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by

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Allan Frendrup\textsuperscript{1}, Preben Dahl Vestergaard\textsuperscript{1}, Anders Yeo\textsuperscript{2}

\textsuperscript{1} Department of Mathematical Sciences, Aalborg University, Denmark
\textsuperscript{2} Department of Computer Science, Royal Holloway, University of London

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Abstract. We present results on total domination in a partitioned graph $G = (V, E)$. Let $\gamma_t(G)$ denote the total dominating number of $G$. For a partition $V_1, V_2, \ldots, V_k$, $k \geq 2$, of $V$, let $\gamma_t(G; V_i)$ be the cardinality of a smallest subset of $V$ such that every vertex of $V_i$ has a neighbour in it and define the following

\begin{align*}
  f_t(G; V_1, V_2, \ldots, V_k) &= \gamma_t(G) + \gamma_t(G; V_1) + \gamma_t(G; V_2) + \ldots + \gamma_t(G; V_k) \\
  f_t(G; k) &= \max\{f_t(G; V_1, V_2, \ldots, V_k) \mid V_1, V_2, \ldots, V_k \text{ is a partition of } V\} \\
  g_t(G; k) &= \max\{\sum_{i=1}^{k} \gamma_t(G; V_i) \mid V_1, V_2, \ldots, V_k \text{ is a partition of } V\}
\end{align*}

We summarize known bounds on $\gamma_t(G)$ and for graphs with all degrees at least $\delta$ we derive the following bounds for $f_t(G; k)$ and $g_t(G; k)$.

\begin{enumerate}
  \item For $\delta \geq 2$ and $k \geq 3$ we prove $f_t(G; k) \leq 11|V|/7$ and this inequality is best possible.
  \item For $\delta \geq 3$ we prove that $f_t(G; 2) \leq (5/4 - 1/372)|V|$. That inequality may not be best possible, but we conjecture that $f_t(G; 2) \leq 7|V|/6$ is.
  \item For $\delta \geq 3$ we prove $f_t(G; k) \leq 3|V|/2$ and this inequality is best possible.
  \item For $\delta \geq 3$ the inequality $g_t(G; k) \leq 3|V|/4$ holds and is best possible.
\end{enumerate}

Key words. Total domination, Partitions and Hypergraphs.

1. Notation

By $G = (V, E)$ we denote a graph $G$ with vertex set $V = V(G)$ and edge set $E = E(G)$. The order of $G$ is $|V(G)| = n$. For $x \in V(G)$ we denote by $N_G(x)$ the set of neighbours to $x$ and $N_G[x] = \{x\} \cup N_G(x)$. Indices may be omitted if clear from context. The degree of $x$ is $d_G(x) = |N_G(x)|$, the number of neighbours to $x$. We let $\delta(G) = \delta$ denote the minimum degree in $G$ and $\Delta(G) = \Delta$ the maximum degree. A hypergraph $H = (V, E)$ has vertex set $V = V(H)$ and its set of hyperedges, or edges for short, is $E = E(H)$. Each hyperedge $e$ is a subset of $V$, $e \subseteq V(H)$. A vertex $v$ is incident with an edge $e$ if $v \in e$, the degree of
\( v \) is the number of hyperedges in \( H \) containing \( v \). We let \( \delta(H) = \delta \) denote the minimum degree in \( H \) and \( \Delta(H) = \Delta \) the maximum degree. \( H \) is \( r \)-regular if each vertex has degree \( r \), i.e. \( d_H(x) = r \), or equivalently, \( x \) is contained in precisely \( r \) edges. \( H \) is \( k \)-uniform if each hyperedge contains exactly \( k \) vertices. Two edges \( e_1 \) and \( e_2 \) are said to be overlapping if \( |V(e_1) \cap V(e_2)| \geq 2 \). Let \( Y \subseteq V(H) \) then \( E(Y) \) denotes all hyperedges, \( e \), contained in \( Y \) (i.e. \( V(e) \subseteq Y \)).

For a hypergraph \( H \) a hitting set or a transversal \( T \) is a set of vertices \( T \subseteq V(H) \) such that \( e \cap T \neq \emptyset \) for each hyperedge \( e \) in \( E(H) \), i.e. each edge \( e \) contains at least one vertex from \( T \). \( T(H) \) denotes the minimum cardinality of a transversal for the hypergraph \( H \). For \( S, T \subseteq V \), in a graph \( G \) the set \( S \text{ totally dominates } T \) if every vertex in \( T \) is adjacent to some vertex of \( S \). The minimum number of vertices needed to totally dominate \( V \) is the total domination number \( \gamma_t(G) \). For a subset \( S \) of \( V \) we let \( \gamma_t(G; S) \) denote the smallest number of vertices in \( G \) which totally dominates \( S \). A partition \( V = (V_1, V_2, \ldots, V_k) \) of \( V(G) \) into \( k \) disjoint sets, \( k \geq 2 \), has \( V = \bigcup_{i=1}^{k} V_i \), \( V_i \cap V_j = \emptyset \), \( 1 \leq i < j \leq k \). For a partition \( (V_1, V_2, \ldots, V_k) \) of \( V \), we define the following.

\[
\begin{align*}
  f_t(G; V_1, V_2, \ldots, V_k) &= \gamma_t(G) + \gamma_t(G; V_1) + \gamma_t(G; V_2) + \ldots + \gamma_t(G; V_k) \\
  g_t(G; V_1, V_2, \ldots, V_k) &= \gamma_t(G; V_1) + \gamma_t(G; V_2) + \ldots + \gamma_t(G; V_k)
\end{align*}
\]

We furthermore define \( f_t(G; k) \) and \( g_t(G; k) \) as follows.

\[
\begin{align*}
  f_t(G; k) &= \max\{f_t(G; V_1, V_2, \ldots, V_k) \mid V_1, V_2, \ldots, V_k \text{ is a partition of } V\} \\
  g_t(G; k) &= \max\{g_t(G; V_1, V_2, \ldots, V_k) \mid V_1, V_2, \ldots, V_k \text{ is a partition of } V\}
\end{align*}
\]

For further notation we refer to Chartrand and Lesniak [1].

2. Introduction

The theory of domination is outlined in two books by Haynes, Hedetniemi and Slater [5, 6]. A combination of domination and partitions is treated by Hartnell and Vestergaard [7], Seager [14], Tuza and Vestergaard [17], Henning and Vestergaard [11]. There has been an upsurge in the study of total domination. New results on total domination are given by Henning, Kang, Shan, Thomassé and Yeo in [10,12,15,18]. In [9] Henning surveys recent results on total domination. Here we shall study total domination in partitioned graphs.

3. Bounds on \( \gamma_t \)

We summarize in Theorem 1 results found by Henning, Thomassé and Yeo. If \( C_{10} : v_1, v_2, \ldots, v_{10}, v_1 \) is the circuit with 10 vertices then let \( G_{10} \) denote the graph obtained from \( C_{10} \) by addition of the edge \( v_1v_6 \) and let \( H_{10} \) denote the graph obtained from \( C_{10} \) by addition of the edges \( v_1v_6 \) and \( v_2v_7 \).

**Theorem 1.** Let \( G \) be a connected graph with \( n \) vertices and minimum degree \( \delta(G) = \delta \). Then

\[
\begin{align*}
  \delta \geq 2 \implies \gamma_t(G) \leq 4n/7 \text{ for } G \notin \{C_3, C_5, C_6, C_{10}, G_{10}, H_{10}\} \quad ([8, \text{ Corollary 6}], [9, \text{ Theorem 27}]). \\
  \delta \geq 3 \implies \gamma_t(G) \leq n/2. \quad ([15]).
\end{align*}
\]
δ ≥ 4 implies \( \gamma_t(G) \leq 3n/7 \) ([15]) and there exists some \( \epsilon > 0 \) such that \( \gamma_t(G) \leq (3/7 - \epsilon)n \) for \( G \neq G_{14} \), where \( G_{14} \) is an incidence bipartite graph of order 14 derived from the Fano plane ([19]).

It is a conjecture that \( \delta \geq 5 \) implies \( \gamma_t(G) \leq 4n/11 \).

**Theorem 2.** ([9, Theorem 29]) Let \( G \) be a connected graph of order \( n > 14 \) with \( \delta \geq 2 \). Then \( \gamma_t(G) = 4n/7 \) if and only if \( G \) can be obtained from a connected graph \( F \) of order at least three by adding \( |V(F)| \) disjoint copies of \( C_6 \), one corresponding to each \( v \in V(F) \), such that either \( v \) is joined by a new edge to a vertex in its corresponding \( C_6 \) or by two new edges to two vertices at distance two apart in its corresponding \( C_6 \).

The family \( \mathcal{G} \cup \mathcal{H} \) is constructed in [3] as follows. Take two copies \( a_1b_1a_2b_2 \ldots a_kb_k \) and \( c_1d_1c_2d_2 \ldots c_kd_k \), of the path \( P_{2k}, k \geq 2 \), and add edges \( a_id_i, b_ic_i \) for \( i = 1, 2, \ldots, k \).

From this the graph of order \( 4k \) belonging to the infinite family \( \mathcal{G} \) is obtained by adding \( a_1c_1 \) and \( b_1d_1 \), while the graph of order \( 4k \) in \( \mathcal{H} \) is obtained by adding \( a_1b_k \) and \( c_1d_k \).

The generalized Petersen graph \( GP_{16} \) is obtained from two circuits \( u_1u_2u_3 \ldots u_7u_8 \) and \( v_1v_2v_3 \ldots v_7v_8 \) by addition of edges \( u_1v_1, u_2v_4, u_3v_7, u_4v_2, u_5v_5, u_6v_8, u_7v_3, u_8v_6 \).

**Theorem 3.** ([12, Theorem 5]) Let \( G \) be a connected graph with \( \delta(G) \geq 3 \). Then \( \gamma_t(G) = n/2 \) if and only if \( G \in \mathcal{G} \cup \mathcal{H} \) or \( G = GP_{16} \).

### 4. \( f_t \) for \( k \)-partitioned graphs with \( \delta \geq 2 \)

We have that \( f_t \) increases with the number of partition classes, i.e., \( f_t(G; k) \leq f_t(G; k + 1) \). Therefore we get a weaker inequality if we partition \( V \) into more than two classes. That is demonstrated in Theorem 4 below.

**Theorem 4.** Let \( G \) be a connected graph of order \( n \) with \( \delta(G) \geq 2 \) and \( G \notin \{C_3, C_5, C_6, C_{10}\} \). If \( k \geq 2 \) then \( f_t(G; k) \leq 11n/7 \).

If \( k = 2 \) then \( f_t(G; k) \leq 3n/2 \). Equality holds if and only if \( G \) is a circuit of length zero modulo four, \( G = C_{4t}, t \geq 1 \).

If \( k = 3 \) then \( f_t(G; k) \leq 11n/7 \). For \( n > 14 \) equality holds if and only if \( G \) can be obtained from a circuit or a path of order at least three by joining each of its vertices by one edge to disjoint copies of \( C_6 \).

If \( k \geq 4 \) then \( f_t(G; k) \leq 11n/7 \) and for \( n > 14 \) equality holds if and only if \( \Delta(G) \leq k \) and \( G \) can be obtained from a connected graph \( F \) having order at least three and \( g_t(F; k) = |V(F)| \) by adding disjoint copies of \( C_6 \), one corresponding to each \( v \in V(F) \), such that either \( v \) is joined by a new edge to one vertex in its corresponding \( C_6 \) or by two new edges to two vertices at distance two apart in its corresponding \( C_6 \).

**Proof.** By Theorem 1 we have \( \gamma_t(G) \leq 4n/7 \) and assigning to each vertex its own class dominator we have \( g_t(G; k) \leq n \). Therefore \( f_t(G; k) = \gamma_t(G) + g_t(G; k) \leq 11n/7 \). The result for \( k = 2 \) is proven by Frendrup, Henning and Vestergaard in [4, Theorem 2]. For \( k \geq 3 \) the equality \( f_t(G; k) = 11n/7 \) implies \( \gamma_t(G) = 4n/7 \) and \( g_t(G; k) = n \) and therefore \( G \) has the structure described in Theorem 2. Since \( g_t(G; k) = n \) each subgraph \( H \) of \( G \) must satisfy \( g_t(H; k) = |V(H)| \) and further \( \Delta(G) \leq k \). Let \( H_1 \) be the graph obtained from
a circuit $C_6: v_1v_2\ldots v_6$ by adding a new vertex $x$ and the edge $xv_1$ and let $H_2 := H_1 + xv_3$. Observe for $k = 3$ that $g_t(H_1; k) = |V(H_1)|$ (obtainable from partitioning $x, v_1, v_2, \ldots, v_6$ into classes indexed 1122133 or 1221133) while $g_t(H_2; k) < |V(H_2)|$. For $k \geq 4$ we can easily show that $g_t(H_3; k) = |V(H_3)|$, $i = 1, 2$. This proves for $k \geq 3$ that $f_t(G; k) = 11n/7$ implies $G$ has the structure described in this theorem. Conversely, assume first that $k = 3$ and that $G$ is obtainable as a disjoint union of $H_1$‘s with edges added between the vertices named $x$, so they span $F$, where $F$ is a path or circuit. We must exhibit a partition of $V(G)$ proving that $f_t(G; k) = 11n/7$, i.e. that $g_t(G; k) = |V(G)|$. It is easy to find a partition $V'_1, V'_2, V'_3$ of $V(F)$ such that $g_t(F; k) = |V(F)|$. If $k = 3$ we can extend this partition to all the $H_i$‘s such that the following holds, which proves that $g_t(G; V'_1, V'_2, V'_3) = n$.

- $N(x) = N_F(x) \cup \{v_1\}$ contains at most one vertex from each $V'_1, V'_2, V'_3$ (just put $v_1$ in the partition set which doesn’t contain any of the two vertices in $N_F(x)$).
- $N(v_1) = \{x, v_2, v_6\}$ contains one vertex from each $V'_1, V'_2, V'_3$ (just put $v_2$ and $v_6$ in the partition sets such that this holds).
- $N(v_3), N(v_5) \subset \{v_2, v_4, v_6\}$, which contains one vertex from each $V'_1, V'_2, V'_3$ (just put $v_4$ in the same set as $x$).
- $N(v_2), N(v_4), N(v_6) \subset \{v_1, v_3, v_5\}$, which contains one vertex from each $V'_1, V'_2, V'_3$ (just put $v_3$ and $v_5$ in the partition sets such that this holds).

Assume next that $k \geq 4$. Then a vertex $x \in F$ may belong to a unit $H_1$ or $H_2$. Again there is a partition $V'_1, V'_2, \ldots, V'_k$ of $V(F)$ such that $g_t(F; k) = |V(F)|$ and similarly to above we can extend this partition to all of $G$, such that the neighbourhood of every vertex in $G$ contains at most one vertex from any partition set. The details are left to the reader. This proves that $g_t(G; k) = n$. \hfill $\square$

5. $g_t$ for two-partitioned graphs with $\delta \geq 3$

Chvátal and McDiarmid [2] and Tuza [16] independently established the following result about transversals in hypergraphs (see also Thomassé and Yeo [15] for a short proof of this result).

**Theorem 5.** ([2,16,15]) If $H$ is a hypergraph with all edges of size at least three, then $\mathcal{T}(H) \leq (|V(H)| + |E(H)|) / 4$.

**Theorem 6.** Let $G$ be a graph of order $n$ with $\delta \geq 3$. Then $g_t(G; 2) \leq 3n/4$.

**Proof.** From the two-partitioned graph $G$, we define for $i = 1, 2$, $H_i$ to be the hypergraph on $n$ vertices and $m_i$ edges where $V(H_i) = V(G)$ and the hyperedges of $H_i$ are the sets of neighbourhoods of class $i$ vertices. In other words, $e \in E(H_i)$ precisely if, for some vertex $v$ in $V_i$, $e = N_G(v)$. Each edge in $H_i$ has at least three vertices because $\delta(G) \geq 3$. In $G$ we see that a set $\mathcal{T}_i$ of vertices totally dominates $V_i$ if and only if $\mathcal{T}_i$ is a transversal of $H_i$. Applying Theorem 5 to $H_1$ and $H_2$ separately we obtain transversals $\mathcal{T}_i$ of $H_i$, $i = 1, 2$, satisfying

$$|\mathcal{T}_1| \leq \frac{m_1+n}{4} \quad |\mathcal{T}_2| \leq \frac{m_2+n}{4}.$$ 

Since $m_1 + m_2 = n$ we obtain $|\mathcal{T}_1| + |\mathcal{T}_2| \leq \frac{m_1+n}{4} + \frac{m_2+n}{4} = \frac{3n}{4}$. This proves Theorem 6. \hfill $\square$

An example of graphs with equality $g_t(G; 2) = 3n/4$ is given in the next section.
6. An infinite family of graphs extremal for Theorem 6

We have the following theorem.

**Theorem 7.** For each integer $r \geq 1$ there exists a connected bipartite graph $G_r$ of order $n = 16r$ with $\delta(G_r) = 3$ such that $g_t(G_r; 2) = 3|V(G_r)|/4$ and $f_t(G_r; 2) \geq 9|V(G_r)|/8$.

**Proof.** We define the graph $G_r$ as follows. Define the vertex set of $G_r$ to be $V(G_r) = W_r \cup A_r \cup B_r$, where

\[
W_r = \{w_0, w_1, w_2, \ldots, w_{8r-1}\} \\
A_r = \{a_0, a_1, a_2, \ldots, a_{4r-1}\} \\
B_r = \{b_0, b_1, b_2, \ldots, b_{4r-1}\}
\]

We define the edge set of $G_r$ such that the following holds, for all $i \in \{0, 1, 2, \ldots, r-1\}$ (where $b_{-1} = b_{4r-1}$ by definition):

\[
N(w_{8i}) = \{a_{4i}, a_{4i+1}, a_{4i+2}, a_{4i+3}\} \\
N(w_{8i+2}) = \{a_{4i}, a_{4i+2}, b_{4i}\} \\
N(w_{8i+4}) = \{a_{4i+2}, b_{4i+1}, b_{4i+2}\} \\
N(w_{8i+6}) = \{a_{4i+3}, b_{4i+1}, b_{4i+3}\}
\]

We now assume $r \geq 1$ is fixed, and therefore omit the subscripts of the above sets and graph. Define $V_1$ and $V_2$ as follows.

\[
V_1 = A \cup \bigcup_{i=0}^{r-1} \{w_{8i+1}, w_{8i+2}, w_{8i+3}, w_{8i+5}\} \\
V_2 = B \cup \bigcup_{i=0}^{r-1} \{w_{8i}, w_{8i+4}, w_{8i+6}, w_{8i+7}\}
\]

We will now show that if $S_i$ is a set such that every vertex in $V_i$ has a neighbour in $S_i$, then $|S_i| \geq 3|V(G)|/8$, for $i = 1, 2$. This would imply that $f_t(G; 2) \geq 9|V(G)|/8$ and $g_t(G) \geq 6|V(G)|/8$ when $k = 2$ (as clearly the above would also imply that $\gamma_t(G) \geq 3|V(G)|/8$). From Theorem 6 follows that $g_t(G) = 3|V(G)|/4$.

Let $S_1$ be a set that totally dominates $V_1$ (i.e. every vertex in $V_1$ has a neighbour in $S_1$). As $w_{8i+5}$ has a neighbour in $S_1$ we note that $|S_1 \cap \{a_{4i+3}, b_{4i+1}, b_{4i+2}\}| \geq 1$, for all $i = 0, 1, 2, \ldots, r-1$. As $w_{8i+1}$, $w_{8i+2}$ and $w_{8i+3}$ all have a neighbour in $S_1$ we note that $|S_1 \cap \{a_{4i}, a_{4i+1}, a_{4i+2}, b_{4i}, b_{4i-1}\}| \geq 2$, for all $i = 0, 1, 2, \ldots, r-1$ (recall that $b_{-1} = b_{4r-1}$).

As the above sets are all disjoint we note that $|S_1 \cap (A \cup B)| \geq 3|A \cup B|/8$.

As $a_{4i+3}$ has a neighbour in $S_1$ we note that $|S_1 \cap \{w_{8i+1}, w_{8i+5}, w_{8i+6}, w_{8i+7}\}| \geq 1$, for all $i = 0, 1, 2, \ldots, r-1$. As $a_{4i}$, $a_{4i+1}$ and $a_{4i+2}$ all have a neighbour in $S_1$ we note that $|S_1 \cap \{w_{8i}, w_{8i+1}, w_{8i+2}, w_{8i+3}, w_{8i+4}\}| \geq 2$, for all $i = 0, 1, 2, \ldots, r-1$. As the above sets are all disjoint we note that $|S_1 \cap W| \geq 3|W|/8$. This implies the desired result for $S_1$.

The fact that if $S_2$ totally dominates $V_2$, then $|S_2| \geq 3|V(G)|/8$ is proved analogously to above. We now just need to show that $G$ is connected. Let $P_i = \{w_{8i}, w_{8i+1}, \ldots, w_{8i+7}\}$ and let $Q_i = \{a_{4i}, a_{4i+1}, a_{4i+2}, a_{4i+3}, b_{4i}, b_{4i+1}, b_{4i+2}, b_{4i+3}\}$ for all $i = 0, 1, 2, \ldots, r-1$. Note that $G[P_i \cup Q_i]$ is connected. As the edges $w_{8i+3}b_{4i-1}$, for all $i = 0, 1, 2, \ldots, r-1$ connects $P_i$ with $Q_{i-1}$ ($Q_{-1} = Q_{r-1}$) we are done.

7. $f_t(G)$ for two-partitioned graphs with $\delta \geq 3$

Let $G$ be a graph of order $n$ with $\delta(G) \geq 3$. 

From Theorems 1 and 6 it follows immediately that $f_t(G; 2) = \gamma_t(G) + g_t(G; k) \leq n/2 + 3n/4 = 5n/4$ when $\delta(G) \geq 3$. We shall in Theorem 8 below prove a slightly stronger result and later pose an even stronger conjecture.

The following result is known (see for example [13]).

**Lemma 1.** ([13]) If $G$ is a 3-regular graph, then there exists a matching $M$ in $G$, such that $|M| \geq \frac{7}{16}|V(G)|$.

**Lemma 2.** Let $H$ be a 2-regular 3-uniform hypergraph with no two edges overlapping. Then $T(H) \leq \frac{|V(H) + E(H)|}{4} - \frac{|V(H)|}{24}$.

*Proof.* Let $H$ be a 2-regular 3-uniform hypergraph with no overlapping edges. Define the graph $G_H$ as follows $V(G_H) = E(H)$ and $E(G_H) = \{e_1e_2 : |V(e_1) \cap V(e_2)| = 1\}$. As there are no overlapping edges and $H$ is 2-regular and 3-uniform, we note that $G_H$ is a 3-regular graph. By Lemma 1, there exists a matching $M$ in $G_H$, such that $|M| \geq \frac{7}{16}|V(G_H)|$.

If $e_1e_2 \in M$, then by the definition of $G_H$ we note that $V(e_1) \cap V(e_2) = \{x_{e_1e_2}\}$ for some $x_{e_1e_2} \in V(H)$. Let $X = \{x_f \mid f \in M\}$ and note that $2|M|$ edges in $H$ contain a vertex from $X$ (as $M$ was a matching). Let $X'$ be a set of vertices of order $|E(H)| - 2|M|$ containing a vertex from every edge in $H$, which does not contain a vertex from $X$. Note that $X \cup X'$ is a transversal of $H$ of order $|M| + (|E(H)| - 2|M|)$. By the above bound on $|M|$ we get the following, as $3|E(H)| = \sum_{x \in V(H)} d(x) = 2|V(H)|$.

$$T(H) \leq |E(H)| - |M| \leq |E(H)| - \frac{7}{16}|E(H)| = \frac{|E(H)|}{4} - \frac{5}{16} \times \frac{2|V(H)|}{3}$$

This completes the proof.

**Lemma 3.** Let $H$ be a 3-uniform hypergraph, where multiple edges are allowed. For each edge and vertex in $H$ we assign a non-empty subset of $\{0, 1, 2\}$. Let this subset be denoted by $L(q)$ for all $q \in V(H) \cup E(H)$. Let $H_i$ be the 3-uniform hypergraph containing vertex-set $V_i = \{v : i \in L(v) \text{ and } v \in V(H)\}$ and edge-set $E_i = \{e : i \in L(v) \text{ and } e \in E(H)\}$, for $i = 0, 1, 2$. Let $Y \subseteq V(H)$ be arbitrary and assume that the following holds.

(a): $\Delta(H_1), \Delta(H_2) \leq 2$
(b): $\Delta(H - E(Y)) \leq 4$.
(c): There are no overlapping edges in $H_i$, $i \in \{1, 2\}$.
(d): If $e \in E(H) - E(Y)$, then $0 \in L(e)$ and $|L(e)| \geq 2$.

This implies that the following holds.

$$\sum_{i=0}^{2} T(H_i) \leq \left(\sum_{i=0}^{2} |V_i| + |E_i|\right) - \frac{|V(H_0) \cap V(H_1) \cap V(H_2) \setminus N_H[Y]|}{372}$$

**Remark.** We assume here in Lemma 3 that the assignment of a set $L(q)$ to each $q$ is done such that $H_0, H_1, H_2$ really are hypergraphs, i.e., such that each hyperedge in $E_i$ consists of vertices from $V_i$, $i = 0, 1, 2$. This requirement will be satisfied in the proof of Theorem 8 where the lemma is applied.

*Proof.* Assume that the lemma is false, and that $H$ is a counterexample with minimum $|E_0| + |E_1| + |E_2|$. Clearly $|E_0| + |E_1| + |E_2| > 0$, as otherwise $\sum_{i=0}^{2} T(H_i) = 0$. For simplicity we will use the following notation:
\[ T^* = \sum_{i=0}^{2} T(H_i) \]
\[ S^* = \sum_{i=0}^{2} \frac{|V_i| + |E_i|}{4} \]
\[ V^* = V(H_0) \cap V(H_1) \cap V(H_2) \]

We recall that \( H \) was assumed to be a “minimal” counterexample to \( T^* \leq S^* - (|V^*| - N_H(Y))/372 \). We will now prove a few claims, which end in a contradiction, thereby proving the lemma. For \( H \) the left hand side of the inequality, \( \ell \), and the right hand side of the inequality, \( r \), in Lemma 3 satisfies \( \ell > r \). We shall construct smaller \( H' \) which also satisfies (a)-(d) and which therefore has \( \ell' \leq r' \) by the minimality of \( H \). \( H' \) is to be constructed such that there exist \( \alpha \leq \beta \) for which \( \ell - \alpha \leq \ell' \) and \( r' \leq r - \beta \). Those inequalities combine to give the desired contradiction \( \ell \leq r \).

**Claim A:** If we add a vertex to \( Y \), then \( N[Y] \) does not increase by more than 9 vertices.

**Proof of Claim A:** This follows from the fact that \( H \) is 3-uniform and \( \Delta(H - E(Y)) \leq 4 \), by (b) in the statement of the lemma.

**Claim B:** There is no \( e = \{v_1, v_2, x\} \in E_i \), such that \( d_{H_i}(v_1) = d_{H_i}(v_2) = 1 \) and \( d_{H_i}(x) = 2 \), for \( i = 0, 1, 2 \).

**Proof of Claim B:** Assume that there is such an edge \( e = \{v_1, v_2, x\} \in E_i \). Let \( e' = \{w_1, w_2, x\} \) be the other edge in \( H_i \) containing \( x \). Now delete \( v_1, v_2, x \) and \( e' \) from \( H_i \) and add \( \{v_1, v_2, x, w_1, w_2\} \) to \( Y \). Note that (a)-(d) still hold and that \( T^* \) decreases by 1 as we simply add \( x \) to any transversal in the new \( H_i \) in order to get a transversal in the old \( H_i \). By Claim A the set \( N[Y] \) does not increase by more than 45 vertices. As \( V^* \) does not decrease by more than 3 vertices and \( S^* \) decreases by 5/4, we are done by the “minimality” of \( H \) (as \( \alpha = 1 \leq 5/4 - 48/372 = \beta \) in the argument above Claim A).

**Claim C:** There is no \( e = \{x, v_1, v_2\} \in E_i \), such that \( d_{H_i}(v_1) = d_{H_i}(v_2) = 2 \) and \( d_{H_i}(x) = 1 \), for \( i = 1, 2 \).

**Proof of Claim C:** Assume that there is such an edge \( e = \{x, v_1, v_2\} \in E_i \). Let \( e_1 = \{w_1, w_2, v_1\} \) be the other edge in \( H_i \) containing \( v_1 \) and let \( e_2 = \{u_1, u_2, v_2\} \) be the other edge in \( H_i \) containing \( v_2 \). As there are no overlapping edges in \( H_i \) (by (c) in the statement of the lemma) we note that \( e_1 \neq e_2 \) and \( |\{w_1, v_2, u_1, u_2\}| \geq 3 \). Let \( S \) be any subset of \( \{w_1, w_2, u_1, u_2\} \) such that \( |S| = 3 \). We now separately consider the cases when addition of \( S \) as a new hyperedge to \( H_i \) causes overlapping edges in \( H_i \), and when it doesn’t.

Assume that adding \( S \) to \( E_i \) does not cause overlapping edges in \( H_i - e_1 - e_2 \). Now delete \( x, v_1, v_2, e, e_1 \) and \( e_2 \) from \( H_i \) and add the edge \( S \) to \( H_i \) (and \( H \)). Furthermore add \( \{x, v_1, v_2, w_1, w_2, u_1, u_2\} \) to \( Y \). Note that (a)-(d) still hold. If \( T' \) is a transversal in the new \( H_i \) then due to the edge \( S \) we either have \( \{u_1, u_2\} \cap T' \neq \emptyset \), in which case \( T' \cup \{v_1\} \) is a transversal in the old \( H_i \), or \( \{w_1, w_2\} \cap T' \neq \emptyset \), in which case \( T' \cup \{v_2\} \) is a transversal in the old \( H_i \). Therefore \( T^* \) decreases by at most one. By Claim A we have that \( N[Y] \) does not increase by more than 63 vertices. As \( V^* \) does not decrease by more than 3 and \( S^* \) decreases by 5/4, we are done by the “minimality” of \( H \) (as \( 1 \leq 5/4 - 66/372 \).

So now assume that the above addition of \( S \) would cause overlapping edges in \( H_i - e_1 - e_2 \). This can only happen if there is an edge \( e' \in E_i \) such that \( |S \cap V(e')| \geq 2 \). Note that by (a) the degree in \( H_i \) is two for all vertices in \( S \cap V(e') \) (they only lie in \( S \) and \( e' \)). Now delete the vertices \( \{x, v_1, v_2\} \cup (S \cap V(e')) \) from \( H_i \) and delete the edges \( e, e_1, e_2 \) and \( e' \) from \( H_i \) (do not add the edge \( S \) to \( H_i \)). Furthermore add \( \{x, v_1, v_2, w_1, w_2, u_1, u_2\} \cup (V(e') \setminus S) \) to \( Y \). Note that (a)-(d) still hold. By a similar argument to above we note that \( T^* \) decreases
by at most two. By Claim A we see that \(N[Y]\) does not increase by more than 72 vertices. As \(V^*\) does not decrease by more than 6 and \(S^*\) decreases by at least \(9/4\), we are done by the “minimality” of \(H\) (as \(2 \leq 9/4 - 78/372\)).

Claim D: There is no edge \(e = \{x, v_1, v_2\} \in E_0\), such that \(d_{H_0}(v_1) = d_{H_0}(v_2) = 2\) and \(d_{H_0}(x) = 1\) and \(|N_{H_0}[V(e)]| \geq 6\).

Proof of Claim D: Assume that there is such an edge \(e = \{x, v_1, v_2\} \in E_0\). Let \(e_1 = \{w_1, w_2, v_1\}\) be the other edge in \(H_0\) containing \(v_1\) and let \(e_2 = \{u_1, u_2, v_2\}\) be the other edge in \(H_0\) containing \(v_2\). If \(e_1 = e_2\), then \(|N_{H_0}[V(e)]| \leq 4\), a contradiction. So assume that \(e_1 \neq e_2\). As \(|N_{H_0}[V(e)]| \geq 6\) we note that \(|\{w_1, w_2, u_1, u_2\}| \geq 3\). We are now done analogously to Claim C.

Claim E: \(\Delta(H_1), \Delta(H_2) \leq 1\).

Proof of Claim E: Assume that \(\Delta(H_1) \geq 2\). By (a) we have \(\Delta(H_1) = 2\). By Claim B and Claim C we note that there is a 2-regular component, \(R\), in \(H_1\). There are no overlapping edges in \(R\) by (c). By Lemma 2 there is a transversal \(T_R\) in \(R\) of order at most \((|V(R)| + |E(R)|)/4 - |V(R)|/24\). So delete all edges and vertices in \(R\) and add all vertices in \(R\) to \(Y\). By Claim A we have that \(N[Y]\) increases by at most \(9|V(R)|/24\) vertices. We now have a contradiction to the “minimality” of \(H\), as \(|V(R)|/24 \geq 9|V(R)|/372\). Analogously we can show that \(\Delta(H_2) \leq 1\).

Claim F: Assume \(e_1, e_2 \in E(H_0)\) overlap and \(e_i = \{x_i, v_1, u_i\}\) for \(i = 1, 2\), where \(u_1 \neq u_2\). If \(d_{H_0}(x_1) = d_{H_0}(x_2) = 2\), then there is an edge \(e' \in E(H_0)\) such that \(\{u_1, u_2\} \subseteq V(e')\).

Proof of Claim F: Let \(e_1\) and \(e_2\) be defined as in the Claim, and assume that there is no edge \(e' \in E(H_0)\) such that \(\{u_1, u_2\} \subseteq V(e')\). Delete \(e_1, e_2, x_1, x_2\) and \(u_1\) from \(H_0\). For every edge, \(e''\), in \(H_0\) that contains \(u_1\), delete \(e''\) and add the edge \((e'' \setminus \{u_1\}) \cup \{u_2\}\) instead. Furthermore add \(\{x_1, x_2, u_1, u_2\}\) and \(V(e'')\) from all transformed edges, to \(Y\). As there is at most \(4\) edges containing \(u_1\) in \(H_0 - E(Y)\) we note that \(Y\) increases by at most \(10\) (the neighbours of \(u_1\) in \(H_0 - E(Y)\) and \(\{u_1, u_2\}\)). Therefore \(V^* - N[Y]\) decreases by at most \(3 + 90\), by Claim A. We also note that \(S^*\) decreases by \(5/4\).

We now show that \(T^*\) decreases by at most one. If \(u_2 \in T^*\) then \(T^* \cup \{u_1\}\) is a transversal in the old \(H_0\). If \(u_2 \not\in T^*\) then \(T^* \cup \{x_1\}\) is a transversal in the old \(H_0\). As (a)-(d) still holds after the above operations, we have a contradiction to the “minimality” of \(H\), as \(1 \leq 5/4 - 93/372\).

Definition G: Let \(x \in V^* - N[Y]\) be arbitrary. The vertex \(x\) exists since otherwise we would be done by Theorem 5.

Claim H: \(d_{H_1}(u) = d_{H_2}(u) = 1\) for all \(u \in N_{H_0}[x]\), where \(x\) is defined in Definition G.

Proof of Claim H: Assume that \(u \in N_{H_0}[x]\) has \(d_{H_2}(u) = 0\) or \(u \not\in V(H_2)\), which are the only possibilities for \(u\), if \(d_{H_0}(u) \neq 1\) (by Claim E). If \(u \in V(H_2)\) and \(d_{H_0}(u) = 0\), then delete \(u\) from \(V(H_2)\). We are now done as \(T^*\) is unchanged, \(S^*\) decreases by \(1/4\) and \(V^* - N[Y]\) does not decrease by more than one. So we may assume that \(u \not\in V(H_2)\). Since \(x \in V^*\) we note that \(x \in V(H_1)\) and \(x \in V(H_2)\), which by the above argument implies that \(d_{H_1}(x) = d_{H_2}(x) = 1\) and \(u \neq x\). Let \(e_1 = \{x, u, q\}\) be the edge in \(H_1\) (and \(H_0\)) containing \(u\) and \(x\). Let \(e_2\) be the edge in \(H_2\) (and \(H_0\)) that contains \(x\). Note that \(d_{H_1}(x) = 2\) and \(d_{H_0}(u) = 1\). If \(d_{H_0}(q) = 1\) then we are done by Claim B. So \(d_{H_0}(q) \geq 2\). However as any edge containing \(q\) must also lie in \(H_1\) or \(H_2\), as \(q \not\in Y\), we note that
$d_{H_0}(q) = 2$. Let $e_q$ be the edge in $H_2$ that contains $q$. Note that $e_q \neq e_2$, by Claim F. As $e_q$ and $e_2$ do not intersect we note that $|N_{H_0}[V(e)]| = 7 \geq 6$, so we are done by Claim D.

Claim I: Let $e_1 \in E_1$ and $e_2 \in E_2$ be the edges containing $x$ (defined in Definition G). They exist by Claim H. Then $V(e_1) \cap V(e_2) = \{x\}$.

Proof of Claim I: Assume for the sake of contradiction that $|V(e_1) \cap V(e_2)| \geq 2$. If $|V(e_1) \cap V(e_2)| = 3$, then we delete $e_1$ from $H_0$ and add $V(e_1)$ to $Y$. This contradicts the "minimality" of $H$, as $T^*$ remains unchanged, $S^*$ decreases by $1/4$ and $N[Y]$ increases from Claim A by at most 27. Therefore assume that $|V(e_1) \cap V(e_2)| = 2$. Let $e_1 = \{x, v, w\}$ and let $e_2 = \{x, y, v\}$ where $w \neq y$. As $d_{H_0}(x) = d_{H_0}(v) = 2$, there is an edge, $e'$, in $H_0$ such that $\{w, y\} \subseteq V(e')$, by Claim F. However $e' \notin E(H_1)$ and $e' \notin E(H_2)$ by Claim E. This is however a contradiction to (d), as $w, y \notin Y$.

Claim J: We now obtain a contradiction.

Proof of Claim J: Let $e_1 \in E_1$ and $e_2 \in E_2$ be the edges containing $x$ (defined in Definition G). They exist by Claim H and $V(e_1) \cap V(e_2) = \{x\}$, by Claim I. Let $e_1' = \{x, v_1, v_2\}$ and let $e_2' = \{x, w_1, w_2\}$. Let $e_1''$ be the edge in $H_1$ containing $w_1$ and let $e_2''$ be the edge in $H_1$ containing $w_2$ (they exist by Claim H). Let $e_2'$ be the edge in $H_2$ containing $v_1$ and let $e_2''$ be the edge in $H_2$ containing $v_2$ (they exist by Claim H).

If $e_1' = e_2''$, then $V(e_1') \cap V(e_2) = \{w_1, w_2\}$ and $e_1' = \{w_1, w_2, r\}$ for some $r \in V(H_0)$. By Claim F, there is an edge in $H_0$ that contains $x$ and $r$. But this is a contradiction, as neither $e_1$ or $e_2$ contain $r$, by Claim H. Therefore $e_1' \neq e_2''$. Analogously we can show that $e_1' \neq e_2''$.

We now delete $e_1, e_1', e_2''$ from $H$, $H_0$ and $H_1$. Delete $e_2, e_2', e_2''$ from $H$, $H_0$ and $H_2$. Delete $V(e_1) \cup V(e_1') \cup V(e_2'')$ from $V(H_1)$ and delete $V(e_2) \cup V(e_2') \cup V(e_2'')$ from $V(H_2)$. Delete $V(e_1) \cup V(e_2)$ from $H$ and $H_0$. Let $S_1$ be any subset of size three in $V(e_1') \cup V(e_2') - \{w_1, w_2\}$ and let $S_2$ be any subset of size three in $V(e_2') \cup V(e_2'') - \{v_1, v_2\}$. Add the edges $S_1$ and $S_2$ to $H$ and $H_0$. Finally add all vertices in $V(e_1') \cup V(e_1') \cup V(e_2) \cup V(e_2') - \{w_1, w_2, v_1, v_2, x\}$ to $Y$.

We first show that $T^*$ decreases by at most 8. It is clear that the transversal size drops by three in both $H_1$ and $H_2$. So assume that $T^*$ is a transversal of the new $H_0$. As in the proof of Claim C we note that one of the three edges $e_1, e_2, e_2''$ are already covered by a vertex in $T^*$ (due to $S_2$) and the other two edges can be covered by one additional vertex. Similarly by adding one more vertex to $T^*$ we can make sure that $e_2, e_1', e_1''$ are all covered. Therefore the transversal size drops by at most two in $H_0$.

Note that $S^*$ drops by $33/4$ as we delete 9 vertices in each of $H_1$ and $H_2$ and we delete 5 vertices in $H_0$. We also delete three edges in each of $H_1$ and $H_2$ and six edges in $H_0$. But we also add two edges in $H_0$.

$N[Y]$ increases by at most 72 vertices by Claim A, as $|V(e_1') \cup V(e_1') \cup V(e_2) \cup V(e_2') - \{w_1, w_2, v_1, v_2, x\}| \leq 8$. As $V^*$ decreases by at most 13, we note that $V^* - N[Y]$ decreases by at most 85. We note that (a)-(d) still holds after the above operations. We therefore have a contradiction to the "minimality" of $H$, as $8 \leq 33/4 - 85/372$.

Theorem 8. If $G$ is a graph with $\delta(G) \geq 3$ then $f_t(G; 2) \leq \left(\frac{3}{4} - \frac{1}{372}\right)|V(G)|$.

Proof. Let $G$ be any graph with $\delta(G) \geq 3$ and let $(W_1, W_2)$ be a partition of $V(G)$. Define the hypergraph $H_G$, such that $V(H_G) = V(G)$ and $E(H_G)$ is obtained by selecting for each $v \in V(G)$ one set of three vertices from $N_G(v)$ to form a hyperedge. $E(H_G) =$
\{e_v : v \in V(G)\}, e_v = \{x_v, y_v, z_v\} \subseteq N_G(v). Furthermore for every hyperedge, e \in E(H_2)
let L(v) be the set \{0, i\} if v \in W_i. For reasons which will be clear later we let L(v) = \{0, 1, 2\} for every v \in V(H_2). Let H_i be the 3-uniform hypergraph containing vertex-set
\(V_i = \{v : i \in L(v) \text{ and } v \in V(H)\}\) and edge-set \(E_i = \{e : i \in L(e) \text{ and } e \in E(H)\}\), for
\(i = 0, 1, 2\). Note that a transversal of \(H_0\) corresponds to a total dominating set in \(G\) and a transversal of \(H_i\) \((i \in \{1, 2\})\) corresponds to a total dominating set in \(G\) of the set \(W_i\). Therefore we would be done if we could show that \(T(H_0) + T(H_1) + T(H_2) \leq\)
\((\frac{2}{5} - \frac{372}{372})|V(G)|\). Let \(Y\) be an empty set. We note that \(|E_1| + |E_2| = |V_0| = |V_1| = |V_2| = |V(H_0) \cap V(H_1) \cap V(H_2) \setminus N_H[Y]| = |V(G)|\) and therefore the inequality above is equivalent to

\[
(*) \quad \sum_{i=0}^{2} T(H_i) \leq \left( \sum_{i=0}^{2} |V_i| + |E_i| \right) - \frac{|V(H_0) \cap V(H_1) \cap V(H_2) \setminus N_H[Y]|}{372}
\]

For simplicity we will use the following notation:

\(T^* = \sum_{i=0}^{2} T(H_i)\)

\(S^* = \sum_{i=0}^{2} \frac{|V_i| + |E_i|}{4}\)

\(V^* = V(H_0) \cap V(H_1) \cap V(H_2)\)

We will now do a few transformations on \(H, H_0, H_1, H_2\).

**Transformation 1:** While there is some vertex \(x \in V(H)\) with \(d_{H_0}(x) \geq 5\) (or equivalently \(d_H(x) \geq 5\)), delete \(x\) and all edges incident with \(x\) from \(H\) (and therefore also from \(H_0, H_1\) and \(H_2\)).

**Claim A:** If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.

**Proof of Claim A:** We note that \(T^*\) drops by at most three, as we may place \(x\) in the transversal of the new \(H_i\)’s in order to get transversals in the old \(H_i\)’s. We note that \(S^*\) decreases by at least 13/4, as we delete \(x\) from \(H_0, H_1, H_2\) and 5 edges from \(H_0\) plus a total of 5 edges from \(H_1\) and \(H_2\). As \(V^*\) decreases by one and \(N_H[Y] = \emptyset\) remains unchanged, we are done.

**Transformation 2:** While there is a vertex \(x \in V(H)\) with \(d_{H_1}(x) \geq 3\), delete \(x\) and all edges incident to \(x\) from \(H_0\) and \(H_1\). Also delete these edges from \(H\) (but do not delete \(x\) or any edges incident to \(x\) in \(H_2\)). If \(d_{H_2}(x) = 0\) then delete \(x\) from \(H_2\) (i.e. delete 2 from \(L(x)\)). If \(d_{H_2}(x) > 0\) then note that \(d_{H_2}(x) = 1\) (as we have performed transformation 1 as long as we could) and put \(N_{H_2}[x]\) in \(Y\).

**Claim B:** If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.

**Proof of Claim B:** We note that \(T^*\) drops by at most two, as we may place \(x\) in the transversal of the new \(H_0\) and \(H_1\) in order to get transversals in the old \(H_0\) and \(H_1\). We note that \(S^*\) decreases by at least 9/4, as we delete 3 edges and 1 vertex from \(H_0\) and \(H_1\) and we either delete a vertex in \(H_2\) or 4 edges from \(H_0\). As \(V^*\) decreases by one and \(N_H[Y]\) increases by at most 21 (as \(\Delta(H) \leq 4\), after Transformation 1), we are done.

**Transformation 3:** While there is a vertex \(x \in V(H)\) with \(d_{H_2}(x) \geq 3\), then do the following. Delete \(x\) and all edges incident to \(x\) from \(H_0\) and \(H_2\). Also delete these edges from \(H\) (but do not delete \(x\) or any edges incident to \(x\) in \(H_1\)). Furthermore delete any
vertices in $H_2$, which get degree zero by the above transformation. If $d_{H_1}(x) = 0$ then delete $x$ from $H_1$. If $d_{H_1}(x) > 0$, then we put $N_{H_1}[x]$ in $Y$.

**Claim C:** If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.

**Proof of Claim C:** We note that $T^*$ drops by at most two, as we may place $x$ in the transversal of the new $H_0$ and $H_2$ in order to get transversals in the old $H_0$ and $H_2$. Let $s$ count any edge, $e$, in $H_1$, which does not lie in $H_0$ as contributing $1 + |V(e) \cap V(H_0)| / 3$ to the sum $S^*$. We note that there are no such edges when we start the transformation 3’s.

We note that $S^*$ now decreases by at least $25/12$, because of the following. For every edge containing $x$ in $H_2$, which does not lie in $H_0$ there is a vertex of degree one in the edge, due to the above transformations. Therefore we either delete an edge in $H_0$ or a vertex in $H_2$ for each of the edges containing $x$ in $H_2$. As we also delete the edges in $H_2$ and the vertex $x$ in $H_0$ and $H_2$ we note that $S^*$ drops by at least $8/4$. So if $d_{H_1}(x) = 0$ then $S^*$ decreases by at least $9/4$ as claimed. If $d_{H_1}(x) > 0$ and the edge, $e$, containing $x$ in $H_1$ also lies in $H_0$, then we are done as we delete an extra edge in $H_0$ and the edge left in $H_1$ is counted as at most $1 + 2/3$. If $d_{H_1}(x) > 0$ and the edge, $e$, containing $x$ in $H_1$ does not lie in $H_0$, then we decrease the value of $e$ by $1/3$ as $1 + |V(e) \cap V(H_0)| / 3$ decreases. This shows that $S^*$ decreases by at least $25/12$.

As $V^*$ decreases by one and $N[Y]$ increases by at most $21$ (as $\Delta(H) \leq 4$, after Transformation 1), we are done.

**Transformation 4:** If $e_1, e_2 \in E(H_i)$ and $|V(e_1) \cap V(e_2)| \geq 2$ for some $i \in \{1, 2\}$, then we do the following.

If $|V(e_1) \cap V(e_2)| = 3$, then if $e_1, e_2 \in E_0$ we delete $e_2$ from both $H_0$ and $H_i$. If $e_j \notin E_0$ $(j \in \{1, 2\})$ then we delete $e_j$ from $H_i$ (in this case $V(e_j) \subseteq Y$). So now assume that $|V(e_1) \cap V(e_2)| = 2$ and $e_1 = (u_1, x, y)$ and $e_2 = (u_2, x, y)$, where $u_1 \neq u_2$.

If $d_{H_i}(u_1) = d_{H_i}(u_2) = 2$, then by the above transformations we note that $e_1, e_2 \in E_0$. We now add a new vertex $q$ to $H$, $H_0$ and $H_i$. We delete $e_1$ and $e_2$ from $H$, $H_i$ and $H_0$ and add the edges $\{q, x, y\}$ to $H$, $H_i$ and $H_0$.

If $d_{H_i}(u_j) = 1$, for some $j \in \{1, 2\}$, then do the following. Delete $e_1$, $e_2$ and the vertices $\{u_j, x, y\}$ from $H_i$. Add the vertices $\{u_1, u_2, x, y\}$ to $Y$.

**Claim D:** If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.

**Proof of Claim D:** In the case when $|V(e_1) \cap V(e_2)| = 3$ we note that $T^*$ remains unchanged, $S^*$ decreases by $1/4$ and $V^* - N[Y]$ remains unchanged. We are now done with this case.

In the case when $d_{H_1}(u_1) = d_{H_1}(u_2) = 2$, we note that $T^*$, $S^*$ and $V^*$ remain unchanged and $N[Y]$ can only grow by adding $q$ to it, but $q \notin V^*$. We also note that the above transformation decreases the number of edges in $H_i$, so it cannot continue indefinitely. We are now done with this case.

In the case when $d_{H_1}(u_j) = 1$, we note that $T^*$ decreases by at most one, $S^*$ decreases by $5/4$, $V^*$ decreases by at most three and $N[Y]$ increases by at most 24 (In $H - e_1 - e - 2$ we note that $u_1$ and $u_2$ have degree at most 3 while $x$ and $y$ have degree at most 2). As $1/4 \geq 27/372$ we are done with this case.
Claim E: \( \Delta(H_1), \Delta(H_2) \leq 2 \) and \( \Delta(H - E(Y)) \leq 4 \) and there are no overlapping edges in \( H_i, i \in \{1, 2\} \).

Proof of Claim E: The fact that \( \Delta(H_1), \Delta(H_2) \leq 2 \) follow from Transformations 2 and 3. As \( \Delta(H) \leq 4 \) after Transformation 1 and no other transformation increases \( \Delta(H) \), we note that \( \Delta(H - E(Y)) \leq \Delta(H) \leq 4 \). There are no overlapping edges in \( H_i, i \in \{1, 2\} \) due to Transformation 4.

Claim F: If \( e \in E(H) - E(Y) \), then \( 0 \in L(e) \) and \( |L(e)| \geq 2 \).

Proof of Claim F: This was true before Transformation 1 as it was true for all edges. Transformation 1 clearly does not change this property. In Transformation 2, we only keep an edge, \( e \), in \( H_i \), where \( i \in \{1, 2\} \) but delete it in \( H_0 \) if we put \( V(e) \) in \( Y \). So the above still holds after Transformation 2. Analogously it also holds after Transformation 3. It is not difficult to check that it also holds after Transformation 4 (note that the above property holds for the edge we might add to \( H \) in Transformation 4).

We now see that (*) holds due to Lemma 3. That implies the theorem.

8. Possible strengthening of Theorem 8

No graph extremal for Theorem 8 is known and probably an inequality \( f_t(G; 2) \leq \alpha |V(G)| \) can be obtained for some \( \alpha \) smaller than \( \frac{5}{4} - \frac{1}{372} \). Certainly \( \alpha \) must be at least \( 9/8 \), that is demonstrated by the graphs of section 6.

There is a graph of order 12 having \( f_t(H_{12}; 2) = 7n/6 \), namely \( H_{12} \) from the family \( \mathcal{H} \) defined after Theorem 2, with the two \( P_6 \)'s as its partition classes. Unless we, e.g., demand that the order of the graphs be large, \( H_{12} \) shows that we cannot get a better inequality than the following conjecture.

Conjecture 1. Let \( G \) be a graph of order \( n \) with \( \delta \geq 3 \) then \( f_t(G; k) \leq 7n/6 \).

9. Three partition classes

Theorem 9. Let \( G \) be a graph of order \( n \) with \( \delta \geq 3 \) then \( f_t(G; 3) \leq 3n/2 \).

For arbitrarily large \( n \), \( n \equiv 0 \pmod{6} \), there exist graphs \( G_n \) with \( g_t(G_n; 3) = n \), \( \gamma_t(G_n) = n/3 \), \( f_t(G; 3) = 4n/3 \).

Proof. By Theorem 1 we have that \( \gamma_t(G) \leq n/2 \), and \( g_t(G; 3) \leq n \) holds trivially, so by addition we get \( f_t(G; 3) \leq 3n/2 \) as desired.

Assume a graph \( G \) has \( g_t(G; 3) = n \). Then \( \Delta(G) \leq 3 \) and as \( \delta(G) \geq 3 \), \( G \) is cubic. Since each vertex has three neighbours, one in each partition class, we see for each \( i = 1, 2, 3 \), that vertices in class \( V_i \) span a matching in \( G \).

Listing the 3 neighbours to each \( V_i \)-vertex we count each vertex of \( G \) once, so \( 3|V_i| = n \) giving \( |V_1| = |V_2| = |V_3| = n/3 \).

Each \( V_1 \)-vertex is adjacent to precisely one \( V_2 \)-vertex and that has no other \( V_1 \)-neighbour, so there is a perfect matching of \( V_1V_2 \)-edges and analogously \( G \) contains perfect matchings of \( V_1V_3 \)- and \( V_2V_3 \)-edges.

One partition class \( V_i \) totally dominates \( G \) so \( \gamma_t(G) \leq n/3 \). In fact, \( \gamma_t(G) = n/3 \) because each vertex in \( G \) can totally dominate at most its three neighbours.
Following the steps above, it is now easy for \( n \equiv 0 \pmod{3} \) to construct a graph \( G_n \) with \( g_t(G_n; 3) = n \). This graph has \( f_t(G_n; 3) = \gamma_t(G_n) + g_t(G_n; 3) = 4n/3 \).

We do not know if there, for \( \delta \geq 3 \), are graphs \( G \) with \( 4n/3 < f_t(G; 3) \leq 3n/2 \), but we pose the following conjecture.

**Conjecture 2.** There exists some positive \( \epsilon \) such that the following holds. If \( G \) is a graph with \( \delta(G) \geq 3 \), then \( f_t(G; 3) \leq (3/2 - \epsilon)|V(G)| \).

**Theorem 10.** Let \( G \) be a graph of order \( n \) with \( \delta \geq 3 \) and let \( k \geq 4 \). \( f_t(G; k) \leq 3n/2 \) and there exists an infinite family of graphs with \( f_t(G; k) = 3n/2 \).

**Proof.** The inequality is proven as in Theorem 9. For a graph with \( f_t(H; k) = 3n/2 \) take \( H \in \mathcal{H} \) (\( \mathcal{H} \) is defined after Theorem 2). Let \( v_1, v_2, \ldots, v_{n/2} \) and \( u_1, u_2, \ldots, u_{n/2} \) be two disjoint paths in \( H \) such that \( \{v_1u_2, v_2u_1, v_{1}v_{n/2}, u_{1}u_{n/2}\} \subseteq E(H) \). Let \( V_1, V_2, V_3, V_4 \) be a partition of \( H \) such that \( l(v_1), l(v_2), \ldots, l(v_{n/2}) = 1, 2, 3, 4, 1, 2, 3, 4, \ldots \) and \( l(u_1), l(u_2), \ldots, l(u_{n/2}) = 4, 3, 2, 1, 4, 3, 2, 1, \ldots \) where \( l(x) = i \) if \( x \in V_i \), then \( f_t(H; V_1, V_2, V_3, V_4) = 3n/2 \).

**References**


19. A. Yeo, Excluding one graph significantly improves bounds on total domination in connected graphs of minimum degree four. *In preperation.*

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