Model Based Fault Diagnosis in a Centrifugal Pump Application using Structural Analysis

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Model Based Fault Diagnosis in a Centrifugal Pump Application using Structural Analysis

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Abstract—A model based approach for fault detection and isolation in a centrifugal pump is proposed in this paper. The fault detection algorithm is derived using a combination of structural analysis, Analytical Redundant Relations (ARR) and observer designs. Structural considerations on the system are used to identify four subsystems each sensitive to a subset of the faults under consideration. Either an ARR or a residual observer is designed for each of the four subsystems. The four obtained residuals are then used for fault isolation. The applicability of the algorithm is illustrated by applying it to an industrial benchmark. The benchmark tests have shown that the algorithm is capable of detection and isolation of five different faults in the mechanical and hydraulic parts of the pump.

I. INTRODUCTION

Centrifugal pumps are used in a variety of different applications. This could for example be in a water supply application where submersible pumps are used in water wells to lift water to the surface. Some of these installations are crucial for a larger system to work. Failures can lead to substantial economic losses and can affect the life comfort of many people when they occur. Therefore detection of faults, if possible in an early stage, and isolation of their causes are of great interest. Especially fault detection, which can be used for predictive maintenance, could save money and increase reliability of the application in which the pump is placed.

Different approaches have been used for fault detection in centrifugal pumps. In [7], [8] current spectrum signatures are used for detection of different faults like blockage, cavitation, and damaged impeller. In [9], [4] model based approaches are used. In [9] the nonlinear system is modelled by a set of fuzzy functions and in [4] a linearized version of the system model is used. Both of these consider both detection and isolation of faults in systems containing centrifugal pumps.

In this work a model-based approach is used for residual generation. The presented approach utilizes a nonlinear model of the submersible application. This makes the obtained algorithm independent of the operating point in which the pump is running. The algorithm utilizes torque, speed, pressure and flow signals to generate the residuals.

The algorithm is derived by first identifying four subsystems using structural analysis [3], [6]. From the structural model of the system it is seen that all of these subsystems includes different subsets of the faults considered in this work. Therefore they can be used for fault isolation. Three of these sets contain differential constraints [3] meaning that derivatives of the output are necessary if an Analytical Redundancy Relation (ARR) is derived. To overcome this problem residual observers are designed in these three cases. An overview of the contribution to observer design for residual generation can be found in [5]. In the single case where the subsystem does not contain differential constraints an ARR is used for residual generation.

As a model-based approach is used in this work, this paper starts by presenting the model of a submersible pump application in section III. The fault detection algorithm is considered in section IV. This includes identifying subsystems using structural analysis, designing the residual observers, and the ARR. Section V presents test results obtained on an industrial benchmark, which has been particularly developed for this purpose. Finally concluding remarks end the paper.

II. NOMENCLATURE

The parameters in the model presented in section III are described in the following.

- $J$ Moment of inertia of the rotor and the impeller.
- $B$ Linear friction.
- $K_j$ Derived moment of inertia of the water in the system.
- $K_p$ Pressure losses inside the pipeline.
- $K_v$ Pressure losses inside the valve.
- $a_{hi}$ Parameters in the pressure model of the pump, $i \in \{1, 2, 3\}$.
- $a_{ti}$ Parameters in the torque model of the pump, $i \in \{1, 2, 3\}$.
- $g$ Gravity constant.
- $\rho$ The density of the liquid in the system.

III. THE SUBMERSIBLE PUMP APPLICATION

This section presents the mathematical model of a submersible pump application including faults and distur-
The submersible pump application is depicted in figure 1.

This figure illustrates a pump placed at the bottom of a well pumping water to the surface. The variables assumed known in the system are the shaft torque, the shaft speed, the pressure produced by the pump, and the volume flow through the pump. In figure 1 the pressure is labeled $H_p$ and the volume flow is labeled $Q$. Moreover the water in the well is lifted from level $z_{in}$ to $z_{out}$ and the volume flow $Q$ can be controlled by a valve $V_1$ at the top of the well. The inlet and outlet pressure of the pipe system are labeled respectively $p_{in}$ and $p_{out}$.

A. Model Without Faults

The equations describing the submersible pump system under no fault conditions are given by the following set of relations,

\[
\begin{align*}
c_1 : & \quad J_0 \frac{d \omega_r}{dt} = T_r - B \omega_r - T_p \\
c_2 : & \quad K_j \frac{d q}{dt} = H_p - p_l \\
c_3 : & \quad H_p = -a_{h2}Q^2 + a_{h1}Q \omega_r + a_{h0} \omega_r^2 \\
c_4 : & \quad T_p = -a_{t2}Q^2 + a_{t1}Q \omega_r + a_{t0} \omega_r^2 \\
c_5 : & \quad y_1 = H_p \\
c_6 : & \quad y_2 = \omega_r \\
c_7 : & \quad y_3 = Q
\end{align*}
\]

Relation $c_1$ and $c_2$ respectively describe the dynamics of the mechanical and the hydraulic system. In these $\omega_r$ is the shaft speed of the pump and $Q$ is the volume flow through the pump. The relation $c_3$ models the pressure delivered by the pump $H_p$ and the relation $c_4$ models the load torque on the shaft generated by the pump $T_p$. Finally the relations $c_5$ to $c_7$ model the sensor system of the application. Here $y_1$ is the differential pressure measurement, $y_2$ is the speed measurement and $y_3$ is the flow measurement. Beside the measurements the input torque $T_c$ of the system is assumed known.

In the model presented in (1) the pressure $p_l$ is the load pressure of the well and is given by,

\[
p_l = (p_{out} - p_{in} + \rho g(z_{out} - z_{in})) - (K_v + K_p)Q^2
\]

This pressure is derived from the depth of the well denoted by $z_{out} - z_{in}$, the inlet and outlet pressure of the pipeline system, and the flow dependent pressure loss in the pipe and valve. All of these are assumed unknown in the following, meaning that the pressure $p_l$ must be assumed unknown in the development of the detection algorithm.

The model presented in (1) is only valid for positive speed and positive flow, since the valve model and the relations $c_4$ and $c_5$ are only valid for positive flow and speed i.e. $\omega_r, Q \in R_+$.

B. Model Including Disturbances and Faults

Five faults are considered in this work, these are,

1) clogging inside the pump,
2) increased friction due to either rub impact or bearing faults,
3) increased leakage flow,
4) performance degradation due to cavitation,
5) dry running.

The first three faults are internal faults caused by respectively impurities in the liquid and wear. The 4th fault, cavitation, is caused by too low inlet pressure, meaning that the fault is external. However, in this work it is treated as an internal fault. Finally, the last fault, dry running, is a phenomenon caused by faults in the surrounding system, hence it is an external fault and is treated as so. Even though it is not a fault in the pump, this fault is important to detect as sealing rings and bearings will be destroyed when the pump is running without water for only a few seconds.

The mentioned faults all affect the hydraulic part of the pump. The performance of the hydraulic part of the pump is in this model described by relation $c_4$, $c_5$ and $c_7$ in (1). These relations respectively describe the pressure and the torque produced by the pump and the flow measurement. Introducing the faults, these relations become,

\[
\begin{align*}
c_3 : & \quad H_p = f_H(Q, \omega_r) - K_f Q^2 - C_{ch} f_c - C_{dh} f_d \\
c_4 : & \quad T_p = f_T(Q, \omega_r) + \Delta B \omega_r - C_{ct} f_c - C_{dt} f_d \\
c_7 : & \quad y_3 = Q - K_i \sqrt{p_p}
\end{align*}
\]

where $f_H(Q, \omega_r)$ and $f_T(Q, \omega_r)$ are given by,

\[
\begin{align*}
f_H(Q, \omega_r) &= -a_{h2}Q^2 + a_{h1}Q \omega_r + a_{h0} \omega_r^2 \\
f_T(Q, \omega_r) &= -a_{t2}Q^2 + a_{t1}Q \omega_r + a_{t0} \omega_r^2
\end{align*}
\]

In this fault model $K_f \in R_+$ represents clogging, $\Delta B \in R_+$ represents rub impact, $K_i \in R_+$ represents increased leakage flow, $f_c \in R_+$ represents cavitation and $f_d \in R_+$ represents dry running. The first three signals model the faults accurately, while the last two terms are linear approximations.

IV. FAULT DETECTION AND ISOLATION

In this section the model presented in the previous section is used to develop a fault detection and isolation algorithm. To do so structural analysis is used to identify over-determined subsystems containing information about different subsets of the faults. When the subsystems are
identified the residuals are obtained using respectively an 
ARR and three simple observers. The simplicity of these 
observers is due to the utilization of structural analysis to 
obtain simple submodels for use in the observer design.

A. Structure Analysis

The system is described by the relations shown in (1). 
These relations can be represented by the graph shown 
in Table I, where the constraints $c_1, \ldots, c_7$ are given by (1) 
and the constraints $d_1$ and $d_2$ are differential constraints, as 
deﬁned in [3], meaning that $\frac{dx}{dt} = \dot{x}$ in this context.

Using the deﬁnitions and procedures described in [6] and 
[3] four over-determined subsystems are identiﬁed. These 
are,

\[ \mathcal{F}_1 = \{ c_1, c_4, d_1, c_5, c_7 \} \]
\[ \mathcal{F}_2 = \{ c_3, c_6, c_2 \} \]
\[ \mathcal{F}_3 = \{ c_1, c_3, c_4, d_1, c_5, c_6 \} \]
\[ \mathcal{F}_4 = \{ c_1, c_3, c_4, d_1, c_6, c_7 \} \]

From these four over-determined subsystems, or matchings, 
it is seen that the constraint $c_2$ is not used in any of 
the matchings. This constraint describes the application 
in which the pump is placed. When this constraint is not used 
in a matching it means that the matching is independent of 
the application model. Therefore the four above matchings 
can be used for fault detection and isolation in centrifugal 
pumps placed in any possible application.

Looking at the column to the right in Table I the faults 
affecting each of the over-determined subsystems $\mathcal{F}_i$ can 
be identiﬁed. The connection between the faults and the 
over-determined subsystems is shown below,

\[ \mathcal{F}_1 : \{ K_1, \Delta B, f_c, f_d \} \]
\[ \mathcal{F}_2 : \{ K_f, K_1, f_c, f_d \} \]
\[ \mathcal{F}_3 : \{ K_f, \Delta B, f_c, f_d \} \]
\[ \mathcal{F}_4 : \{ K_f, K_1, \Delta B, f_c, f_d \} \]

This connection is a necessary, but not sufﬁcient, condition 
for a given $\mathcal{F}_i$ to be sensitive to a given fault. These 
connections show that the faults $f_c$ and $f_d$ are indistinguishable 
from a structural point of view, meaning that isolation of 
these faults is impossible for almost all set of parameters 
in (1).

From the connection between faults and relations pre-

sented in (3) it is seen that no additional information is 
added using $\mathcal{F}_4$. Therefore the set,

\[ \{ \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \} \]

contains the obtainable information about the faults in 
the system. The last relation $\mathcal{F}_4$ could be used for validation 
in a robust fault detection scheme.

B. The Residual Generators

Looking at the relations forming the matching $\mathcal{F}_2$ it is 
seen that no differential constraints are included in this.

Therefore an ARR obtained from this matching does not 
include derivatives. The ARR is given by,

\[ r_2 = -a_1 y_2^2 - a_1 y_2 y_3 - a_0 y_1^2 - y_2 \]

It is also possible to obtain ARR’s from the sets $\mathcal{F}_1$, 
$\mathcal{F}_3$ and $\mathcal{F}_4$, but as differential constraint is used in each 
of these matchings it is necessary to use derivatives of the 
output in these cases. To avoid this, three residual observers 
are developed in the following.

The three matchings $\mathcal{F}_1$, $\mathcal{F}_3$ and $\mathcal{F}_4$ are all on a form 
given by deﬁnition 1,

Definition 1 A system on the form,

\[ \dot{x} = ax + f(x, z, u) + e_1(x, z) f_1 \]
\[ y_1 = h_1(x, z) + e_2(x, z) f_2 \]
\[ y_2 = h_2(x, z) + e_3(x, z) f_3 \]

is said to be on over-measured form. In (5) $x, y_1, y_2, u, z \in$ 
$R^1$ and $e_1, e_2$ and $e_3$ are nonlinear functions of $x$ and $z$.

Assumption 1 It is assumed that in the case where no 
sensor faults have occurred, i.e. $f_2 = f_3 = 0$, the output 
maps $h_1$ and $h_2$ in deﬁnition 1 can by solved for $x$ and $z$ 
locally. The solutions are given by the following expressions,

\[ x = g_1(y_1, y_2) \]
\[ z = g_2(y_1, y_2) \]

The implicit function theorem [1] can for example be used 
to show that a solution exist locally. Using the above 
asumption the following lemma describes a residual observer 
for the system deﬁned in deﬁnition 1.

Lemma 1 Under assumption 1 the following observer is a 
residual observer for systems described by deﬁnition 1,

\[ \dot{x} = a \dot{x} + f(g_1(y_1, y_2), g_2(y_1, y_2), u) + k(g_1(y_1, y_2) - \dot{x}) \]
\[ r = q(g_1(y_1, y_2) - \dot{x}) \]

The residual observer is asymptotical stable if $a - k < 0$.
The fault input to this observer is given by,

\[ f_f = (f(x, z) - f(x - \delta x, z - \delta z)) \]
\[ + e_1(x, z) f_1 - k \delta x_f \]

where $f_f$ is a derived fault signal, which is strongly 
detectable. In the expression of $f_f$ the signals $\delta x_f$ and $\delta z_f$ 
are given by,

\[ \delta x_f = g_1(y_1 - e_2(x, z) f_2, y_2 - e_3(x, z) f_3) - g_1(y_1, y_2) \]
\[ \delta z_f = g_2(y_1 - e_2(x, z) f_2, y_2 - e_3(x, z) f_3) - g_2(y_1, y_2) \]

The proof of the lemma is given in appendix I.

Remark 1 The derived fault $f_f$ is strongly detectable using 
this observer. This is not the case for the faults $f_1$, $f_2$ and 
$f_3$, as the nonlinear expression of $f_f$ can equal zero even 
though one of the faults $f_1$, $f_2$ or $f_3$ is different from zero.
matchings. The obtained observers are given by, lemma 1 can be used to obtain residual observers for the
containing two elements, fulfills assumption 1. Therefore
given in (1).

to obtain the expression for \( y \). Each of the matchings utilizes different subsets of
in (1). The observer described by lemma 1 is designed
Remark 2 The observer described by lemma 1 is designed
under the assumption that a perfect model exists, and that
the measurements are not affected by noise. This is of cause
not fulfilled in a real life applications. To overcome this the
gain \( k \) of the observer is chosen such that errors due to
small model mismatchs and noise will be suppressed.

The matchings \( \mathcal{F}_1, \mathcal{F}_3 \) and \( \mathcal{F}_4 \) are all on the form
defined in definition 1 and fulfill assumption 1. Therefore
lemma 1 can be utilized for observer design for these three
matchings. The dynamics of the matchings are in all three
cases govern by the following differential equation,

\[
J \frac{d\omega}{dt} = T_e - B \omega + a_{12} Q^2 - a_{11} \omega Q - a_{10} \omega^2
\]  

This equation is formed by using the constrains \( c_1 \) and \( c_4 \)
in (1). Each of the matchings utilizes different subsets of the
following set of output maps,

\[
\begin{align*}
y_1 &= \omega \\
y_2 &= -a_{12} Q^2 + a_{11} \omega Q + a_{10} \omega^2 \\
y_3 &= Q
\end{align*}
\]  

The output maps are formed by using respectively constraint
\( c_5 \) to obtain the expression for \( y_1 \), the constraints \( c_3 \) and \( c_6 \)
to obtain the expression for \( y_2 \), and finally the constraint
\( c_7 \) to obtain the expression for \( y_3 \). The constraints are all
given in (1).

From (9) it is seen that each subset of the output maps,
containing two elements, fulfills assumption 1. Therefore
lemma 1 can be used to obtain residual observers for the
matchings. The obtained observers are given by,

\[
\begin{align*}
O_1 : \left\{ J \frac{d}\omega}{dt} &= -B \omega - f_r(y_3, y_1) + T_e + k_1 (y_1 - \dot{\omega}) \\
r_1 &= q_1 (y_1 - \dot{\omega})
\end{align*}
\]  

where \( k_i \) is designed according to lemma 1 and \( q_i \) is chosen
such that the residuals have a reasonable value in the case of
faults. The function \( f_r \) is given in (2) and the functions
\( g_3 \) and \( g_4 \) are derived from the output maps in (9), and are
given by,

\[
\begin{align*}
g_3(y_1, y_2) &= \frac{a_{11} y_1 + \sqrt{a_{11}^2 y_1^2 - 4 a_{12} (y_2 - a_{10} y_1)}}{2 a_{12}} \\
g_4(y_2, y_3) &= \frac{-a_{11} y_3 + \sqrt{a_{11}^2 y_3^2 + 4 a_{10} (y_2 + a_{12} y_3)}}{2 a_{10}}
\end{align*}
\]  

These expressions are valid for \( y_1, y_3 \in \mathbb{R}_+ \) when using
the parameters of the pump used in the test described in
the following section. Therefore the expressions are valid
in the state space \( \omega, Q \in \mathbb{R}_+ \), which is exactly the state
space in which the model is valid, see section III-A.

V. TEST RESULTS

The detection algorithm, derived in the previous sections,
is in this section tested on a Grundfos 1.5 (KW) CR5-10
pump. This pump placed in a tank system as depicted in
figure 2. The measurements used in the detection algorithm

![Fig. 2. Sketch of the test setup. The measurements are the shaft torque
\( T_s \), the differential pressure \( H_p \) delivered by the pump and the volume
flow through the pump \( Q \).](image)

are the torque on the shaft \( T_e \), the differential pressure \( H_p \)}
delivered by the pump and the volume flow through the pump \( Q \). The valve \( V_1 \) is used to model disturbances in the system. Clogging inside the pump is modelled by the valve \( V_c \) and dry running is modelled by closing \( V_2 \) and opening \( V_3 \). Rub impact is modelled adding an extra force to the shaft and cavitation is modelled by closing valve \( V_2 \) gradually. Leakage flow is modelled by opening \( V_l \).

Test results have shown that the sensitivity to the faults \( f_c \) and \( f_d \) of the observer \( O_4 \) is very low. Infact it is so low that changes due to the faults are smaller than changes due to noise and parameter variations. Moreover in section IV-A it is shown that the obtainable fault information is included in the residual \( r_1 \), \( r_2 \), and \( r_3 \). Therefore only these residuals are considered in the test presented in this section.

Since the tests are performed on a real system, noise is expected on the residuals. To overcome this problem a CUSUM algorithm [2] is used to detect changes in the mean of the residuals and thereby detect the faults. In the following, outputs of the CUSUM algorithms are denoted \( D_1 \) to \( D_3 \), where \( D_1 \) is the decision signal of \( r_1 \) and so forth.

All test results are shown in figure 3. First robustness with respect to the operating point is tested. In this test both the position of the valve \( V_1 \) and the speed of the pump are changed during operation. During the test the valve is changed in three steps from medium to maximum opened. The speed of the pump is changed between 2380 and 2910 (rpm) each 2 (sec) during the test. The result of this test is shown in figure 3(a), where \( r_1 \) to \( r_3 \) is shown in the top figure and the decision signals \( D_1 \) to \( D_3 \) in the bottom figure. The test shows that the three residual generatores are robust with respect to the tested operating points, but also that there are some dependency to the operating point, see top figure of figure 3(a), This is partly due to problems with the flow sensor at zero flow and partly due to dependency between the parameters and the operating point.

Figure 3(b) to 3(f) shows test results concerning isolability of the five faults of interest in this work. All these tests are performed with \( V_1 \) half opened and an angular speed of approximately 2650 (rpm). Comparing the five figures 3(e) and 3(f) it is seen that the faults are distinguishable except for cavitation and dry running. This was expected as the structural analysis in section IV-A already had foreseen this.

VI. CONCLUSION

The topic of this work is fault detection and isolation in a centrifugal pump placed in a submersible application. An algorithm is developed, which is capable of detection and isolation of the faults in a centrifugal pump. The proposed algorithm is independent of the application in which the pump is placed. This makes the algorithm robust and usable in a wide range of applications including the submersible application under consideration in this work.

Tests have shown that it is possible to distinguish between the four of the five faults under consideration with the three chosen residuals. But it is also shown that the algorithm is sensitive to the operating point. This is partly due to dependency between the operating point and the parameters in the model and partly due to flow sensor problems. Even though there are some dependencies between the operating point and the performance of the algorithm, the algorithm still performs considerably better than algorithms build on a linearized model.

REFERENCES


APPENDIX I

PROOF OF LEMMA 1

From assumption 1 the inverse of the output maps of the system in definition 1 exists, meaning the following function can be obtained,

\[
x = g_1(y_1, y_2)
\]
\[
z = g_2(y_1, y_2)
\]

where it is assumed that the faults \( f_2 = f_3 = 0 \).

Choosing the observer dynamics as a copy of the dynamics of the system defined by definition 1, and using the the inverse of the output maps the observer becomes,

\[
\dot{x} = a\dot{x} + f(g_1(y_1, y_2), g_2(y_1, y_2), u) + k(g_1(y_1, y_2) - \dot{x})
\]

when it is assumed that the fault \( f_1 = 0 \). Using the expression for the system the error equation of the observer becomes,

\[
\dot{e} = (ax + f(x, z, u)) - (a\dot{x} + f(x, z, u) + k(x - \dot{x}))
\]

\[
\dot{e} = (a - k)e
\]

where (13) is used in the observer expression (14), meaning that \( g_1(y_1, y_2) = x \) and \( g_2(y_1, y_2) = z \). Equation (15) shows that the error dynamic of the observer is asymptotical stable if \( a - k < 0 \).
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Fig. 3. Test results from test of the developed algorithms on the test setup. The top figures shows the obtained residuals and the bottom figures shows decision signals obtained from CUSUM algorithms.
The expression of the derived fault signal $f_f$ is obtained in the following by introducing the fault signals in the error equation of the observer. Before this can be done, an expression of the fault when mapped through the $g_1$ and $g_2$ must be obtained. First the signals $y_{1f}$ and $y_{2f}$ are defined as,

\[ y_{1f} = y_1 - e_2(x, z) \]
\[ y_{2f} = y_2 - e_3(x, z) \]

From these expression it is seen that the signals $y_{1f}$ and $y_{3f}$ must be used in the maps $g_1$ and $g_2$ to obtain the correct value of $x$ and $z$, e.i.

\[ x = g_1(y_{1f}, y_{2f}) \]
\[ z = g_2(y_{1f}, y_{2f}) \]

Then $\delta x_f$ and $x_f$ is defined as,

\[ \delta x_f = x - x_f \]
\[ \delta x_f = g_1(y_{1f}, y_{2f}) - g_1(y_1, y_2) \]

and $\delta z_f$ and $z_f$ is defined likewise. Using these signals the error equation, including the faults, becomes,

\[
\dot{e} = a e + f(x, z, u) - f(x_f, z_f, u) + e_1(x, z)f_1 - k(x_f - \hat{x})
\]
\[
\dot{e} = (a - k)e + (f(x, z, u) - f(x - \delta x_f, z - \delta z_f, u)) + e_1(x, z)f_1 - k\delta x_f
\]

From this expression the following nonlinear expression of the fault can be identified,

\[ f_f = (f(x, z, u) - f(x + \delta x_f, z + \delta z_f, u)) + e_1(x, z)f_1 - k\delta x_f \]

Including this derived fault signal into the error equation in (16) it becomes,

\[
\dot{e} = (a - k)e + f_f
\]
\[ r = qe \]

From this expression it is seen that the derived fault signal $f_f$ is strongly detectable. This is not the case for the faults $f_1$, $f_2$ or $f_3$ due to the nonlinearities of the expression $f_f$ making it possible that $f_f = 0$ even though one of the faults $f_1$, $f_2$ or $f_3$ is different from zero.