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by

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Mean-variance portfolio selection and efficient frontier for defined contribution pension schemes*

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June 22, 2007

Abstract

We solve a mean-variance portfolio selection problem in the accumulation phase of a defined contribution pension scheme. The efficient frontier, which is found for the 2 asset case as well as the $n + 1$ asset case, gives the member the possibility to decide his own risk/reward profile. The mean-variance approach is then compared to other investment strategies adopted in DC pension schemes, namely the target-based approach and the lifestyle strategy. The comparison is done both in a theoretical framework and based on simulations. As a result, it turns out that the target-based approach can be formulated as a mean-variance optimization problem. It is shown that the corresponding mean and variance of the final fund belong to the efficient frontier and also the opposite, that each point on the efficient frontier corresponds to a target-based optimization problem. Furthermore, numerical results indicate that the largely adopted lifestyle strategy seems to be very far from being efficient in the mean-variance setting.

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1 Introduction

Nowadays defined contribution pension schemes have become increasingly important, due to the everywhere observed shift from defined benefit (DB) schemes to defined contribution (DC) schemes. In many countries, reforms for the support and implementation of the second pillar have been undertaken, mainly pressed by the recognized fact that public systems will not be able to provide a sufficient income in the retirement age for most of the pensioners, due to the aging population problem. The passage from PAYG systems to funded systems is therefore a common characteristic of many developed countries and the choice of DC schemes in preference for DB plans is mainly motivated by the transfer of financial risk from the sponsor to the member, and the consequent elimination of insolvency risk in DC schemes. The drawback is evidently a major risk for the pensioner, whose income is no more guaranteed and depends on two main factors: investment performance in the accumulation phase and annuity prices at retirement age (see for instance Knox (1993)). Great attention has been posed in recent academic literature to the management and control of financial risk in the accumulation phase of DC plans. In particular, the problem of optimal investment allocation according to some criterion has been treated by many authors, see, for instance, Boulier, Huang and Taillard (2001), Haberman and Vigna (2002), Deelstra, Grasselli and Koehl (2003), Battocchio and Menoncin (2004) and Cairns, Blake and Dowd (2006). With a different and more empirical approach, other authors have been looking for suitable, "ad-hoc" investment strategies for defined contribution pension schemes, supporting their proposals by means of simulations, see, among others, Knox (1993), Ludvik (1994), Booth and Yakoubov (2000), Blake, Cairns and Dowd (2001) and Arts and Vigna (2003).

In none of the above mentioned papers the optimal (or suitable) investment allocation has been found by solving a mean-variance optimization problem, nor has the efficient frontier of portfolios been found. The main reason lies perhaps in the difficulty inherent in the extension from single-period to multi-period or continuous-time framework. Namely, in solving a stochastic optimal control problem one typically uses the "smoothing" property of the expectation operator, property that is not satisfied by the variance operator. Hence, a multi-period or continuous-time optimization problem with an objective function that contains the variance is not immediate to solve. Recently, Li and Ng (2000) in a discrete-time multi-period framework and Zhou and Li (2000) in a continuous-time model show how to transform the difficult problem into a tractable one: they embed the original problem into a stochastic linear-quadratic control problem, that can then be solved through standard methods. These seminal papers have been followed by a number of extensions, for instance Bielecky, Jin, Pliska and Zhou (2005), who solve a mean-variance portfolio problem in the continuous-time with a constraint against ruin. In the context of actuarial literature other extensions are Chiu and Li (2006), Wang, Xia and Zhang (2007) and Delong and Gerrard (2007). In the context of pension schemes and independently from the current paper, Delong, Gerrard and Haberman (2007) solve a mean-variance optimization problem in the accumulation phase of a defined benefit pension plan. Up to our knowledge, a mean-variance portfolio selection problem in the accumulation phase of a defined contribution pension scheme has not yet been considered and this paper attempts to fill up this gap in the literature.

Following the work by Zhou and Li (2000), we define and solve a mean-variance portfolio selection problem in a defined contribution pension scheme and find the optimal policy and the efficient frontier of feasible portfolios in closed form. This is done first in a standard Black & Scholes financial market with a risky asset and a riskless one, then in a financial market with n risky assets and a riskless one and it is shown that, due to the mutual fund theorem, the solutions are formally identical. We then compare the efficient investment allocation found with some investment strategies adopted in DC plans. The most important result is that the optimal investment strategy found via the so-called "target-based approach" (using a quadratic utility function in the stochastic control problem) is a point of the efficient frontier and, vice versa, each point on the efficient frontier can be found by solving a target-based optimization problem. On the contrary, the widespread lifestyle strategy is not efficient in the mean-variance setting, for it provides a too high standard deviation of the final wealth.

The remainder of the paper is as follows. In sections 2 and 3 we define and solve the mean-variance optimization problem in a financial market with a riskless and a risky asset. In section 4 we generalize the model considering a market in which n risky assets and a riskless one operate. In section 5 we compare the strategy found via the mean-variance approach with other alternative investment strategies for DC pension schemes. Section 6 concludes.

2 The problem

We consider a financial market that consists of two assets, a riskless one, with constant force of interest r, and a risky one, whose price follows a geometric Brownian motion with drift λ and diffusion σ. The constant contribution rate payed in the unit time in the fund is c. The proportion of portfolio invested in the risky asset at time t is denoted by $y(t)$. The fund at time t, $X(t)$, grows according to the following SDE:

$$\begin{array}{l} dX(t) = \{X(t)[y(t)(\lambda - r) + r] + c\}dt + X(t)y(t)\sigma dW(t) \\ X(0) = x_0 \geq 0 \end{array} \tag{1}$$

where $W(t)$ is a standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$, with $\mathcal{F}_t = \sigma\{W(s) : s \leq t\}$.

The amount x_0 is the initial fund paid in the member's account, which can also be null, if the member has just joined the scheme with no transfer value from another fund. The retiree enters the plan at time 0 and contributes for T years, after which he retires and withdraws all the money (or converts it into annuity). The temporal horizon T is supposed to be fixed, e.g. T can be 20, 30 or even more, dependently on the member's age at entry. The two conflicting objectives of maximum expected final wealth together with minimum variance of final wealth are pursued by the investor, who thus seeks to minimize the vector

$$[-E(X(T)), Var(X(T))]$$

Definition 1 *An investment strategy $y(\cdot)$ is said to be* admissible *if $y(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbf{R})$.*

Definition 2 *The mean-variance optimization problem is defined as*

$$\begin{array}{ll} Minimize & (J_1(y(\cdot)), J_2(y(\cdot))) \equiv (-E(X(T)), Var(X(T))) \\ subject\ to & \left\{ \begin{array}{ll} y(\cdot) & admissible \\ X(\cdot), y(\cdot) & satisfy\ (1) \end{array} \right. \end{array} \tag{2}$$

An admissible strategy $\overline{y}(\cdot)$ is called an efficient strategy *if there exists no admissible strategy $y(\cdot)$ such that*

$$J_1(y(\cdot)) \leq J_1(\overline{y(\cdot)}) \quad J_2(y(\cdot)) \leq J_2(\overline{y}(\cdot)) \tag{3}$$

and at least one of the inequalities holds strictly. In this case, the point $(J_1(\overline{y}(\cdot)), J_2(\overline{y}(\cdot))) \in \mathbf{R^2}$ is called an efficient point *and the set of all efficient points is called the* efficient frontier.

Problem (2) is a multiobjective optimization problem (MOP). The traditional approach to solving a MOP problem

$$\min_{x \in \mathcal{X}} (f_1(x), f_2(x)...f_n(x)) \tag{4}$$

consists in its reformulation as a single objective optimization problem of the type (see for instance Ehrgott (2005)):

$$\min_{x \in \mathcal{X}} \sum_{k=1}^{n} \alpha_k f_k(x) \tag{5}$$

with non-negative weights: $\alpha_i \in \mathbf{R}^+ := \{y \in \mathbf{R} : y \geq 0\}$. Instead of solving problem (2) we will therefore address the problem

$$\min_{y(\cdot)}[-\alpha_1 E(X(T)) + \alpha_2 Var(X(T))]$$

or equivalently

$$\min_{y(\cdot)}[-E(X(T)) + \alpha Var(X(T))] \tag{6}$$

where $\alpha > 0$. It is not so straightforward to tackle problem (6) with standard stochastic control techniques. In fact, when applying stochastic control theory one typically uses the "smoothing" property of the expectation operator

$$E[E(\cdot|\mathcal{F}_s)|\mathcal{F}_t] = E(\cdot|\mathcal{F}_t)$$

where $s > t$. However, when the objective criterion contains the variance one encounters the problem that the variance operator does not satisfies the smoothness property, in that

$$Var[Var(\cdot|\mathcal{F}_s)|\mathcal{F}_t] \neq Var(\cdot|\mathcal{F}_t)$$

Zhou and Li (2000) and Li and Ng (2000) show that it is possible to transform the difficult problem (6) into a tractable one. They show that (6) is equivalent to the problem

$$\min_{y(\cdot)} E[\alpha X(T)^2 - \beta X(T)] \tag{7}$$

which is a linear-quadratic (LQ) control problem. In particular, they show that if $\overline{y}(\cdot)$ is a solution of (6), then it is a solution of (7) with

$$\overline{\beta} = 1 + 2\alpha E(\overline{X}(T)) \tag{8}$$

Then, we now want to solve

$$\begin{array}{ll} Minimize & (J(y(\cdot)), \alpha, \beta) \equiv E[\alpha X(T)^2 - \beta X(T)] \\ \text{subject to} & \left\{ \begin{array}{ll} y(\cdot) & \text{admissible} \\ X(\cdot), y(\cdot) & \text{satisfy (1)} \end{array} \right. \end{array} \tag{9}$$

3 Solution of the problem

In solving problem (9) we partially follow the approach presented in Zhou and Li (2000) (see also Yong and Zhou (1999)). Let us set

$$\gamma = \frac{\beta}{2\alpha} \qquad \text{and} \qquad Z(t) = X(t) - \gamma$$

It turns out that our problem is equivalent to solve

$$\min_{y(\cdot)} \mathbf{E}\left[\frac{1}{2}\alpha Z(T)^2\right] = \min_{y(\cdot)} J(y(\cdot); \alpha) \tag{10}$$

where the process $Z(t)$ follows the SDE

$$\begin{array}{l} dZ(t) = \{(Z(t) + \gamma)[y(t)(\lambda - r) + r] + c\}dt + (Z(t) + \gamma)\sigma y(t) dW(t) \\ Z(0) = x_0 - \gamma \end{array} \tag{11}$$

This is a standard stochastic optimal control problem and we follow the dynamic programming approach to solve it. To this end, let us define the value function

$$V(t, z) = \inf_{y(\cdot)} \mathbf{E}_{t,z}\left[\frac{1}{2}\alpha Z(T)^2\right] = \inf_{y(\cdot)} J(y(\cdot); \alpha). \tag{12}$$

Then V satisfies the Hamilton-Jacobi-Bellmann (HJB) equation

$$\begin{array}{l} \inf_{y\in\mathbf{R}}\left\{\frac{\partial V}{\partial t}+[(z+\gamma)(y(\lambda-r)+r)+c]\,\frac{\partial V}{\partial z}+\frac{1}{2}(z+\gamma)^2\sigma^2y^2\frac{\partial^2 V}{\partial z^2}\right\}=0 \\ V(T,z)=\frac{1}{2}\alpha z^2 \end{array} \tag{13}$$

Assuming V to be a convex function of z, then first order conditions lead to the optimal fraction of portfolio to be invested in the risky asset at time t, $\overline{y}(t,z)$:

$$\overline{y}(t,z)=-\frac{(\lambda-r)}{\sigma^2}\frac{1}{(z+\gamma)}\frac{V_z}{V_{zz}} \tag{14}$$

with obvious notation for the partial derivatives. By replacing the optimal control (14) in the HJB equation (13), and setting

$$\delta=\frac{(\lambda-r)}{\sigma}$$

we get the following non-linear PDE for the value function

$$V_t+[(z+\gamma)r+c]V_z-\frac{1}{2}\delta^2\frac{V_z^2}{V_{zz}}=0 \tag{15}$$

As in previous work, we try a solution of the form

$$V(t,z)=A(t)z^2+B(t)z+C(t) \tag{16}$$

Replacing the partial derivatives of V in (15), we get the following system of ODE's:

$$\left\{\begin{array}{lcl} A'(t) & = & (\delta^2-2r)A(t), \\ B'(t) & = & (\delta^2-r)B(t)-2(\gamma r+c)A(t) \\ C'(t) & = & \frac{\delta^2 B(t)^2}{4A(t)}-(\gamma r+c)B(t) \end{array}\right. \tag{17}$$

with boundary conditions

$$A(T)=\frac{1}{2}\alpha \quad B(T)=0 \quad C(T)=0 \tag{18}$$

Solving system (17-18) yields:

$$\left\{\begin{array}{lcl} A(t) & = & \frac{1}{2}\alpha e^{-(\delta^2-2r)(T-t)} \\ B(t) & = & \frac{\alpha(\gamma r+c)}{r}e^{-(\delta^2-2r)(T-t)}[1-e^{-r(T-t)}] \\ C(t) & = & \int_T^t[\frac{\delta^2B(s)^2}{4A(s)}-(\gamma r+c)B(s)]ds \end{array}\right. \tag{19}$$

We notice that the assumption of convexity of V turns out to be true, as

$$V_{zz}=2A(t)>0$$

since $\alpha>0$. Replacing partial derivatives of V in (14) and replacing $(z+\gamma)$ with x yields

$$\overline{y}(t,x)=-\frac{\lambda-r}{\sigma^2 x}\left[x-\gamma e^{-r(T-t)}+\frac{c}{r}(1-e^{-r(T-t)})\right] \tag{20}$$

The evolution of the fund under optimal control $\overline{X}(t)$ can be easily obtained:

$$\begin{array}{l} d\overline{X}(t)=\left[(r-\delta^2)\overline{X}(t)+e^{-r(T-t)}(\delta^2\gamma+\frac{\delta^2c}{r})+(c-\frac{\delta^2c}{r})\right]dt+ \\ +\left[-\delta\overline{X}(t)+e^{-r(T-t)}(\delta\gamma+\frac{\delta c}{r})-\frac{\delta c}{r}\right]dW(t) \end{array} \tag{21}$$

By application of Ito's lemma to (21), we obtain the SDE that governs the evolution of $\overline{X}^2(t)$:

$$\begin{array}{l} d\overline{X}^2(t)=[(2r-\delta^2)\overline{X}^2(t)+2c\overline{X}(t)+\delta^2((\gamma+\frac{c}{r})e^{-r(T-t)}-\frac{c}{r})^2]dt+ \\ -2\delta\{\overline{X}^2(t)-[(\gamma+\frac{c}{r})e^{-r(T-t)}-\frac{c}{r}]\overline{X}(t)+\frac{c}{r}\}dW(t) \end{array} \tag{22}$$

If we take expectations on both sides of (21) and (22), we find that the expected value of the optimal fund and the expected value of its square follow the linear ODE's:

$$\begin{aligned} &dE(\overline{X}(t)) = [(r-\delta^2)E(\overline{X}(t)) + e^{-r(T-t)}\delta^2(\gamma + \tfrac{c}{r}) + (c - \tfrac{\delta^2 c}{r}]dt \\ &E(\overline{X}(0)) = x_0 \end{aligned} \tag{23}$$

$$\begin{aligned} &dE(\overline{X}^2(t)) = [(2r-\delta^2)E(\overline{X}^2(t)) + 2cE(\overline{X}(t)) + \delta^2\left((\gamma + \tfrac{c}{r})e^{-r(T-t)} - \tfrac{c}{r}\right)^2]dt \\ &E(\overline{X}^2(0)) = x_0^2 \end{aligned} \tag{24}$$

By solving the ODE's we find that the expected value of the fund under optimal control at time t is

$$E(\overline{X}(t)) = \left(x_0 + \frac{c}{r}\right)e^{-(\delta^2-r)t} + \left(\gamma + \frac{c}{r}\right)e^{-r(T-t)} - \left(\gamma + \frac{c}{r}\right)e^{-r(T-t)-\delta^2 t} - \frac{c}{r} \tag{25}$$

and the expected value of the square of the fund under optimal control at time t is:

$$\begin{aligned} &E(\overline{X}^2(t)) = \left(x_0 + \tfrac{c}{r}\right)^2 e^{-(\delta^2-2r)t} - \left(\gamma + \tfrac{c}{r}\right)^2 e^{-2r(T-t)-\delta^2 t} - \tfrac{2c}{r}\left(\gamma + \tfrac{c}{r}\right)e^{-r(T-t)} \\ &+\tfrac{2c}{r}\left(\gamma + \tfrac{c}{r}\right)e^{-r(T-t)-\delta^2 t} - \tfrac{2c}{r}\left(x_0 + \tfrac{c}{r}\right)e^{-(\delta^2-r)t} + \left(\gamma + \tfrac{c}{r}\right)^2 e^{-2r(T-t)} + \tfrac{c^2}{r^2} \end{aligned} \tag{26}$$

At terminal time T we have:

$$E(\overline{X}(T)) = \left(x_0 + \frac{c}{r}\right)e^{-(\delta^2-r)T} + \gamma\left(1 - e^{-\delta^2 T}\right) - \frac{c}{r}e^{-\delta^2 T} \tag{27}$$

and

$$E(\overline{X}^2(T)) = \left(x_0 + \frac{c}{r}\right)^2 e^{-(\delta^2-2r)T} + \gamma^2\left(1 - e^{-\delta^2 T}\right) - \frac{2c}{r}\left(x_0 + \frac{c}{r}\right)e^{-(\delta^2-r)T} + \frac{c^2}{r^2}e^{-\delta^2 T} \tag{28}$$

Observe, from (8), (27) and the definition of γ, that γ is a decreasing function of α:

$$\gamma = \frac{e^{\delta^2 T}}{2\alpha} + x_0 e^{rT} + \frac{c}{r}(e^{rT} - 1) \tag{29}$$

The expected optimal final fund can be rewritten in terms of α:

$$E(\overline{X}(T)) = x_0 e^{rT} + c\frac{e^{rT}-1}{r} + \frac{e^{\delta^2 T}-1}{2\alpha} \tag{30}$$

It is easy to see that the expected optimal final fund is the sum of the fund that one would get investing the whole portfolio always in the riskless asset plus a term, $\frac{e^{\delta^2 T}-1}{2\alpha}$ that depends both on the goodness of the risky asset w.r.t. the riskless one and on the weight given to the minimization of the variance. Thus, the higher the Sharpe ratio of the risky asset, δ, the higher the expected optimal final wealth, everything else being equal; the higher the importance given to the minimization of the variance of the final wealth, α, the lower its mean. These are intuitive results. One can also write the optimal proportion to be invested in the risky asset in terms of α:

$$\overline{y}(t,x) = -\frac{\lambda - r}{\sigma^2 x}\left[x - \left(x_0 e^{rt} + \frac{c}{r}(e^{rt} - 1)\right) - \frac{e^{-r(T-t)+\delta^2 T}}{2\alpha}\right] \tag{31}$$

The amount $x\overline{y}(t,x)$ invested in the risky asset at time t is proportional to the difference between the fund x at time t and the fund available investing always only in the riskless asset, minus a term that depends on δ^2, α and the time to retirement. Again, the higher the weight given to the minimization of the variance, the lower the amount invested in the risky asset, and vice versa, which is an obvious result. It is clear that a necessary and sufficient condition for the fund to be invested at any time t in the riskless asset is $\alpha = +\infty$: the (extreme) strategy of investing the whole portfolio in the riskless asset is optimal if and only if the weight given to the minimization of the variance is infinite, or, in other words, if and only if one allocates zero importance to the maximization of the expected final wealth.

Using (30) and (31) one can express the optimal investment strategy in terms of the expected final wealth in the following way:

$$\overline{y}(t,x) = -\frac{\lambda - r}{\sigma^2 x}\left[x - \left(E[\overline{X}(T)]e^{-r(T-t)} - \frac{c}{r}(1-e^{-r(T-t)})\right) - \frac{e^{-r(T-t)}}{2\alpha}\right] \tag{32}$$

Now the amount $x\overline{y}(t,x)$ invested in the risky asset at time t is proportional to the difference between the fund x at time t and the amount that would be sufficient to guarantee the achievement of the target by adoption of the riskless strategy until retirement, minus a term that depends on α and the time to retirement.

In realistic situations, when the minimization of the variance plays a role in the investor's decisions, expressions (27) and (28) allow one to choose his own profile risk/reward. In fact, as in classical mean-variance analysis, it is possible to express the variance - or the standard deviation - of the final fund in terms of its mean. Once one has chosen a certain couple mean-variance it is then straightforward to find the value of α via (30) and the optimal quote of portfolio to be invested in the risky asset at any time t via (31). The subjective choice of the profile risk/reward becomes easier if one is given the efficient frontier of feasible portfolios.

3.1 The efficient frontier

To find the efficient frontier of portfolios let us introduce the following notation:

$$\begin{array}{ll} y_0 \equiv x_0 + c & \theta \equiv 1 - e^{-\delta^2 T} \\ \rho \equiv e^{-(\delta^2 - r)T} & \phi \equiv e^{-(\delta^2 - 2r)T} \end{array} \tag{33}$$

From (27) and (28), we have:

$$E(\overline{X}(T)) = y_0\rho - \frac{c}{r}(1-\theta) + \gamma\theta \tag{34}$$

and

$$E(\overline{X}^2(T)) = y_0^2\phi - 2\frac{c}{r}y_0\rho + \frac{c^2}{r^2}(1-\theta) + \gamma^2\theta \tag{35}$$

Therefore

$$\begin{array}{l} Var(\overline{X}(T)) = E(\overline{X}^2(T)) - E(\overline{X}(T))^2 = y_0^2\phi - 2\frac{c}{r}y_0\rho + \frac{c^2}{r^2}(1-\theta) + \gamma^2\theta - \\ -y_0^2\rho^2 - \frac{c^2}{r^2}(1-\theta)^2 - \gamma^2\theta^2 + 2y_0\rho\frac{c}{r}(1-\theta) - 2y_0\rho\gamma\theta + 2\frac{c}{r}(1-\theta)\gamma\theta \end{array} \tag{36}$$

After a few passages and noticing that

$$\phi - \rho^2 = \phi\theta$$

we have

$$Var(\overline{X}(T)) = y_0^2\theta\phi + \theta(1-\theta)\left(\gamma + \frac{c}{r}\right)^2 - 2y_0\rho\theta\left(\gamma + \frac{c}{r}\right)$$

From (34), we have

$$\theta\left(\gamma + \frac{c}{r}\right) = E(\overline{X}(T)) - y_0\rho + \frac{c}{r}$$

Therefore

$$\begin{array}{rcl} Var(\overline{X}(T)) & = & y_0^2\theta\phi + \theta(1-\theta)\frac{\left(E(\overline{X}(T)) - y_0\rho + \frac{c}{r}\right)^2}{\theta^2} - 2y_0\rho\left(E(\overline{X}(T)) - y_0\rho + \frac{c}{r}\right) \\ & = & \frac{1-\theta}{\theta}\left[\frac{y_0^2\phi\theta^2}{1-\theta} + \left(E(\overline{X}(T)) - y_0\rho + \frac{c}{r}\right)^2 - \frac{2y_0\rho\theta}{1-\theta}\left(E(\overline{X}(T)) - y_0\rho + \frac{c}{r}\right)\right] \\ & = & \frac{1-\theta}{\theta}\left[\frac{\phi\theta^2 + \rho^2 + \rho^2\theta}{1-\theta}y_0^2 + 2E(\overline{X}(T))\frac{c}{r} + \frac{c^2}{r^2} + E(\overline{X}(T))^2 - 2y_0\frac{\rho}{1-\theta}\left(E(\overline{X}(T)) + \frac{c}{r}\right)\right] \end{array} \tag{37}$$

Now notice that

$$\frac{\phi\theta^2 + \rho^2 + \rho^2\theta}{1-\theta} = e^{2rT} \qquad \text{and} \qquad \frac{\rho}{1-\theta} = e^{rT}$$

So that

$$\begin{aligned} Var(\overline{X}(T)) &= \tfrac{1-\theta}{\theta}\left[y_0^2 e^{2rT} + \left(E(\overline{X}(T)) + \tfrac{c}{r}\right)^2 - 2y_0 e^{rT}\left(E(\overline{X}(T)) + \tfrac{c}{r}\right)\right] \\ &= \tfrac{1-\theta}{\theta}\left[\left(E(\overline{X}(T)) + \tfrac{c}{r}\right) - y_0 e^{rT}\right]^2 \\ &= \tfrac{e^{-\delta^2 T}}{1-e^{-\delta^2 T}}\left[E(\overline{X}(T)) - \left(x_0 e^{rT} + c\tfrac{e^{rT}-1}{r}\right)\right]^2 \end{aligned} \tag{38}$$

where in the last equality we have used (33). It is possible to express the variance of the final fund in terms of α and δ. In fact, applying (30) in the expression above we have:

$$Var(\overline{X}(T)) = \frac{e^{-\delta^2 T}}{1-e^{-\delta^2 T}}\left(\frac{e^{\delta^2 T}-1}{2\alpha}\right)^2 = \frac{e^{\delta^2 T}-1}{4\alpha^2} \tag{39}$$

The variance is increasing if the Sharpe ratio increases, which is an expected result: in this case the investment in the risky asset is heavier, leading to higher variance. Obviously, the higher the importance α given to minimize the variance, the lower the variance of the final fund, which is null if and only if $\alpha = +\infty$: in this case, the portfolio is entirely invested in the riskfree asset and

$$\overline{X}(T) = E(\overline{X}(T)) = x_0 e^{rT} + c\frac{e^{rT}-1}{r}$$

From (38), we get the expected final fund as a function of the standard deviation:

$$E(\overline{X}(T)) = x_0 e^{rT} + c\frac{e^{rT}-1}{r} + \sqrt{\frac{1-e^{-\delta^2 T}}{e^{-\delta^2 T}}}\sigma(\overline{X}(T)) \tag{40}$$

The efficient frontier in the mean-standard deviation diagram is a straight line with slope

$$\sqrt{\frac{1-e^{-\delta^2 T}}{e^{-\delta^2 T}}} = \sqrt{e^{\delta^2 T}-1}$$

which is called "price of risk" (see Luenberger (1998)): it indicates by how much the mean of the final fund increases if the volatility of the final fund increases by one unit.

4 The $n+1$ asset case

The model presented in the previous sections can be generalized to consider a market in which n risky assets and one riskfree are available. Let us assume, as before, that the riskfree asset has force of interest equal to r. The price of the i^{th} asset follows the SDE

$$dS_i(t) = \lambda_i S_i(t)dt + S_i(t)\sum_{j=1}^{n}\sigma_{ij}dW_j(t) \tag{41}$$

where the drift λ_i is a constant, $\mathbf{W}(t) \equiv (W_1(t), W_2(t), ..., W_n(t))^T$ is a standard $\{\mathcal{F}_t\}_{t\geq 0}-$adapted n-dimensional Brownian motion defined on the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$, $W_i(t)$ and $W_j(t)$ are mutually independent for $i \neq j$ and the volatility matrix $\Sigma = \{\sigma_{ij}\}_{i,j=1}^n$ is nonsingular and supposed to be constant over time.

The fund $X(t)$ is invested in the $n+1$ assets, denote with $y_i(t)$ the quote of portfolio invested in the i^{th} risky asset at time t. The remaining part $1-\sum_{i=1}^n y_i(t)$ is invested in the riskless asset. The fund grows according to the SDE:

$$\begin{aligned} &dX(t) = \{X(t)[\mathbf{B}\mathbf{y}(t) + r] + c\}dt + X(t)\textstyle\sum_{j=1}^n \sigma_j \mathbf{y}(t)dW_j(t) \\ &X(0) = x_0 \geq 0 \end{aligned} \tag{42}$$

where

$$\mathbf{B} = (\lambda_1 - r, \lambda_2 - r, ..., \lambda_n - r)$$

collects the extra expected returns of the risky assets w.r.t. the riskless one,

$$\sigma_j = (\sigma_{1j}, \sigma_{2j}, ..., \sigma_{nj})$$

is the j^{th} row of the matrix Σ^T (the transpose of the volatility matrix) and

$$\mathbf{y}(t) = (y_1(t), y_2(t), ..., y_n(t))^T$$

reports the portfolio composition at time t.

Definition 3 *An investment strategy* $\mathbf{y}(\cdot)$ *is said to be* admissible *if* $\mathbf{y}(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbf{R^n})$.

The problem we want to solve is

$$\begin{array}{ll} Minimize & (J(\mathbf{y}(\cdot)), \alpha, \beta) \equiv E[\alpha X(T)^2 - \beta X(T)] \\ \text{subject to} & \left\{ \begin{array}{l} \mathbf{y}(\cdot) \quad \text{admissible} \\ X(\cdot), \mathbf{y}(\cdot) \quad \text{satisfy (42)} \end{array} \right. \end{array} \tag{43}$$

Problem (43) is a stochastic linear quadratic (SLQ) problem, which can be solved in different ways. One way is to use the so-called "mutual fund theorem" (see Karatzas, Lehoczky, Sethi and Shreve (1986)), whereby the $n+1$ asset problem is transformed in a 2 asset problem. Another possible way is to consider the general theory of SLQ problems, which is thoroughly treated in Yong and Zhou (1999). We follow here the first method and report in the appendix how to find the solution with the second method.

The idea is to construct a "corresponding" 2-asset case and then use, in a simple way, its solution (which is easily found via section 3) to formulate the solution of the $n+1$ asset case. In order to do this, let us introduce some quantities:

$$\begin{array}{rcl} \Delta^2 : & = & \mathbf{B}(\Sigma\Sigma^T)^{-1}\mathbf{B}^T \\ \lambda : & = & r + \Delta^2 \\ \sigma^2 : & = & \Delta^2 \end{array} \tag{44}$$

It can be proved that with this choice of λ and σ^2, problem (43) is equivalent to problem (9) with wealth equation (1) where the Brownian motion $W(t)$ is replaced by the following linear combination of the Brownian motions $W_j(t)$:

$$\widehat{W}(t) = \frac{\sigma}{\lambda - r}\mathbf{B}(\Sigma\Sigma^T)^{-1}\Sigma\mathbf{W}(t) \tag{45}$$

Then, it can be shown that the optimal investment strategy at time t, $\overline{\mathbf{y}}(t, x)$ is given by:

$$\overline{\mathbf{y}}(t, x) = \overline{y}(t, x)\frac{\sigma^2}{\lambda - r}(\Sigma\Sigma^T)^{-1}\mathbf{B}^T \tag{46}$$

where $\overline{y}(t, x)$ is given by (20). It turns out that the optimal investment strategy is:

$$\overline{\mathbf{y}}(t, x) = -\frac{1}{x}\left[x - \gamma e^{-r(T-t)} + \frac{c}{r}(1 - e^{-r(T-t)})\right](\Sigma\Sigma^T)^{-1}\mathbf{B}^T \tag{47}$$

Therefore the optimal fund evolves according to

$$\left\{ \begin{array}{rcl} d\overline{X}(t) & = & \left[(r - \Delta^2)\overline{X}(t) + e^{-r(T-t)}\Delta^2(\gamma + \frac{c}{r}) + c - \Delta^2\frac{c}{r}\right]dt + \\ & & + \left[\gamma e^{-r(T-t)} - \overline{X}(t) - \frac{c}{r}(1 - e^{-r(T-t)})\right]\mathbf{B}(\Sigma\Sigma^T)^{-1}\Sigma d\mathbf{W}(t) \\ \overline{X}(0) & = & x_0 \end{array} \right. \tag{48}$$

We now apply the multidimensional Ito's lemma to (48) to find the evolution of $\overline{X}^2(t)$:

$$\begin{array}{rcl} d\overline{X}^2(t) & = & \{2\overline{X}(t)\left[(r - \Delta^2)\overline{X}(t) + e^{-r(T-t)}\Delta^2(\gamma + \frac{c}{r}) + c - \Delta^2\frac{c}{r}\right] + \\ & & + \left[\gamma e^{-r(T-t)} - \overline{X}(t) - \frac{c}{r}(1 - e^{-r(T-t)})\right]^2 Tr\left[(\mathbf{B}(\Sigma\Sigma^T)^{-1}\Sigma)(\mathbf{B}(\Sigma\Sigma^T)^{-1}\Sigma)^T\right]\}dt \\ & & +\{...\}d\mathbf{W}(t) \end{array} \tag{49}$$

where $Tr(H)$ denotes the trace of the square H. Thus, after some passages we see that $\overline{X}^2(t)$ evolves according to the SDE:

$$\begin{cases} d\overline{X}^2(t) &= [(2r-\Delta^2)\overline{X}^2(t)+2c\overline{X}(t)+\Delta^2\left((\gamma+\frac{c}{r})e^{-r(T-t)}-\frac{c}{r}\right)^2]dt+\{...\}d\mathbf{W}(t) \\ \overline{X}^2(0) &= x_0^2 \end{cases} \tag{50}$$

It is easy to see that the drifts of the stochastic differential equations governing the evolution of the optimal fund and its square (48-50) are essentially the same of those of the 2 asset case (21-22), with $\delta^2=\Delta^2$. Due to the mutual fund theorem, this result is expected: namely Δ^2 is the corresponding of δ^2 in the transformed 2-asset case (see (44)). Thus, the differential equations for $E(\overline{X}(t))$ and $E(\overline{X}^2(t))$ are the same, and proceeding in the same way as before we find that the efficient frontier for the $n+1$ asset case in the mean-variance diagram is the curve

$$Var(\overline{X}(T))=\frac{e^{-\Delta^2T}}{1-e^{-\Delta^2T}}\left[E(\overline{X}(T))-\left(x_0e^{rT}+c\frac{e^{rT}-1}{r}\right)\right]^2 \tag{51}$$

and in the mean-standard deviation diagram is the line

$$E(\overline{X}(T))=x_0e^{rT}+c\frac{e^{rT}-1}{r}+\sqrt{\frac{1-e^{-\Delta^2T}}{e^{-\Delta^2T}}}\sigma(\overline{X}(T)) \tag{52}$$

where now the price of risk is

$$\sqrt{\frac{1-e^{-\Delta^2T}}{e^{-\Delta^2T}}}=\sqrt{e^{\Delta^2T}-1}$$

We notice that these results generalize in a very natural way those found by Zhou and Li (2000), and coincide with them if the contribution paid in the fund is null.

5 Comparison with other strategies

It may be of interest for both academic and professional purposes to compare the mean-variance optimal portfolio here proposed with other investment strategies typically adopted in defined contribution pension schemes. We compare our optimal portfolio with two alternatives: the widespread lifestyle strategy and the optimal investment strategy found via the mean-square error approach, or target-based approach. The lifestyle strategy (see for instance Booth and Yakoubov (2000) or Cairns et al. (2006)) consists in investing the whole portfolio in equities at the beginning of the membership and switch it gradually into bonds and cash in the last years before retirement. The mean-square error approach is a target-based approach (see, for instance, Haberman and Vigna (2002) or Gerrard, Haberman and Vigna (2004) in the accumulation and in the decumulation phase of a DC scheme, respectively): the general form of this approach consists in setting pre-determined periodic targets - in the discrete time - or a target function - in the continuous time - for the size of the fund and finding the optimal investment strategy that minimizes the sum of the squares of the deviations of the fund from the target. In order to facilitate comparisons we base our work on a 2-asset world.

5.1 Comparison with the target-based approach

In Gerrard et al. (2004) the following problem is considered: for a given target function $F(t)$ choose the optimal investment strategy that minimizes

$$E\left[\int_0^T e^{-\varrho t}\varepsilon_1(X(t)-F(t))^2dt+\varepsilon_2e^{-\varrho T}(X(T)-F(T))^2\right]$$

Since the mean-variance approach is only concerned of the values at time T, we choose here for the comparisons $\varepsilon_1=0$ and $\varepsilon_2=1$. Furthermore, let us notice that for the problem to be interesting the final target $F(T)$ should be chosen big enough, i.e. such that

$$F(T)>x_0e^{rT}+\frac{c}{r}(e^{rT}-1) \tag{53}$$

We observe that when $\varepsilon_1 = 0$ the discount rate ϱ does not influence the optimal strategy, and therefore we choose $\varrho = r$ for simplicity. We can see from Gerrard et al. (2004) that the optimal investment strategy for the target-based (T-B) approach is given on the following form[1]

$$y_{tb}(t,x) = -\frac{\lambda - r}{\sigma^2 x}(x - G(t)) \tag{54}$$

where

$$G(t) = \frac{-B(t)}{2A(t)}.$$

The functions $A(t), B(t)$ solve the following ODE's

$$\begin{cases} A'(t) = (\delta^2 - r)A(t) \\ B'(t) = \delta^2 B(t) - 2cA(t) \end{cases}$$

with boundary conditions

$$A(T) = 1, \;\; B(T) = -2F(T)$$

These have solution

$$\begin{cases} A(t) = e^{-(\delta^2 - r)(T-t)} \\ B(t) = -2(F(T) + \frac{c}{r})e^{-\delta^2(T-t)} + \frac{2c}{r}e^{-(\delta^2 - r)(T-t)} \end{cases} \tag{55}$$

so that

$$G(t) = F(T)e^{-r(T-t)} - \frac{c}{r}(1 - e^{-r(T-t)}) \tag{56}$$

and

$$y_{tb}(t,x) = -\frac{\lambda - r}{\sigma^2 x}\left[x - \left(F(T)e^{-r(T-t)} - \frac{c}{r}(1 - e^{-r(T-t)})\right)\right] \tag{57}$$

Let us notice that the function $G(t)$ represents a sort of target level for the fund at time t: should the fund $X(t)$ reach $G(t)$ at some point of time $t < T$, then the final target $F(T)$ could be easily achieved by adoption of the riskless strategy until retirement. However, as will be shown in the next paragraph, the achievement of $G(t)$ and therefore the sure achievement of the target, is prevented under optimal control by the construction of the solution.

5.1.1 Aiming for the target

For consistent comparisons, in the mean-variance (M-V) approach we choose α such that $E(\overline{X}(T)) = F(T)$ and we call $\overline{\alpha}$ this particular choice of α. Comparing (32) with (57) we see that the optimal investment allocations in the risky asset at time t are very similar. However, the $\overline{y}(t,x)$ in the mean-variance approach is riskier than $y_{tb}(t,x)$ due to the additional term

$$\frac{e^{-r(T-t)}}{2\overline{\alpha}} \tag{58}$$

which clearly will add a higher variance to the final wealth in the M-V approach than in the T-B approach, with equality only if $\overline{\alpha} = +\infty$. It is, however, important to notice here that $\overline{\alpha}$ is no longer a free parameter. This higher variance should apparently violate the concept of the M-V approach, but one should notice that the expected final wealth in the M-V approach is forced to be equal to the target value $F(T)$ which is not the case for the T-B approach. To calculate the expected value of the final fund in the T-B approach we let $X^*(t)$ denote the optimal wealth function for this case. Then in Gerrard et al. (2004) it can be seen that $X^*(t)$ satisfies the following SDE:

$$dX^*(t) = [rG(t) + c + (\delta^2 - r)(G(t) - X^*(t))]dt + \delta(G(t) - X^*(t))dW(t). \tag{59}$$

[1]Notice that Gerrard et al. (2004) consider the decumulation phase of a DC scheme. The difference in the wealth equation is that in that case there are periodic withdrawals from the fund whereas here we have periodic inflows into the fund. Formally the equations are identical if one sets $-b_0 = c$.

As in previous work, let us define the process

$$U(t) = G(t) - X^*(t)$$

Then

$$dU(t) = G'(t)dt - dX^*(t) = (r - \delta^2)U(t)dt - \delta U(t)dW(t) \tag{60}$$

where in the last equality we have applied (56) and (59). We can see that the process $U(t)$ follows a geometric Brownian motion and is given by:

$$U(t) = U(0)e^{(r-\frac{3}{2}\delta^2)t-\delta W(t)} \tag{61}$$

Noting that

$$G(T) = F(T)$$

one has

$$U(T) = F(T) - X^*(T)$$

Thus

$$\begin{aligned} E(X^*(T)) &= F(T) - E(U(T)) = F(T) - (x_0 - G(0))e^{-(\delta^2 - r)T} \\ &= e^{-\delta^2 T}[x_0 e^{rT} + \tfrac{c}{r}(e^{rT} - 1)] + (1 - e^{-\delta^2 T})F(T) \end{aligned} \tag{62}$$

The expected final fund turns out to be a weighted average of the target and of the fund that one would have by investing fully in the riskless asset. Furthermore, the difference

$$E(X^*(T)) - E(\overline{X}(T)) = -e^{-\delta^2 T}(F(T) - x_0 e^{rT} - \frac{c}{r}(e^{rT} - 1)) = \frac{e^{-\delta^2 T} - 1}{2\overline{\alpha}} < 0 \tag{63}$$

where the last equality follows by rearranging terms in (30). This means that the T-B strategy is a 'biased' strategy in the sense that the expected final wealth is not equal to the target, but the variance of the final wealth is smaller than in the unbiased M-V approach.

Furthermore, it is straightforward to see that in the T-B approach the final target cannot be reached. In fact, from (61), one can see that $U(T) > 0$ if $U(0) > 0$. Let us notice that

$$U(0) = G(0) - x_0 = F(T)e^{-rT} - \frac{c}{r}(1 - e^{-rT}) - x_0$$

Notice that $U(0) > 0$ is the condition (53) that makes the problem interesting. Therefore, the final fund is always lower than the target. This result is not new. A similar result was already found by Gerrard et al. (2004) and by Gerrard, Haberman and Vigna (2006) in the decumulation phase of a DC scheme: with a different formulation of the optimization problem and including a running cost, in both works they find that there is a "natural" time-varying target that acts as a sort of safety level for the needs of the pensioner and that cannot be reached under optimal control. Previously, in a different context, a similar result was found by Browne (1997): in a problem where the aim is to maximize the probability of hitting a certain upper boundary before ruin, when optimal control is applied the safety level (the minimum level of fund that guarantees fixed consumption by investing the whole portfolio in the riskless asset) can never be reached. From (54) we can see that another direct consequence of the positivity of $U(t)$ is the fact that the amount invested in the risky asset under optimal control is always positive and this is obviously the case also for the M-V approach.

5.1.2 Aiming for the same expected final wealth

The obvious question that arises is what happens if one chooses the mean of the final fund in the M-V approach equal to the expected final fund in the T-B approach, namely if one chooses $E(\overline{X}(T)) = E(X^*(T))$. The - perhaps unsurprising - answer is that in this case the two optimal investment strategies coincide. In fact, from (62) we have

$$e^{\delta^2 T}E(X^*(T)) = x_0 e^{rT} + \frac{c}{r}(e^{rT} - 1) + F(T)(e^{\delta^2 T} - 1)$$

Then, applying (30) with $E(\overline{X}(T)) = E(X^*(T))$ yields

$$e^{\delta^2 T} E(X^*(T)) = E(X^*(T)) - \frac{e^{\delta^2 T} - 1}{2\alpha^*} + F(T)(e^{\delta^2 T} - 1)$$

where α^* is the choice of α that makes $E(\overline{X}(T)) = E(X^*(T))$ in the M-V approach. Collecting terms we have

$$E(X^*(T)) = F(T) - \frac{1}{2\alpha^*} \tag{64}$$

From (29) and (30) we notice that in this case, interestingly,

$$\gamma = F(T) \tag{65}$$

Inserting (64) into (32) yields

$$\begin{aligned}
\overline{y}(t,x) &= -\frac{\lambda - r}{\sigma^2 x}\left\{x - \left[\left(F(T) - \frac{1}{2\alpha^*}\right)e^{-r(T-t)} - \frac{c}{r}(1 - e^{-r(T-t)}) + \frac{e^{-r(T-t)}}{2\alpha^*}\right]\right\} \\
&= -\frac{\lambda - r}{\sigma^2 x}\left\{x - \left[F(T)e^{-r(T-t)} - \frac{c}{r}(1 - e^{-r(T-t)})\right]\right\} \\
&= -\frac{\lambda - r}{\sigma^2 x}(x - G(t)) = y_{tb}(t,x)
\end{aligned} \tag{66}$$

This result shows that the target-based approach is a particular case of the mean-variance approach by choosing a specific value of α, namely α^*. In particular, applying (64), we have

$$\alpha^* = \frac{1}{2(F(T) - E(X^*(T)))} \tag{67}$$

This enables us to compare the two approaches in a more detailed way, as they belong to the same family. Using (63) and (64), we find that the relationship between α^* and $\overline{\alpha}$ is

$$\alpha^* = \frac{\overline{\alpha}}{1 - e^{-\delta^2 T}} \tag{68}$$

from which we can see that $\alpha^* > \overline{\alpha}$. This gives us the opportunity also to compare the variance of the final fund in the T-B approach, $Var(X^*(T))$, with the variance in the M-V approach (with expected wealth equal to the target), $Var(\overline{X}(T))$. In fact we have:

$$Var(X^*(T)) = \frac{e^{\delta^2 T} - 1}{4\alpha^{*2}} = \frac{(e^{\delta^T} - 1)(1 - e^{-\delta^2 T})^2}{4\overline{\alpha}^2} = (1 - e^{-\delta^2 T})^2 Var(\overline{X}(T)) \tag{69}$$

where we have applied (39) and (68). From (69) it is easy to see that the variance of the final fund in the T-B approach is smaller than that of the M-V approach (when the expected wealth equals the target), which is a result already mentioned earlier and here quantified. Furthermore, the fact that

$$E(X^*(T)) < E(\overline{X}(T)) \quad \text{and} \quad Var(X^*(T)) < Var(\overline{X}(T))$$

is obvious, since the two points

$$MV_{X^*} :\equiv (\sigma(X^*(T)), E(X^*(T))) \quad \text{and} \quad MV_{\overline{X}} :\equiv (\sigma(\overline{X}(T)), E(\overline{X}(T)))$$

belong to the same efficient frontier of portfolios.

We have just proved that the target-based approach is a particular case of the mean-variance approach. It is straightforward to see that also the reverse statement is valid: each point of the efficient frontier can be found by solving a target-based optimization problem. In fact, noticing (39), the efficient frontier of portfolios (40) can be written in the following form

$$E(\overline{X}(T)) = x_0 e^{rT} + \frac{c}{r}(e^{rT} - 1) + \frac{e^{\delta^2 T}(1 - e^{-\delta^2 T})}{2\alpha}$$

Thus, as expected, there is a one-to-one correspondence between points on the efficient frontier and values of the parameter α. This allows us to say that chosen a point $(E(\overline{X}(T)), \sigma(\overline{X}(T)))$ on the efficient frontier, we can find the corresponding $\overline{\alpha}$ which in turn defines the target

$$\overline{F}(T) = E(\overline{X}(T)) + \frac{1}{2\overline{\alpha}}$$

It is then obvious that the point $(E(\overline{X}(T)), \sigma(\overline{X}(T)))$ chosen on the efficient frontier can be found by solving the target-based optimization problem with target equal to $\overline{F}(T)$. Thus, each point on the efficient frontier corresponds to a target-based optimization problem.

The fact that the target-based approach is a particular case of the mean-variance approach should put an end to the criticism of the quadratic utility function, that penalizes deviations above the target as well as deviations below it. The intuitive motivation for supporting such a utility function: "The choice of trying to achieve a target and no more than this has the effect of a natural limitation on the overall level of risk for the portfolio: once the target is reached, there is no reason for further exposure to risk and therefore any surplus becomes undesirable" finds here full justification in a rigorous setting.
We notice that a similar result was mentioned, without proof, by Bielecky et al. (2005). They noticed, however, that the portfolio's expected return would be unclear to determine a priori. In contrast, here we provide the exact expected return and variance of the optimal portfolio via optimization of the quadratic utility function. We are thus able to determine completely the exact point on the efficient frontier of portfolios.

5.2 Simulation results

In this section we report results from the simulations that we have run in order to compare the M-V optimal investment strategy with the T-B optimal investment strategy and with the lifestyle strategy. Since the target-based approach is a mean-variance approach with a particular choice of α, we expect results for the first two cases to be qualitatively equal. The main reason why we perform the comparison between these two points of the same efficient frontier is to show how a different formulation of the same problem can help explaining what could appear as disappointing results. For the lifestyle strategy the assumption is that the fund is invested fully in the risky asset until 10 years prior to retirement, and then is gradually switched into the riskless asset by switching 10% of the portfolio from risky to riskless asset each year. For the target-based approach, we choose a final target on the following form (see Haberman and Vigna (2002)):

$$F(T) = x_0 e^{RT} + \frac{c}{R}(e^{RT} - 1)$$

where

$$R = \frac{1}{2}(\lambda + r) + \frac{1}{8}\sigma^2$$

In other words, the final target $F(T)$ is the fund that would be available after T years if the fund and the contributions were invested in a riskless asset with rate of return equal to R (target return), which is chosen to be a certain average of the returns of the rikless and risky assets available in the market. The values chosen for the parameters are $r = 0.03, \lambda = 0.08, \sigma = 0.15, c = 0.1, x_0 = 1, T = 20$. Therefore, the Sharpe ratio is $\delta = 0.33$, the target return is $R = 0.0578$, the target is $F(T) = 6.945$, implying $\overline{\alpha} = 1.726$, $Var(\overline{X}(T)) = 0.69$, $\sigma(\overline{X}(T)) = 0.831$, $\alpha^* = 1.936, E(X^*(T)) = 6.687, Var(X^*(T)) = 0.548$ and $\sigma(X^*(T)) = 0.741$. So, in our setting, we have

$$MV_{X^*} = (0.741, 6.687) \quad \text{and} \quad MV_{\overline{X}} = (0.831, 6.945)$$

We have carried out 1000 Monte Carlo simulations with discretization done on a weekly basis and in each scenario have found the optimal investment strategy and the corresponding final wealth. For consistent comparisons, in each approach we have applied the same stream of pseudo random numbers. Figure 1 reports the optimal investment strategies over time of the M-V and the T-B approaches in one particular scenario. The x-axis represents the time in years since joining the scheme, the y-axis represents the value of the optimal investment allocation in the risky asset in the two approaches.

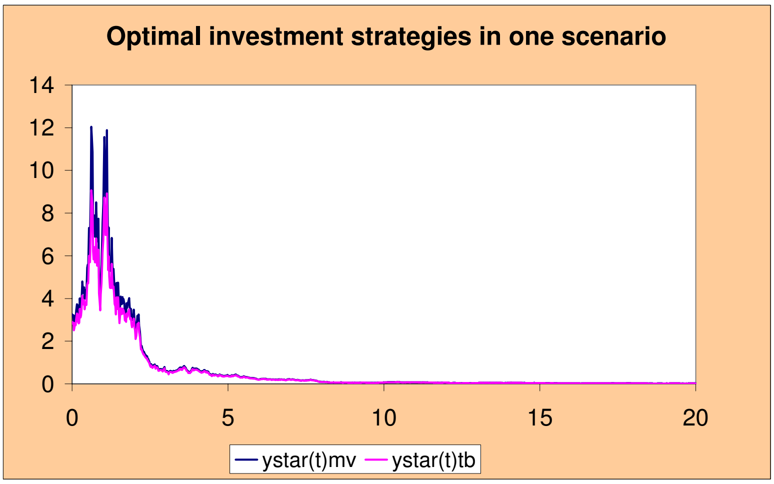

Figure 1.

From Figure 1 we see that both strategies tend to apply a remarkable amount of borrowing for small values of x and we therefore also introduce the suboptimal strategies $\overline{y}^{cut}(t,x)$ and $y_{tb}^{cut}(t,x)$ which are cut off at 0 or 1 if the optimal strategy goes beyond the interval $[0,1]$. It must be mentioned that suboptimal policies of the same type were applied by Gerrard et al. (2006) in the decumulation phase of a DC scheme, and proved to be satisfactory: with respect to the unrestricted case, the effect on the final results turned out to be negligible and the controls resulted to be more stable over time. Clearly, imposing restriction on the controls would change substantially the formulation of the problem and would make it very difficult to tackle mathematically. Up to our knowledge, the only work where an optimization problem with constraints has been thoroughly treated in the accumulation phase of a DC scheme is Di Giacinto and Gozzi (2007) by means of viscosity solutions. In that case the optimal policy is written in feedback form and can be given in explicit form only in a special case.

We investigate the behaviour of the cut-off strategies and compare the reached final wealth with the wealth reached by the optimal strategies and the wealth obtained by the lifestyle strategy. Figure 2 reports the mean and standard deviation of the cut-off strategies over the 1000 scenarios and the behaviour of the lifestyle strategy (whose mean obviously coincides with the strategy itself and has null standard deviation). The x-axis, as before, represents the time in years since joining the scheme, the y-axis represents the value of the investment allocation in the risky asset in each strategy.

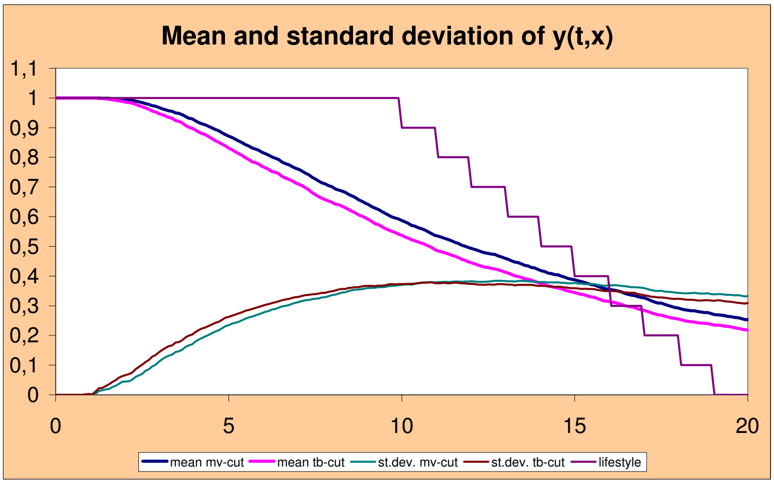

Figure 2.

On average, the suboptimal investment strategy $y(t,x)$ in the M-V and T-B approaches is decreasing over time, showing initial investment in the risky asset that is gradually and partially switched into the riskless one when retirement approaches, like the lifestyle. This result was found also by Haberman and Vigna (2002). However, unlike the lifestyle strategy, the average portfolio is invested fully in the risky asset only for a few years after joining the scheme, and is never invested fully in the riskless asset. From a deeper inspection of the percentiles of the suboptimal investment strategies (not reported here) investing the whole portfolio in the riskless asset never occurs (this is expected because for the corresponding optimal strategies this result comes directly from the theoretical analysis, since α is finite), while in 50% of the cases the strategy imposes to start disinvesting from the risky asset between 2 and 8 years after joining the scheme. The tb-cut strategy lies always below the mv-cut strategy, and again this result is expected due to the observations done in section 5.1.1.

Table 1 reports, for the five strategies considered, some percentiles of the distribution of the final wealth, its mean and standard deviation, the probability of reaching the target and the mean shortfall, defined as the mean of the deviation of the fund from the target, given that the target is not reached.

	mv-cut	mv-not cut	tb-cut	tb-not cut	lifestyle
5th perc.	3.678	5.909	3.822	5.763	3.803
25th perc.	6.333	6.901	6.251	6.648	5.13
50th perc.	6.946	7.11	6.716	6.834	6.607
75th perc.	7.137	7.191	6.869	6.906	8.72
95th perc.	7.213	7.224	6.927	6.935	13.574
mean	6.445	6.89	6.294	6.638	7.316
st.dev.	1.132	0.793	0.998	0.707	3.058
prob reaching target	0.502	0.715	0	0	0.448
mean shortfall	0.589	0.196	0.651	0.307	0.939

Table 1. Final fund: some statistics of the simulation results (target = 6.945).

We notice that the notcut-optimal strategies dominate in all possible ways the corresponding cut-suboptimal strategies, providing higher mean, lower standard deviation, higher probability of reaching the target (in the mv strategies) and lower mean shortfall. However, the price that one has to pay is twofold. Firstly, a much higher variability of the investment allocation, that can

take values significantly far away the interval $[0, 1]$. Secondly, the possibility of ruin: we observe that in some cases with the not-cut strategies the fund goes below zero, causing a negative value of the optimal investment allocation. This does not happen with the cut strategies that provide always positive values for the fund and for the proportion invested in the risky asset.
Whereas in the M-V approach the target is reached in about 50% or 71% of the cases (depending whether restrictions are applied or not), in the T-B approach the probability of reaching the target is 0 for both strategies with and without restrictions. This apparently disappointing result is a consequence of the fact that in the target-based approach the target can never be reached under optimal control, as shown in 5.1.1. We recall that in the T-B approach the expected final fund is lower than the nominal target and a correct comparison should be done with the probability of reaching the expected final fund: this probability turns out to be 53.1% for the tb-cut strategy and 71.5% for the not-cut one, figures that are, expectedly, on the same level of those for the M-V approach.
The comparison of the first four strategies with the lifestyle shows that the final fund for the lifestyle strategy has the highest mean but also the highest standard deviation, which turns out to be significantly high, as a result of the heavy and prolonged investment in the risky asset; in addition, it has the highest mean shortfall from the target and a probability of reaching the target of 45% that is lower than in the M-V approach.

Figure 3 shows the efficient frontier of portfolios in the mean-standard deviation plan and reports the points $(\sigma(X(T)), E(X(T)))$ for each strategy considered so far.

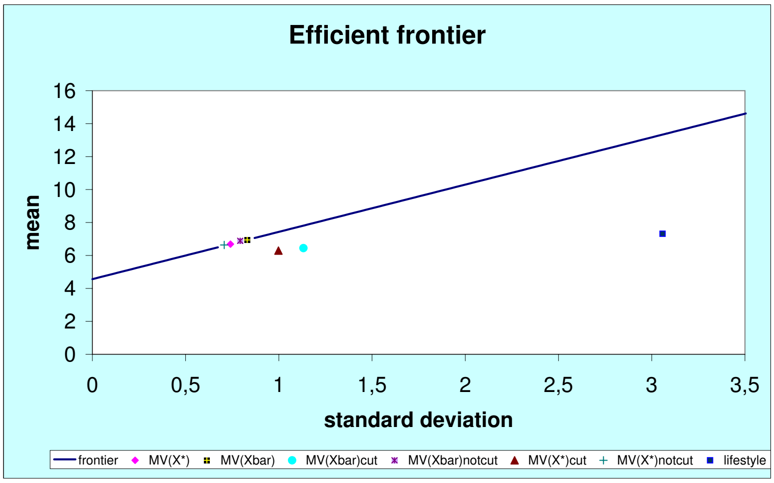

Figure 3.

It is evident that the lifestyle strategy, though it gives a higher mean than the other strategies, proves to be very far from being efficient. In particular, for being efficient it should provide either a standard deviation of about 0.96 (instead of 3.06) with same level of mean, or a mean of 13.34 (instead of 7.32) with the same level of standard deviation. As expected, the empirical values found by applying the not-cut strategies ($MV(\overline{X})notcut$, $MV(X^*)notcut$) lie on the efficient frontier and are not very far from their theoretical counterparts ($MV(\overline{X})$, $MV(X^*)$). However, the cut versions ($MV(\overline{X})cut$, $MV(X^*)cut$), though inefficient in strict sense, seem to be not too far from the efficient frontier. This confirms results previously found (see Gerrard et al. (2006)) that suboptimal policies prove to be satisfactory in terms of final results achieved.

5.2.1 Changing the Sharpe ratio

In Gerrard et al. (2004) it was shown that if the riskless rate of return r does not change, the evolution of the fund under optimal control is invariant for assets with the same Sharpe ratio. Here the same result applies and it is therefore interesting to compare assets with different Sharpe ratio to see how this quantity affects the distribution of the final wealth. It is obvious that also the slope of the efficient frontier in the mean-standard deviation plan changes accordingly. In the following we report simulation results with a lower and with a higher Sharpe ratio w.r.t the previous section, namely we choose $\sigma = 20\%$, leading to $\delta = 0.25$ and $\sigma = 10\%$, leading to $\delta = 0.5$. Table 2 reports results for the case $\delta = 0.25$, table 3 the case $\delta = 0.5$. For consistent comparisons, we have applied the same stream of pseudo random numbers of the previous section.

	mv-cut	mv-not cut	tb-cut	tb-not cut	lifestyle
5th perc.	2,545	4,224	2,843	4,32	3,045
25th perc.	5,637	6,805	5,691	6,162	4,396
50th perc.	7,149	7,551	6,548	6,694	6,091
75th perc.	7,763	7,925	6,899	6,961	8,731
95th perc.	8,083	8,132	7,091	7,109	15,878
mean	6,411	7,018	6,018	6,314	7,225
st.dev.	1,803	1,736	1,317	1,239	4,184
prob reaching target	0,487	0,652	0	0	0,373
mean shortfall	1,049	0,56	1,168	0,872	1,481

Table 2. Final fund distribution, Sharpe ratio = 0.25 (target = 7.187).

The probability of reaching the expected final fund when $\delta = 0.25$ is 54.3% for the tb-cut strategy and 65.2% for the not-cut one.

	mv-cut	mv-not cut	tb-cut	tb-not cut	lifestyle
5th perc.	5,328	6,728	5,34	6,714	4,814
25th perc.	6,667	6,785	6,656	6,77	5,945
50th perc.	6,771	6,791	6,757	6,776	7,073
75th perc.	6,789	6,792	6,774	6,777	8,512
95th perc.	6,792	6,793	6,777	6,778	11,319
mean	6,566	6,775	6,555	6,76	7,409
st.dev.	0,556	0,091	0,55	0,091	2,009
prob reaching target	0,441	0,83	0	0	0,568
mean shortfall	0,216	0,012	0,222	0,017	0,465

Table 3. Final fund distribution, Sharpe ratio = 0.5 (target = 6.778).

The probability of reaching the expected final fund when $\delta = 0.5$ is 44.5% for the tb-cut strategy and 83% for the not-cut one.

Similar comments to those made for Table 1 apply here. As expected, all the results improve when the risky asset has a higher Sharpe ratio and worsen when the Sharpe ratio is lower.
The graphs of the efficient frontier of portfolios (not reported here) show similar results to that of Figure 3 of the previous paragraph, as do the graphs of mean and standard deviation of the investment strategy $y(t, x)$. The tendency observed is that increasing the Sharpe ratio the inefficient points tend to become more inefficient (in the sense of distance from the efficient frontier) and vice versa. The intuition is that by increasing the goodness of the risky asset w.r.t. the riskless one, the investment in the risky asset becomes heavier, leading to higher variance of the final fund. We also notice that the mv- and the tb- investment strategies (both cut and notcut) tend to be closer to each other with higher values of δ and be more distant from each other with

lower values of δ. This can be explained by observing from (68) that the higher δ the closer the values of α^* and $\overline{\alpha}$ and therefore the closer the two portfolios on the same efficient frontier.

6 Concluding comments

In this paper we have defined and solved a mean-variance portfolio selection problem in the accumulation phase of a defined contribution pension scheme. The solution has been found by transforming the mean-variance problem in a linear-quadratic control problem, which has been solved through standard techniques of stochastic optimal control theory. The optimal investment strategy and the efficient frontier of portfolios are given in closed form first in a financial market with two assets and then in a financial market with $n+1$ assets: using previous results, we show that the $n+1$ assets case is formally identical to the two assets case.

We then compare the optimal strategy found via the mean-variance approach with two alternative investment strategies typically adopted in DC plans: the lifestyle strategy and the optimal investment strategy found solving an optimization problem where the aim is to minimize the square of the distance of the final fund from a final target (target-based approach). The comparison has been done in a theoretical framework and by means of simulations. The main finding is that, not only the target-based approach is a particular case of the mean-variance approach and therefore the optimal portfolio associated to a target-based problem is efficient, but there is a one-to-one correspondence between points of the efficient frontier and optimal portfolios found via the target-based approach. Another interesting result is the fact that the lifestyle strategy seems to be very far from being efficient, providing a standard deviation of the final fund that is too high compared to its mean. The implementation of suboptimal strategies (found by cutting the optimal investment allocation at 0 or 1 whenever the optimal value goes outside the interval $[0,1]$) gives satisfactory results, in that the suboptimal portfolios, though inefficient in strict sense, do not seem to lie too far away from the efficient frontier. We prove that an efficient portfolio is never invested completely in the riskless asset, unless the investor allocates zero importance to the maximization of the final fund: furthermore, the amount invested in the risky asset is always strictly positive.

The difficulty of the problem that arises when passing from single-period to multi-period or to continuous-time in the mean-variance formulation, here solved using fairly recent results that link portfolio selection problems with standard stochastic control models, had prevented the construction of an efficient frontier in a long term investment setting like pension funds. We think that the optimal strategies and the efficient frontier presented here can have a practical impact on the investment decisions of investment managers of DC pension schemes. The criterium of minimizing the square of the difference between final fund and desired target - apparently criticizable because the deviations above the target are penalized - finds here solid justification, whereas the more widespread lifestyle strategy seems to be less appropriate due to its inefficiency in the mean-variance setting. The efficient frontier can also find a direct application on the member's decisions regarding his own risk/reward profile, for he can select his own couple desired mean-maximum variance tolerated and therefore choose the corresponding optimal investment strategy.

This work can be extended in different directions. The efficient investment strategy has been compared only with two possible investment strategies for DC pension schemes. This choice has been done in order to limit the length of the paper. In further research, it would be of interest to extend the comparison to other investment allocations proposed in the literature and to assess their efficiency. Furthermore, the problem has been solved without restrictions on the optimal investment allocation, due to the difficulty that arises when constraints are introduced in the model. It would be of interest to solve the problem with constraints, and make the comparison with the restricted suboptimal policy applied here. This is left to future research.

Appendix

A Alternative derivation of the solution of the $n+1$ asset case

The general theory of SLQ problems is treated in great details in Yong and Zhou (1999): the general solution is given according to three different approaches, namely the stochastic maximum principle, the dynamic programming and a completion of square technique and the equivalence among these methods is shown. The solution of the $n+1$ asset problem here is found following the dynamic programming approach, in line with the first part of the paper. We omit the derivation of the relevant formulae and refer the interested reader to Yong and Zhou (1999) for full understanding of all details.

Problem (43) is equivalent to solve

$$\min_{\mathbf{y}(\cdot)} \mathbf{E}\left[\frac{1}{2}\alpha Z(T)^2\right] = \min_{\mathbf{y}(\cdot)} J(\mathbf{y}(\cdot);\alpha) \tag{70}$$

with

$$\gamma = \frac{\beta}{2\alpha} \qquad \text{and} \qquad Z(t) = X(t) - \gamma$$

where the process $Z(t)$ follows the SDE

$$\begin{array}{l} dZ(t) = \{(Z(t)+\gamma)[\mathbf{B}\mathbf{y}(t)+r]+c\}dt + (Z(t)+\gamma)\sum_{j=1}^{n}\sigma_j\mathbf{y}(t)dW_j(t) \\ Z(0) = x_0 - \gamma \end{array} \tag{71}$$

The value function is defined as

$$V(t,z) = \inf_{\mathbf{y}(\cdot)} \mathbf{E}_{t,z}\left[\frac{1}{2}\alpha Z(T)^2\right] = \inf_{\mathbf{y}(\cdot)} J(\mathbf{y}(\cdot),\alpha) \tag{72}$$

V satisfies the HJB equation

$$\inf_{\mathbf{y}\in\mathbf{R}^n}\{V_t + V_z[(z+\gamma)r + (z+\gamma)\mathbf{B}\mathbf{y} + c] + \frac{1}{2}V_{zz}(z+\gamma)^2(\sum_{j=1}^{n}\sigma_j\mathbf{y})^2\} = 0 \tag{73}$$

Then, it turns out that the value function V is quadratic in z and is given by

$$V(t,z) = \frac{1}{2}P(t)z^2 + Q(t)z + R(t) \tag{74}$$

where the functions P, Q and R satisfy the system of ODE's:

$$\left\{\begin{array}{lcl} P'(t) & = & (\Delta^2 - 2r)P(t) \\ Q'(t) & = & (\Delta^2 - r)Q(t) - (\gamma r + c)P(t) \\ R'(t) & = & \frac{\Delta^2 Q^2(t)}{2P(t)} - (\gamma r + c)Q(t) \end{array}\right. \tag{75}$$

with boundary conditions

$$P(T) = \alpha \qquad Q(T) = 0 \qquad R(T) = 0 \tag{76}$$

where Δ^2 is given by (44). It is clear, as expected, that the two systems (17)-(18) and (75)-(76) are essentially the same, with $P(t) = 2A(t)$, $Q(t) = B(t)$, $R(t) = C(t)$ and $\delta^2 = \Delta^2$. Thus the solution is identical:

$$\left\{\begin{array}{lcl} P(t) & = & \alpha e^{-(\Delta^2-2r)(T-t)} \\ Q(t) & = & \frac{\alpha(\gamma r+c)}{r}e^{-(\Delta^2-2r)(T-t)}[1 - e^{-r(T-t)}] \\ R(t) & = & \int_T^t [\frac{\Delta^2 Q(s)^2}{2P(s)} - (\gamma r + c)Q(s)]ds \end{array}\right. \tag{77}$$

The optimal investment strategy at time t is given by

$$\overline{\mathbf{y}}(t,z) = -\frac{1}{z+\gamma}\left[z + \left(\gamma + \frac{c}{r}\right)(1-e^{-r(T-t)})\right](\Sigma\Sigma^T)^{-1}\mathbf{B}^T \tag{78}$$

and replacing $z+\gamma$ with x:

$$\overline{\mathbf{y}}(t,x) = -\frac{1}{x}\left[x - \gamma e^{-r(T-t)} + \frac{c}{r}(1-e^{-r(T-t)})\right](\Sigma\Sigma^T)^{-1}\mathbf{B}^T \tag{79}$$

which, as expected, coincides with (47).

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