Subdigraphs of almost Moore digraphs induced by fixpoints of an automorphism

by

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Department of Mathematical Sciences
Aalborg University
Fredrik Bajers Vej 7G, 9220 Aalborg East, Denmark
anita@math.aau.dk

Abstract

The degree/diameter problem for directed graphs is the problem of de-
termining the largest possible order for a digraph with given maximum out-
degree $d$ and diameter $k$. An upper bound is given by the Moore bound
$M(d,k) = \sum_{i=0}^{k} d^i$ and almost Moore digraphs are digraphs with maximum
out-degree $d$, diameter $k$ and order $M(d,k) - 1$.

In this paper we will look at the structure of subdigraphs of almost Moore
digraphs, which are induced by the vertices fixed by some automorphism
$\varphi$. If the automorphism fixes at least three vertices, we prove that the
induced subdigraph is either an almost Moore digraph or a diregular $k$-
geodetic digraph of degree $d' \leq d - 2$, order $M(d',k) + 1$ and diameter
$k + 1$.

As it is known that almost Moore digraphs have an automorphism $r$,
these results can help us determine structural properties of almost Moore
digraphs, such as how many vertices of each order there are with respect to
$r$. We determine this for $d = 4$ and $d = 5$, where we prove that except in
some special cases, all vertices will have the same order.

1. Introduction

Let $G$ be a digraph and $u$ be a vertex of maximum out-degree $d$ in $G$,
and let $n_i$ denote the number of vertices in distance $i$ from $u$. Then we have
$n_i \leq d^i$ for $i = 0,1,\ldots,k$, and thus the order $n$ of $G$ is bounded by

$$ n = \sum_{i=0}^{k} n_i \leq \sum_{i=0}^{k} d^i. $$

(1)
If equality is obtained in (1) we say that $G$ is a Moore digraph of degree $d$ and diameter $k$, and the right-hand side of (1) is called the Moore bound denoted by $M(d, k) = \sum_{i=0}^{k} d^i$. Moore digraphs are known to be diregular and exist only when $d = 1$ (cycles of length $(k + 1)$) or $k = 1$ (complete digraphs with order $d + 1$), see [1] or [2]. So we are interested in knowing how close the order can get to the Moore bound for $d > 1$ and $k > 1$. Let $G$ be a digraph of maximum out-degree $d$, diameter $k$ and order $M(d, k) - \delta$, then we say $G$ is a $(d, k, -\delta)$-digraph or alternatively a $(d, k)$-digraph of defect $\delta$. When $\delta < M(d, k - 1)$ we have out-regularity, see [3], whereas it in general is not known if we also have in-regularity. Of special interest is the case $\delta = 1$, and a $(d, k, -1)$-digraph is also denoted as an almost Moore digraph. Almost Moore digraphs do exist for $k = 2$ as the line digraphs of $K_{d+1}$ for any $d \geq 2$, see [4], whereas $(2, k, -1)$-digraphs for $k > 2$, $(3, k, -1)$-digraphs for $k > 2$, $(d, 3, -1)$-digraphs for $d > 1$ and $(d, 4, -1)$-digraphs for $d > 1$ do not exist, see [5], [6], [7] and [8]. We do know that almost Moore digraphs are diregular for $d > 1$ and $k > 1$, see [3].

In the last section of the paper, we will be needing the following theorem which summarises some of the above results.

**Theorem 1 ([5], [6]).** Almost Moore digraphs of degree 2 and 3 and diameter $k > 2$ do not exist.

Furthermore, almost Moore digraphs satisfies the following properties, where $a \leq k$-walk is a walk of length at most $k$.

**Lemma 1 ([9]).** Let $G$ be an almost Moore digraph, then

- for each pair of vertices $u, v \in V(G)$ there is at most one $< k$-walk from $u$ to $v$,
- for every vertex $u \in V(G)$ there exist a unique vertex $r(u)$ such that there are two $\leq k$-walks from $u$ to $r(u)$.

The mapping $r : V(G) \rightarrow V(G)$ is in fact an automorphism, see [9] and thus the two $\leq k$-walks from $u$ to $r(u)$ are internally disjoint. The vertex $r(u)$ is said to be the repeat of $u$. If we have $u = r(u)$, thus $u$ has order 1 with respect to $r$, $u$ is said to be a selfrepeat. If there is a selfrepeat in $G$, then there are exactly $k$ selfrepeats, which lie on a $k$-cycle, see [10].

In this paper we will give some conditions for the existence of an almost Moore digraph $G$ with respect to some automorphism $\varphi : V(G) \rightarrow V(G)$. These results can then be used to investigate the orders of the vertices with respect to the automorphism $r$. Before stating the core result of this paper,
we will introduce another type of digraph which shows to be important when characterizing induced subdigraphs of almost Moore digraphs.

Let $D$ be a digraph such that for each pair of vertices $u, v \in V(D)$ we have at most one $k$-walk from $u$ to $v$, then we say $D$ is $k$-geodetic. Let $u$ be a vertex of minimum out-degree $d$, and let $n_i$ be the number of vertices in distance $i$ from $u$ for $i = 0, 1, \ldots, k$. Then $n_i \geq d^i$ and the order $n$ of $D$ is bounded by

$$n \geq \sum_{i=0}^{k} n_i \geq \sum_{i=0}^{k} d^i. \quad (2)$$

Notice that the right-hand side is the Moore bound, $M(d, k)$ and that the diameter for a $k$-geodetic digraph is at least $k$. As we already know, Moore digraphs do only exist for $d = 1$ or $k = 1$, we wish to know how close the order of a $k$-geodetic digraph can get to the Moore bound. By a $(d, k, \epsilon)$-digraph we understand a $k$-geodetic digraph of minimum out-degree $d$ and order $M(d, k) + \epsilon$. Alternatively we say that we have a $(d, k)$-digraph of excess $\epsilon$. The first case which is interesting is when $\epsilon = 1$. A $(d, k, 1)$-digraph has diameter $k + 1$, and for each vertex $u$ there is exactly one vertex, the outlier $o(u)$ such that $\text{dist}(u, o(u)) = k + 1$, see [11].

A $(d, k, 1)$-digraph is diregular if and only the mapping $o : V(D) \mapsto V(D)$ is an automorphism, see [11]. From [11] we also have the following theorem.

**Theorem 2** ([11]). No diregular $(2, k, 1)$-digraphs exist for $k > 1$.

### 2. Results

For simplicity, we will, in the remaining part of this paper, let a $(d, k, -1)$-digraph (almost Moore digraphs) denote any digraph which has degree $d > 0$, diameter $k > 0$ and order $M(d, k) - 1$, thus we will let $k$-cycles be included in this class. Similar, a $(d, k, 1)$-digraph will denote any $k$-geodetic digraph of minimum out-degree $d > 0$ and order $M(d, k) + 1$.

The scope of this paper is to prove the following theorem.

**Theorem 3.** Let $G$ be an almost Moore digraph of degree $d \geq 4$ and diameter $k \geq 3$ and let $H$ be a subdigraph induced by the vertices which are fixed by some automorphism $\varphi : V(G) \mapsto V(G)$. Then $H$ is either

- the empty digraph,
- two isolated vertices,
- an almost Moore digraph of degree $d' \leq d$ and diameter $k$ or
a directed $(d', k, 1)$-digraph where $d' \leq d - 2$.

In the remaining part of this paper we will assume $G$ to be an almost Moore digraph of degree $d \geq 4$ and diameter $k \geq 3$, and $H$ to be a subdigraph of $G$ induced by the fixpoints of some automorphism $\varphi : V(G) \rightarrow V(G)$.

We start by stating some properties of the fixpoints of $G$.

**Lemma 2.** Let $u$ and $v$ be fixpoints of $G$ with respect to the automorphism $\varphi$, then

- $r(u)$ is a fixpoint,
- if there is a $\leq k$-walk $P$ from $u$ to $v$ and $v \neq r(u)$, all vertices $w \in P$ are fixpoints
- if $v = r(u)$ and $P$ and $Q$ are the two $\leq k$-walks from $u$ to $v$, either all internal vertices on $P$ and $Q$ are fixpoints, or none of them are. Furthermore, if $\text{dist}(u, r(u)) < k$, then all vertices on $P$ and $Q$ are fixpoints.

**Proof.**
- We know there are two $\leq k$-walks, $P$ and $Q$, from $u$ to $r(u)$. Now, $\varphi(P)$ and $\varphi(Q)$ are two $\leq k$-walks from $u$ to $\varphi(r(u))$, and hence $\varphi(r(u))$ is a repeat of $u$. As $u$ only has one repeat, the statement follows.
- Let $P$ be the unique $\leq k$-walk from $u$ to $v$. Then $\varphi(P)$ will also be a $\leq k$-walk from $u$ to $v$, and hence $P = \varphi(P)$.
- Assume not all vertices on the $\leq k$-walk $P$ are fixpoints, hence there exist a vertex $w \in P$ such that $w \neq \varphi(w)$ and thus $\varphi(P) \neq P$ is also a $\leq k$-walk from $u$ to $v = r(u)$. As there are only two $\leq k$-walks from $u$ to $v = r(u)$, we must have $\varphi(P) = Q$ and thus none of the internal vertices of $P$ are fixpoints, as $P$ and $Q$ are internally disjoint. Now if $\text{dist}(u, r(u)) < k$, then $P$ and $Q$ are obviously of different length, so we must have all vertices on $P$ and $Q$ as fixpoints.

**Corollary 1.** Let $\varphi$ be an automorphism of $G$, then all $\leq k$-walks among the fixpoints of $\varphi$ in $G$ are preserved to $H$, except for possibly the $k$-walks from a vertex to its repeat.
Notice, that if \( u \) and \( v \) are selfrepeats fixed by \( \varphi \), then there are exactly \( d \) internally disjoint \( \leq (k+1) \)-walks from \( u \) to \( v \), \( (u, u_i, \ldots, v_i, v) \) for \( i = 1, 2, \ldots, d \). Hence if the order of \( u_i \) with respect to \( \varphi \) is \( p \), and the order of \( v_i \) with respect to \( \varphi \) is \( q \), then \( (u, u_i = \varphi^p(u_i), \ldots, v_i = \varphi^q(v_i), v) \) and \( (u, u = \varphi^q(u_i), \ldots, v_i = \varphi^q(v_i), v) \) are both \( \leq (k+1) \)-walks, and thus we must have \( p = q \). Said in another way, the permutation cycles with respect to some automorphism \( \varphi \) of the vertices in \( N^+(u) \) and \( N^-(v) \) are the same when \( u \) and \( v \) are selfrepeats.

The following lemma is a more general result than that of [12].

**Lemma 3.** If \( G \) has a selfrepeat which is fixed by \( \varphi \), then \( H \) is an almost Moore digraph with selfrepeats of degree \( d' \leq d \) and diameter \( k \).

**Proof.** Let \( z = r(z) = \varphi(z) \), then according to Lemma 2 we must have all vertices on the two \( \leq k \)-walks from \( z \) to \( r(z) \) as fixpoints, and all the selfrepeats lie on the non-trivial walk from \( z \) to \( z \), so \( H \) contains a \( k \)-cycle.

Notice that \( d^+_H(z) = d^-_H(z) = d' \leq d \) for all \( z = r(z) \in V(H) \), as the permutation cycles in \( N^+(z) \) and \( N^-(z) \) are the same. Now, if we have a vertex \( u = \varphi(u) \neq r(u) \), then we can pick a selfrepeat \( z \) such that \( r(u) \notin N^-(z) \), as otherwise we would have \( r(u) \in N^-(z') \) for all selfrepeats \( z' \) of \( G \), and therefore \( r(r(u)) \) would be a selfrepeat, a contradiction as \( u \) is not a selfrepeat. Thus for this \( u \) and \( z \) we have \( d \) internally disjoint \( \leq (k+1) \)-walks \( (u, u_i, \ldots, z_i, z) \) in \( G \). Then \( d' \) of the internally disjoint \( \leq (k+1) \)-walks from \( u \) to \( z \) will also be in \( H \), due to Lemma 2, and thus \( d^+(u) \geq d' \). Assume that \( d^+(u) > d' \), then there exists a \( j \in \{1, 2, \ldots, d\} \) such that \( u_j = \varphi(u_j) \) and \( z_j \neq \varphi(z_j) \). But then \( (u_j, \ldots, z_j, z) \) and \( (u_j, \ldots, \varphi(z_j), z) \) are two distinct \( \leq k \)-walks from \( u_j \) to \( z \), a contradiction as \( z \) is a selfrepeat.

So \( H \) is a diregular digraph of degree \( d' \). Now, assume \( H \) has diameter \( k+1 \), this implies that there exists a vertex \( v \) such that \( \text{dist}_H(v, r(v)) = k+1 \) thus the order of \( H \) is \( n = 1 + d' + d'^2 + \ldots + d'^k + 1 = M(d', k) + 1 \) according to Corollary 1. However, looking at a selfrepeat \( z \in H \), we get the order as \( n = 1 + d' + d'^2 + \ldots + d'^k - 1 = M(d', k) - 1 \), a contradiction.

So \( H \) must be diregular with degree \( d' \leq d \), diameter \( k \) and its order must be \( M(d, k) - 1 \), hence it is an almost Moore digraph with selfrepeats, as the girth of \( H \) is \( k \). \( \square \)

**Lemma 4.** Let \( \varphi \) fix at least three vertices, then \( H \) is diregular of degree \( d' \) and either

- \( H \) is an almost Moore digraph of degree \( d' \leq d \) and diameter \( k \), or
- \( H \) is a \( (d', k, 1) \)-digraph of degree \( d' \leq d - 2 \).
Proof. If \( \varphi \) fixes a selfrepeat, then we have the first case of the statement according to Lemma 3. Thus we can assume \( \varphi \) does not fix any selfrepeats.

Let \( u \) and \( v \) be any two fixed vertices in \( G \), thus they are not selfrepeats, and let \( N^+(u) = \{u_1, u_2, \ldots, u_d\} \) and \( N^-(v) = \{v_1, v_2, \ldots, v_d\} \). Assume \( r(u) \neq v_j \) for \( j = 1, 2, \ldots, d \). Then in \( G \) we have internally disjoint \( (k+1) \)-walks \((u, u_i, \ldots, v, v)\) for \( i = 1, 2, \ldots, d \). As \( r \) is an automorphism, we get \( r(u_i) \neq v \) for \( i = 1, 2, \ldots, d \). Now, we have \( u_i = \varphi(u_i) \) if and only if \( v_i = \varphi(v_i) \) due to Lemma 2, hence \( d_H^+(u) = d_H^+(v) \). As we could have \( v = r(u) \), we see that each vertex in \( H \) is balanced, as \( d_H^+(u) = d_H^+(r(u)) \) and \( d_H^-(u) = d_H^-(r(u)) \).

Now, assume \( H \) is not diregular, thus for each vertex \( u \in V(H) \) we must have a vertex \( v \in N^+(r(u)) \cap V(H) \) such that \( d_H^+(u) \neq d_H^+(v) \). Let \( u \in V(G) \) be a vertex of minimum degree \( d_1 \leq d \) in \( H \), and let \( v \in V(H) \) be a vertex with \( d_H^-(v) > d_1 \). Then \( d_H^+(v) = d_1 + 2 \) as we must have \( v \in N^+(r(u)) \) with \( \text{dist}_H(u, r(u)) = k + 1 \) and \( \text{dist}_H(r^-(v), v) \leq k \). But then there must be at most \( d_1 \) vertices of degree different from \( d_1 \) in \( H \) and at most \( d_1 + 2 \) vertices of degree different from \( d_1 + 2 \), hence \( |V(H)| \leq d_1 + (d_1 + 2) \). This is a contradiction to the fact that \( |V(H)| \geq d_1 + d_1^2 + \ldots + d_1^k \) as the diameter of \( H \) is at least \( k \geq 3 \). So, obviously \( H \) is diregular. If \( \text{dist}(u, r(u)) = k + 1 \), then each vertex in \( H \) must have at least two out-neighbours of order two with respect to \( \varphi \) and thus the statement follows.

Theorem 3 now follows directly from Lemmas 3 and 4.

3. Almost Moore digraphs of degree 4 and 5

In this section we will look at almost Moore digraphs of degree 4 and 5 and specify the order of the vertices with respect to the automorphism \( r \).

Lemma 5. Let \( u \in V(G) \) be a vertex with \( \varphi(u) = u \neq r(u) \), then if \( H \) is two isolated vertices or has diameter \( (k+1) \) we must have two vertices in \( N_H^+(u) \) which have order 2 with respect to \( \varphi \).

Proof. In \( G \) we have two \( \leq k \)-paths, \( P \) and \( Q \) from \( u \) to \( r(u) \). If \( H \) is either two isolated vertices or has diameter \( k + 1 \), we must have that the internal vertices on \( P \) and \( Q \) are not in \( H \). Thus \( \varphi(P) = Q \) and \( \varphi(Q) = P \), and hence \( \varphi^2(v) = v \) and \( \varphi(v) \neq v \) for all internal vertices \( v \) on \( P \) and \( Q \).

The following theorem is a more general result than that of [13] and [12].

Theorem 4. Let \( G \) be an almost Moore digraph of degree 4, then the vertices of \( G \) have orders with respect to the automorphism \( r \) according to one of the following:

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• there are $k$ vertices of order 1 and $M(4, k) - 1 - k$ of order 3 or
• all vertices are of the same order $p \geq 2$.

Proof. Assume throughout that not all vertices are of the same order. Let $u$ be a vertex of $G$ of the smallest order $p$ with respect to $r$ in $G$. Let $N^+(u) = \{u_1, u_2, u_3, u_4\}$, then we can split $N^+(u)$ into permutation cycles with respect to $r^p$ in one of the following ways: $(u_1)(u_2)(u_3, u_4)$, $(u_1)(u_2, u_3, u_4)$ or $(u_1, u_2)(u_3, u_4)$. Notice however that the splitting $(u_1)(u_2)(u_3, u_4)$ is not possible, as there according to Theorem 3 where $\varphi = r^p$ would exist a $(2, k, -1)$- or $(2, k, 1)$-digraph as an induced subdiagram of $G$, a contradiction to Theorems 1 and 2.

First assume there is a vertex $u$ of order 1, thus $u$ is a selfrepeat and hence there are exactly $k$ vertices of order 1 inducing a $k$-cycle in $G$. Thus among the above ways of having permutation cycles, the only possibility is $(u_1)(u_2, u_3, u_4)$. Then all vertices which are not selfrepeats must have order 3 according to Lemma 3 by letting $\varphi = r^3$.

Now assume $u \in V(G)$ has the smallest possible order $p \geq 2$, then according to Lemma 5 the only possible permutation cycles are $(u_1, u_2)(u_3, u_4)$. In turn, this is only possible if $p = 2$, as there will always be at least $p$ vertices of order $p$ in $G$.

Thus $G$ will contain $M(4, k) - 3$ vertices of order 4, thus 4 should divide $M(4, k) - 3$. But in fact

$$M(4, k) - 3 \equiv -2 + 4 + 4^2 + \ldots 4^k \equiv 2 \mod 4,$$

a contradiction. \qed

Theorem 5. Let $G$ be an almost Moore digraph of degree 5, then one of the following is true regarding the orders with respect to the automorphism $r$ of the vertices in $G$:

• there are $M(3, k) + 1$ vertices of order $p \geq 2$ and $M(5, k) - M(3, k) - 2$ of order $2p$
• there are $k + 2$ vertices of order $p \geq 2$ and $M(5, k) - 3 - k$ of order $2p$
• there are $k$ vertices of order 1 and either $M(5, k) - 1 - k$ of order 2 or $M(5, k) - 1 - k$ of order 4
• all vertices are of the same order $p \geq 2$. 

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Proof. Assume throughout that not all vertices are of the same order. Let \( u \) be a vertex of \( G \) of the smallest order \( p \). Let \( N^+(u) = \{u_1, u_2, u_3, u_4, u_5\} \), then we can split \( N^+(u) \) into permutation cycles with respect to \( r^p \) in one of the following ways: \((u_1)(u_2, u_3, u_4, u_5)\), \((u_1)(u_2)(u_3)(u_4, u_5)\) or \((u_1)(u_2, u_3)(u_4, u_5)\) due to Lemma 5 and Theorems 1 and 2.

If the permutation cycles are \((u_1)(u_2, u_3, u_4, u_5)\), then due to Lemma 5 we must have \( u \) is a selfrepeat, hence there is \( k \) vertices of order 1 and \( M(5, k) - k - 1 \) of order 4. If instead the permutation cycles are \((u_1)(u_2, u_3)(u_4, u_5)\), then we could have \( k \) vertices of order 1 and \( M(5, k) - k - 1 \) of order 2 or \( k + 2 \) vertices of order \( p \geq 2 \) and \( M(5, k) - k - 3 \) of order 2.

Finally, if the permutation cycles are \((u_1)(u_2)(u_3)(u_4, u_5)\), then if \( \varphi = r^p \), we would have \( H \) to be either a \((3, k, -1)\)-digraph or a \((3, k, 1)\)-digraph. But \((3, k, -1)\)-digraphs do not exist according to Theorem 1, thus we must have \( M(3, k) + 1 \) vertices of order \( p \geq 2 \) and \( M(5, k) - M(3, k) - 2 \) of order 2p.

References


