$x_i(0-\tau) = \phi_i(\tau), \ i=1,\ldots,n \quad (14)$$

where $\phi_i : [-\max_i(\alpha_i/\bar{q}_i),0] \to \mathbb{R}$ are continuous functions (known as history functions) defining the states prior to time $t=0$.

C. Preliminary design for delay-free system

Before proceeding, we will briefly derive a stabilizing control law for a delay-free version of the system (11)–(13) for future reference. For notational convenience, in the rest of the paper we ignore current-time dependency; that is, we write $\dot{x}_1,\dot{x}_2,x_1,x_2$ for $\dot{x}_1(t),\dot{x}_2(t),x_1(t),x_2(t)$, etc., but retain $x_n(t-d)$ where appropriate.

We consider the system (11)–(13) with $d_i \equiv 0, i = 1,\ldots,n-1$ along with the positive definite, radially unbounded function

$$V(x_1,\ldots,x_n) = \frac{1}{2}x_1^2 + \cdots + \frac{1}{2}x_n^2 \quad (15)$$

The time derivative of this Lyapunov function candidate along the state trajectories is

$$\begin{aligned}
\dot{V} &= x_1\dot{x}_1 + \cdots + x_n\dot{x}_n \\
&= x_1(a_1(x_n-x_1)-(b_1+x_1-x_n)u_1)+\ldots \\
&\quad +x_{n-1}(a_{n-1}(x_n-x_{n-1})-(b_{n-1}+x_{n-1}-x_n)u_{n-1}) \\
&\quad +x_n\sum_{i=1}^{n-1} a_i(x_i-x_n)+x_n\sum_{i=1}^{p}(b_i+x_i-x_n)u_i+x_nu_n \\
&= -a_1(x_1^2-x_1x_n)+a_1(x_1x_n-x_n^2)-\ldots \\
&\quad -a_{n-1}(x_{n-1}^2-x_{n-1}x_n)+a_{n-1}(x_{n-1}x_n-x_n^2) \\
&\quad -(b_1x_1+x_1^2-x_1x_n)u_1+(b_1x_n+x_1x_n-x_n^2)u_1-\ldots \\
&\quad -(b_{n-1}x_{n-1}+x_{n-1}^2-x_{n-1}x_n)u_{n-1} \\
&\quad +(b_{n-1}x_n+x_{n-1}x_n-x_n^2)u_1+x_nu_n \\
&= -\sum_{i=1}^{n-1}\left(a_i(x_i-x_n)^2+(x_i-x_n)(b_i+x_i-x_n)u_i\right) \\
&\quad +x_nu_n \quad (16)
\end{aligned}$$

Thus, we see that choosing

$$u_i = \mu_i(x_i-x_n)(b_i+x_i-x_n), \ i=1,\ldots,n-1 \quad (17)$$
$$u_n = -kx_n \quad (18)$$

with $\mu_i,k > 0$ results in

$$\dot{V} = -\sum_{i=1}^{p}(x_i-x_n)^2(a_i+\mu_i(b_i+x_i-x_n)^2)-kx_n^2 \quad (19)$$

which is clearly negative definite for all x_i,x_n and positive a_i,b_i. This construction thus gives a stabilization result with a global region of attraction in the absence of delays. Note that the larger x_i-x_n gets, the more dominant the bilinear effects become. Hence, if only linear feedback were applied for u_i, an uncompensated perturbation of the form $-(x_1-x_2)^3$ arises, which is detrimental to stability when x_2 is a lot larger than x_1; the quadratic part of the control law has the effect of compensating directly for this.

In fact, we can state the following result:

Theorem 1: Consider the system (11)–(13) with $d_1,\ldots,d_{n-1} = 0$ and with the control law (17), (18). The origin of the closed-loop system is globally asymptotically stable. Furthermore, the control law

$$u_i = \beta(x_i-x_n)(b_i+x_i-x_n), \quad i=1,\ldots,n-1 \quad (20)$$
$$u_n = -\beta x_n, \quad (21)$$

with $\beta \geq 2$, is the minimizer of the cost functional

$$J(u) = \int_0^\infty \left[l(x(t))+u_1^2(t)+\cdots+u_{n-1}^2(t)+u_n^2(t)\right] dt \quad (22)$$

where

$$\begin{aligned}
l(x) &= 2\beta\sum_{i=1}^{n-1} a_i(x_i-x_n)^2+\beta^2 x_n^2 \\
&\quad +\beta^2\sum_{i=1}^{n-1}(x_i-x_n)^2(b_i+x_i-x_n)^2. \quad (23)
\end{aligned}$$

For further details, including a proof, see [11] and [12]. On a side note, in the original temperature coordinates, (22) becomes

$$\begin{aligned}
J(u) &= \int_0^\infty \Bigg[2\beta\sum_{i=1}^{p}\frac{\bar{q}}{V_i}(T_i(t)-T_{\text{in}}(t)-\bar{T}_i+\bar{T}_{\text{in}})^2 \\
&\quad +2\beta^2\sum_{i=1}^{p}(T_i(t)-T_{\text{in}}(t)-\bar{T}_i+\bar{T}_{\text{in}})^2(T_i(t)-T_{\text{in}}(t))^2 \\
&\quad +(T_{\text{in}}(t)-\bar{T}_{\text{in}})^2\Bigg] dt,
\end{aligned}$$

again assuming all delays are negligible.

III. CONTROL DESIGN FOR SYSTEM WITH DELAYS

We now move on to the case with non-zero delays.

A. Global design

Recognizing that keeping the flows to each consumer constant will also cause the delays to be constant, we are able to state the main result of this paper.

Theorem 2: Consider the system (11)–(13) with history functions $\phi_1,\ldots,\phi_n : [-\max(\alpha_i/\bar{q}_i),0) \to \mathbb{R}$; the control law

$$u_1(t) = u_2(t) = \cdots = u_{n-1}(t) \equiv 0 \ \forall t \geq 0 \quad (24)$$
$$u_n(t) = -kx_n(t) \quad (25)$$

with

$$k > \sum_{i=1}^{n-1} \frac{1}{3} a_i e^{\delta_i \alpha_i / \bar{q}_i}, \; \delta_i > 0$$

renders the origin globally exponentially stable.

Proof: Consider the Lyapunov-Krasovskii function candidate (weighted L_2-norm):

$$V(x) = \frac{1}{2} \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n-1} \varepsilon_i \int_{-d_i}^{0} e^{\delta_i \zeta_i} x_n(t + \zeta_i)^2 d\zeta_i \quad (26)$$

with $\varepsilon_i, \delta_i \in \mathbb{R}_+$ constants to be determined and fixed $d_i = \alpha_i / \bar{q}_i, i = 1, \ldots, n-1$.

Inserting (11)–(13) into (26), substituting $u_i = 0, i = 1, \ldots, n-1$, $u_n = -kx_n$ and using Leibniz' Rule followed by integration by substitution, the time derivative of V along the trajectories of (11)–(13) is found to be

$$\dot{V} = \sum_{i=1}^{n} x_i \dot{x}_i + \sum_{i=1}^{n-1} \frac{\partial}{\partial t} \varepsilon_i \int_{-d_i}^{0} e^{\delta_i \zeta_i} (x_n(t + \zeta_i)^2) d\zeta_i$$

$$= \sum_{i=1}^{n} x_i \dot{x}_i + \sum_{i=1}^{n-1} \varepsilon_i \int_{-d_i}^{0} e^{\delta_i \zeta_i} \frac{\partial}{\partial t} (x_n(t + \zeta_i)^2) d\zeta_i$$

$$= \sum_{i=1}^{n-1} x_i a_i(-x_i + x_n(t - d_i)) + \sum_{i=1}^{n-1} x_n a_i(x_i - x_n) - kx_n^2$$

$$+ \sum_{i=1}^{n-1} \varepsilon_i \int_{-d_i}^{0} e^{\delta_i \zeta_i} \frac{\partial}{\partial \zeta_i} (x_n(t + \zeta_i)^2) d\zeta_i \quad (27)$$

As can be seen, the exponential weighting in the L_2-norm of the delayed state ensures that the same scaled L_2-norm also appears in $\dot{V}$. Rearranging and integrating by parts, we get

$$\dot{V} = \sum_{i=1}^{n-1} \left(-a_i x_i^2 + a_i x_i x_n(t - d_i) + a_i x_i x_n - a_i x_n^2 \right) - kx_n^2$$

$$+ \sum_{i=1}^{n-1} \varepsilon_i e^{\delta_i \zeta_i} x_n(t + \zeta_i)^2 \Big|_{\zeta_i = -d_i}^{\zeta_i = 0}$$

$$- \sum_{i=1}^{n-1} \delta_i \varepsilon_i \int_{-d_i}^{0} e^{\delta_i \zeta_i} x_n(t + \zeta_i)^2 d\zeta_i$$

$$= \sum_{i=1}^{n-1} \left(-a_i x_i^2 + a_i x_i x_n(t - d_i) + a_i x_i x_n - a_i x_n^2 \right) - kx_n^2$$

$$+ \sum_{i=1}^{n-1} \varepsilon_i x_n^2 - \sum_{i=1}^{n-1} \varepsilon_i e^{-\delta_i d_i} x_n(t - d_i)^2$$

$$- \sum_{i=1}^{n-1} \delta_i \varepsilon_i \int_{-d_i}^{0} e^{\delta_i \zeta_i} x_n(t + \zeta_i)^2 d\zeta_i \quad (28)$$

Here we see that the $\varepsilon_i e^{-\delta_i d_i} x_n(t - d_i)^2$-terms provide the opportunity to match the $x_i x_n(t - d_i)$-terms in the following completion of squares:

$$\dot{V} = \sum_{i=1}^{n-1} \left(-a_i x_i^2 + a_i x_i x_n - a_i x_n^2 \right) - (k - \sum_{i=1}^{n-1} \varepsilon_i) x_n^2$$

$$+ \sum_{i=1}^{n-1} \frac{a_i^2}{4\varepsilon_i} e^{\delta_i d_i} x_i^2 - \sum_{i=1}^{n-1} \delta_i \varepsilon_i \int_{-d_i}^{0} e^{\delta_i \zeta_i} x_n(t + \zeta_i)^2 d\zeta_i$$

$$- \sum_{i=1}^{n-1} \varepsilon_i e^{-\delta_i d_i} \left(\frac{a_i^2}{4\varepsilon_i^2} e^{2\delta_i d_i} x_i^2 - \frac{a_i}{\varepsilon_i} e^{\delta_i d_i} x_i x_n(t - d_i) \right)$$

$$- \sum_{i=1}^{n-1} \varepsilon_i e^{-\delta_i d_i} x_n(t - d_i)^2 \quad (29)$$

$$= - \sum_{i=1}^{n-1} a_i \left(\frac{3}{4} - \frac{a_i}{4\varepsilon_i} e^{\delta_i d_i} \right) x_i^2 - \sum_{i=1}^{n-1} a_i \left(\frac{1}{2} x_i - x_n \right)^2$$

$$- (k - \sum_{i=1}^{n-1} \varepsilon_i) x_n^2 - \sum_{i=1}^{n-1} \delta_i \varepsilon_i \int_{-d_i}^{0} e^{\delta_i \zeta_i} x_n(t + \zeta_i)^2 d\zeta_i$$

$$- \sum_{i=1}^{n-1} \varepsilon_i e^{-\delta_i d_i} \left(\frac{a_i}{2\varepsilon_i} e^{\delta_i d_i} x_i - x_n(t - d_i) \right)^2 \quad (30)$$

The second, fourth and fifth terms are clearly non-positive. The first and third terms will be negative if the expressions in parantheses are negative. $\dot{V}$ can thus be rendered strictly negative by any choice of $\delta_i > 0, \varepsilon_i > \frac{1}{3} a_i e^{\delta_i d_i}$ and $k \geq \sum_{i=1}^{n-1} \varepsilon_i$. ∎

Thus, even in the presence of large delays, this proportional controller is able to stabilize the system for constant flows and disturbances. We also note that even if the flow rates in the system are small, the feedback gain k may not necessarily have to be chosen very large if transient performance is not an important issue. This is because the lower bound on k in (25) can be made close to proportional to a_i by choosing $\delta_i << d_i$, and a_i is in turn proportional to $\bar{q}_i$ as well.

B. Variable delay

Having established a design that works for constant flow rates, the next question is whether we can exploit $u_i, i < n$ actively to drive the system state close to the origin. First off, recall that

$$d_i = \frac{\alpha_i}{q_i} = \frac{\alpha_i}{\bar{q}_i + u_i} \quad (31)$$

Thus, if $u_i > 0$ the delay will always be smaller than $\alpha_i / \bar{q}_i$, whereas if we allow the input to become less than zero, the delay will grow essentially without bound. Inspired by the nominal controller in Section II-C, we suggest the heuristic control law

$$u_i = \begin{cases} (x_i - x_n)(b_i + x_i - x_n) & \text{if } x_i > x_n, i = 1, \ldots, n-1 \\ 0 & \text{if } x_i \leq x_n \end{cases} \quad (32)$$

$$u_n = -kx_n \quad (33)$$

as illustrated in Figure 2. Note that u_i is non-negative everywhere, ensuring that $d_i \leq \alpha_i / \bar{q}_i \forall t \geq 0$.

The idea behind this choice is that, whenever $x_i >> x_n$, $u_i >> 0$ and the i'th delay would be small according to (31), implying that the dynamical response of the i'th consumer would be "close" to the delay-free case considered in Section II-C. Conversely, when $x_i \approx x_n$, u_i would approach 0 and the delay would approach the value $\alpha_i / \bar{q}_i$, the case treated in Theorem 1. However, as noted in the introduction, a stability proof has not been derived for this control law yet, so in this paper we will only investigate the 'design' through some simulation examples.
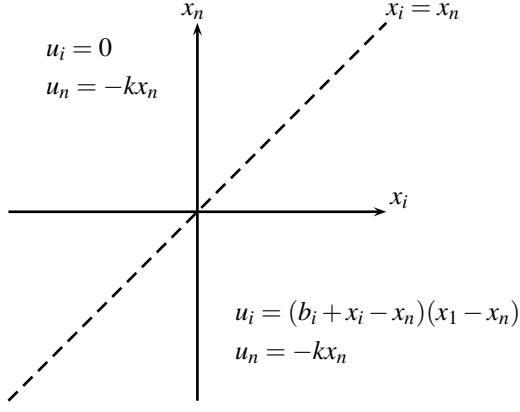
Fig. 2.   Illustration of the control law (32)

Fig. 3.   Simulation with the stabilizing control law from Theorem 2. Top plot: $x_1(-\cdot-)$, $x_2(--)$, x_3 $(\ldots)$, x_4 (—); bottom plot: $u_1(-\cdot-)$, $u_2(--)$, u_3 $(\ldots)$, u_4 (—)

IV. SIMULATION EXAMPLES

In this section, we present some simulation examples to study the feasibility and relative performance of the designs considered above. We simulate a system with two consumers with significantly different dynamics:

$$\dot{x}_1 = 0.5(-x_1(t)+x_4(t-\frac{4.0}{0.5+u_1(t)}))$$
$$+(-5.0-x_1(t)+x_4(t-\frac{4.0}{0.5+u_1(t)}))u_1(t)\ (34)$$

$$\dot{x}_2 = 10(-x_2(t)+x_4(t-\frac{8.0}{10+u_2(t)}))$$
$$+(-10-x_2(t)+x_4(t-\frac{8.0}{10+u_2(t)}))u_2(t)\ \ (35)$$

$$\dot{x}_3 = 6.0(-x_3(t)+x_4(t-\frac{2.0}{6.0+u_3(t)}))$$
$$+(-8.0-x_3(t)+x_4(t-\frac{2.0}{6.0+u_3(t)}))u_3(t)\ (36)$$

$$\dot{x}_4 = 0.5(x_1(t)-x_4(t))+10(x_2(t)-x_4(t))$$
$$+6.0(x_3(t)-x_4(t))$$
$$+(5.0+x_1(t)-x_4(t))u_1(t)$$
$$+(10+x_2(t)-x_4(t))u_2(t)$$
$$+(8.0+x_3(t)-x_4(t))u_3(t)-7.5x_4(t)\ \ \ \ \ (37)$$

The simulations were carried out using Matlab's ODE45 (Dormand-Prince) solver. Note that the feedback gain $k = 7.5$ was chosen to satisfy the condition stated in Theorem 2 for small values of δ_i.

Figure 3 shows a simulation with $x_1(0) = -1, x_2(0) = 5, x(3) = -5, x_4(0) = 2$ and $\phi_1(\tau) = \phi_2(\tau) = \phi_3(\tau) \equiv x_4(0)$. As can be seen, the state trajectories converge to 0 although the effects of the delays mean that it takes a while to drive especially $x_1(t)$ to 0.

Next, Figure 4 shows a simulation with the same initial states, but this time with $u_1(t) - u_3(t)$ computed according to (32). It is clear from the figure that the control action is much more aggressive now, forcing x_1 to 0 much more
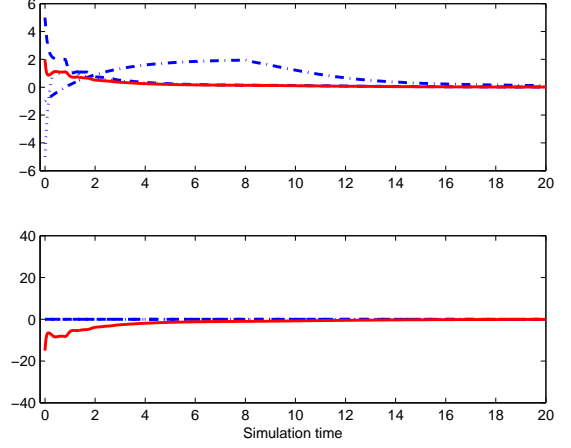
quickly than in Figure 3 at the cost of a large spike in u_1. Figure 5 shows a zoomed-in view of the first ten seconds of the same simulation to illustrate the active use of $u_1(t) - u_3(t)$, while Figure 6 shows how the delays develop during this simulation. Since u_2 and u_3 are mostly inactive, d_2 and d_3 remain almost constant at 0.8 and 0.33, respectively, whereas d_1 varies with u_1.
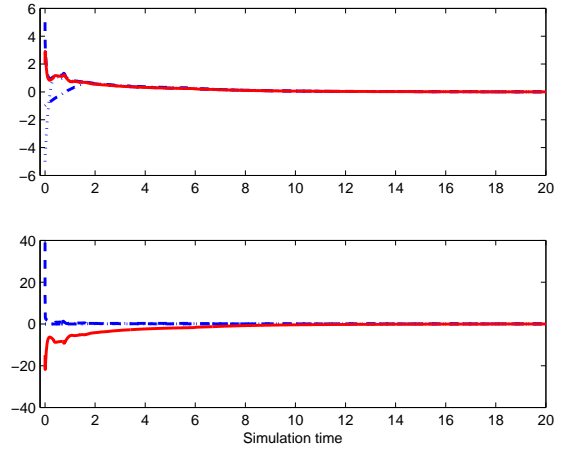

Fig. 4.   Simulation with the control law (32). Top plot: $x_1(-\cdot-)$, $x_2(--)$, x_3 $(\ldots)$, x_4 (—); bottom plot: $u_1(-\cdot-)$, $u_2(--)$, u_3 $(\ldots)$, u_4 (—)

Finally, the large spike in u_1 in Figure 4 might give rise to concerns regarding sensitivity to noise. Figure 7 shows a simulation under similar conditions as above, but with Gaussian measurement noise with a standard deviation of 0.2 added to all three states (the noise is added before the measurements are feed back to the controller). As can be seen, the amplitude of the control signals is not significantly different from the previous simulation, but some filtering of u would probably be desirable to avoid excessive wear of
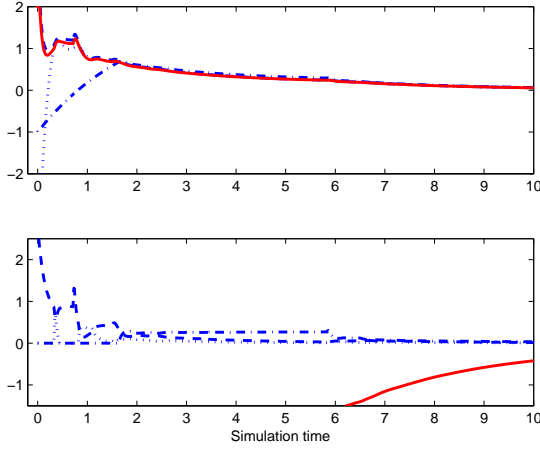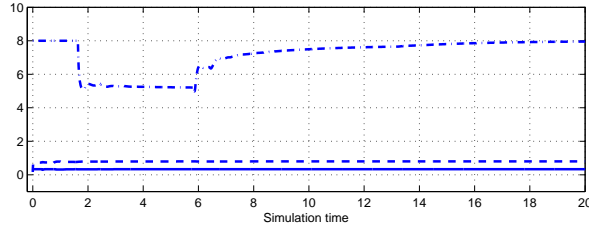
Fig. 5.   Zoom on Figure 4

Fig. 7.   Simulation with the control law (32) and measurement noise. Top plot: $x_1(-\cdot-)$, $x_2(--)$, x_3 (—); bottom plot: $u_1(-\cdot-)$, $u_2(--)$, u_3 (—)

Fig. 6.   Delays during the simulation shown in Figure 4. $d_1(-\cdot-)$, $d_2(--)$, d_3 (—)

the actuators.

V. CONCLUSION

We considered a simplified model of a single-phase cooling system with non-negligible coolant transport between a central cooler and a number of consumers that require heat to be removed in real time. The system was written as a bilinear system with delays in the states that depend on the input signal. We overcame this difficulty by fixing the flow rate and deriving a Lyapunov-stable feedback law for the remaining input. In addition, we presented a preliminary, heuristic design which approaches an optimal nonlinear feedback design in some regions of state space and converges to our Lyapunov-stable design as the state approaches the origin. Although slightly naive, both designs appear to work well in simulations.

One of the interesting points about the proposed control laws is that they limit the delays to pre-computable intervals, although they do not limit the rate of variation of the delays. Note also that the proposed control laws do not actually depend on the delayed states. While this means that the control signals can be computed easily, it also complicates the stability analysis. Future work will thus involve developing analysis tools for input-dependent delays that may vary quickly, but within bounded intervals.
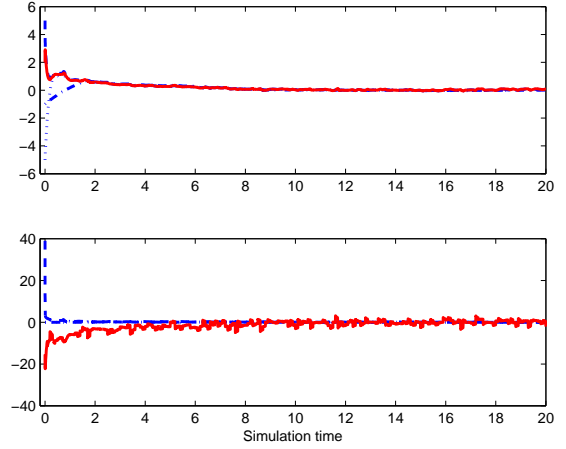
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