# Power Distributions Revisited

Andersen, Jørgen Bach; Kovacs, Istvan Zsolt

*Published in:*
COST 273 TD-02-04

*Publication date:*
2002

*Document Version*
Publisher's PDF, also known as Version of record

EUROPEAN COOPERATION
IN THE FIELD OF SCIENTIFIC
AND TECHNICAL RESEARCH

EURO-COST

COST 273 TD(02) 004
Guildford, UK
2002/January 17-18

SOURCE: Center for Personkommunikation
Aalborg University
Denmark

# Power Distributions Revisited

J. Bach Andersen, Istvan Kovacs
Center for Personkommunikation
Aalborg University
Niels Jernes Vej 12
9220 Aalborg, Denmark
Phone: +45 9635 8641
Fax: +45 9815 1583
Email: (jba,istvan)@cpk.auc.dk

# Power Distributions Revisited

J. Bach Andersen, Istvan Kovacs
Center for Personkommunikation
Aalborg University, Denmark

*Abstract*

*A new model utilizing multiple scattering replaces the traditional subdivision of the total fading into a slow lognormal and a fast Rayleigh component. The distribution agrees well with experimental results from a forest and an urban environment. It is shown that the slow fading in general may not be due to shadowing, but rather the slow variation of the coupling between scatterers, when the mobile is moving. This means that the slow fading is just as unpredictable as the fast fading, since it originates from the same scatterers. Shadowing will still exist behind major changes in the environment.*

## *1. Introduction*

Due to a multitude of physical phenomena like reflection, scattering, diffraction and guiding effects the field strength will vary with position. Traditionally ([1], [2]) the narrowband fading has been split into two parts, the fast fading and the slow or shadow fading. The slow fading is supposed to give the local mean of the fast Rayleigh fading, and experimentally the division has been made by choosing a proper averaging length for the fast fading. An example of the total fading along a street is shown in Figure 1

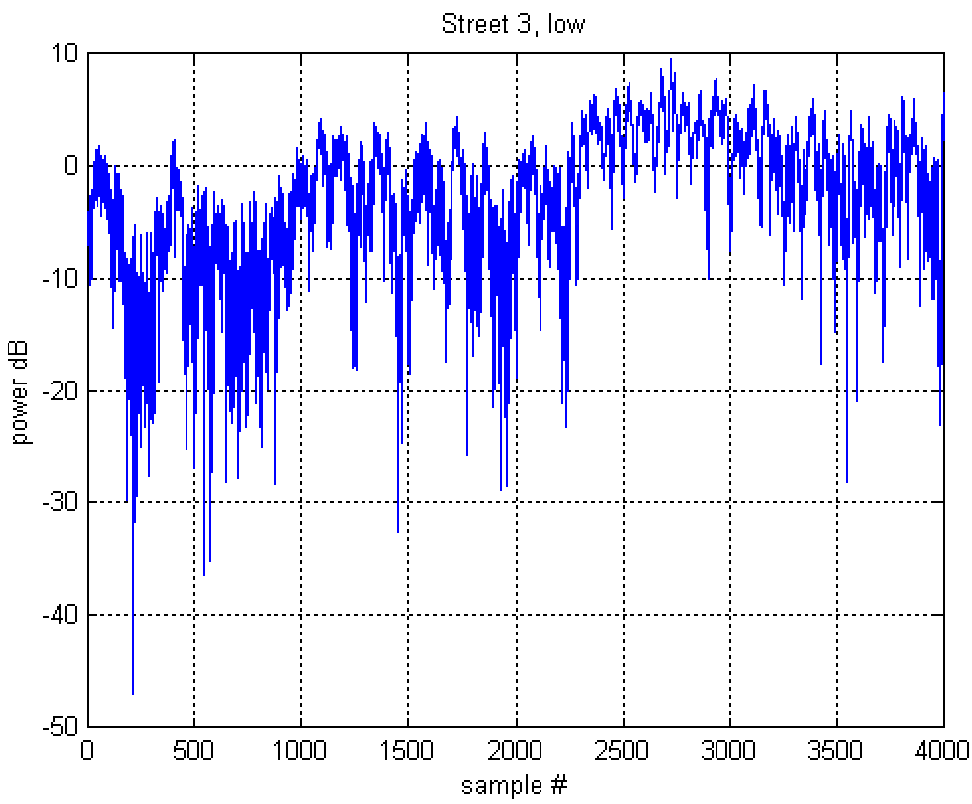

*Figure 1. Samples of narrowband power in a street in Aarhus, Denmark with low antenna height showing typical Rayleigh and lognormal behavior. Frequency 1800 MHz, and total length 100 m. Measurements are described in [3].*

The amplitude *r(x)* is usually separated into two parts

$$r(x) = m(x) r_0(x) \tag{1}$$

the long-term (or slow) fading *m(x)* and the short-term (or fast) fading $r_0(x)$. Typically *m* is described by a lognormal distribution and $r_0$ by a Rayleigh distribution. This mixed distribution is also called a Suzuki distribution [4]
The pdf's are given by

$$p(r_0) = \frac{r_0}{\sigma} e^{-r_0^2/2\sigma} \tag{2}$$

for the Rayleigh distribution, and

$$p(m) = \frac{10}{\log(10)\sqrt{2\pi\mu}m} e^{-(10\log_{10}(m)-\mu)^2/2\mu} \tag{3}$$

for the lognormal with $\rho$ and $\mu$.as variance and mean, respectively.
In case there is a constant part, the Rayleigh distribution is replaced by the Rice-distribution,

$$p(r_0) = \frac{r_0}{\sigma} e^{-(r_0^2+K^2)/2\sigma} I_0\left(\frac{Kr_0}{\sigma}\right) \tag{4}$$

with *K* as the constant part.

The resulting distributions of *m* and $r_0$ depend to some extent on how the subdivision is made, i.e. what averaging lengths are chosen. If chosen too short there are not sufficient samples to describe the fast fading, and if chosen too long, the average becomes inaccurate. Lee [2] recommends an averaging length of 20 to 40 wavelengths. From a practical point of view the division into the two types of fading has been successful, but theoretically it is unsatisfactory and arbitrary. There is in the general case no good explanation of the slow fading distribution (the lognormal distribution), and it is of interest to see if there are other distributions, which satisfy experimental results and give better explanations.

The lognormal distribution has also been used for satellite propagation, especially through foliage [5], and as a combination with the Rice-distribution where the 'constant' part is assumed to obey a lognormal distribution [6].

The main intention of this paper is to give a physically justified distribution without averaging over the fast fading. By comparison with experimental results from various environments these distributions will be used to explain the propagation mechanisms of fast and slow fading. First the classical distributions are introduced and the physical arguments behind them. Next, a new set of multiple-reflection distributions is introduced, and finally some comparisons with forest and urban measurements are made.

*2. The justification of the classical distributions*

As an example consider the distribution of power along a street with no line-of-sight to the base station. The waves are supposed to reach the street by over-rooftop propagation like in the Walfisch-Ikegami model, and then scattered and possibly re-scattered in the local environment of the street. The fine-structure or fast fading is justified by the sum of a large number of waves coming locally from fixed scattering centers on roof tops or sides of buildings. Mathematically we get the following for the antenna voltage

$$V_1(x) = \sum_i A_i / R_i e^{-jkx\cos\theta_i} \tag{5}$$

with $R_i$ the distance to the i'th scatterer, $\theta_i$ the angle between the x-axis and the direction to the i'th scatterer, and $x$ the distance moved. When the number of scatterers is large both the real part and the imaginary part will approach a zero mean normal distribution with the subsequent Rayleigh distribution of the absolute value, and exponential distribution of the power, as is well known. Here in essence the central limit theorem has been invoked, and thus the Rayleigh distribution is well justified theoretically. The addition of a non-varying part leads exactly to the Rice-distribution, which is then also well justified.

The lognormal distribution is harder to justify. Since it is a normal distribution of dB values it is obvious that a *product* of a large number of amplitudes lead to a lognormal distribution by again using the central limit theorem. It is difficult to see, however, how such a product would appear in practice for urban propagation, since if it was a multiple forward scattering phenomenon where a new scattering cross-section is multiplied by the previous one as a factor, this would essentially lead to exponential decay of the power in contrast to the well established $d^{-n}$ power law with distance. If the entire scattering occurred locally at street level, it would be strange only to observe the multiply reflected parts, and not the single, dual et cetera reflections. Seen from a statistical point of view the variance of the sum will equal the sum of variances, so for a large number of sums (in dB) the variance will rapidly grow. This is illustrated in Figure 2 which shows a set of distributions of $n$ Rayleigh fading signals multiplied together as in eq. 6, i.e.

$$V_N = \prod_1^N V_i \tag{6}$$

They are normalized to their own mean power, and it is noted that they rapidly deviate from the Rayleigh case ($n$=1). The ordinate is the logarithm of the cumulative probability, so –2 corresponds to $10^{-2}$. The distributions are simulated by using 100.000 realizations. Also shown is the lognormal case with a standard deviation of 8 dB, typical in urban areas, and it shows no similarity to the other curves. Even if they are normalized to have the same standard deviation the case of $n=4$ is far from the lognormal distribution, in fact simulations show that 40-50 factors in the product are needed to get close to the lognormal distribution at the $10^{-3}$ cumulative probability. This agrees with the observations by Coulson et al [8].

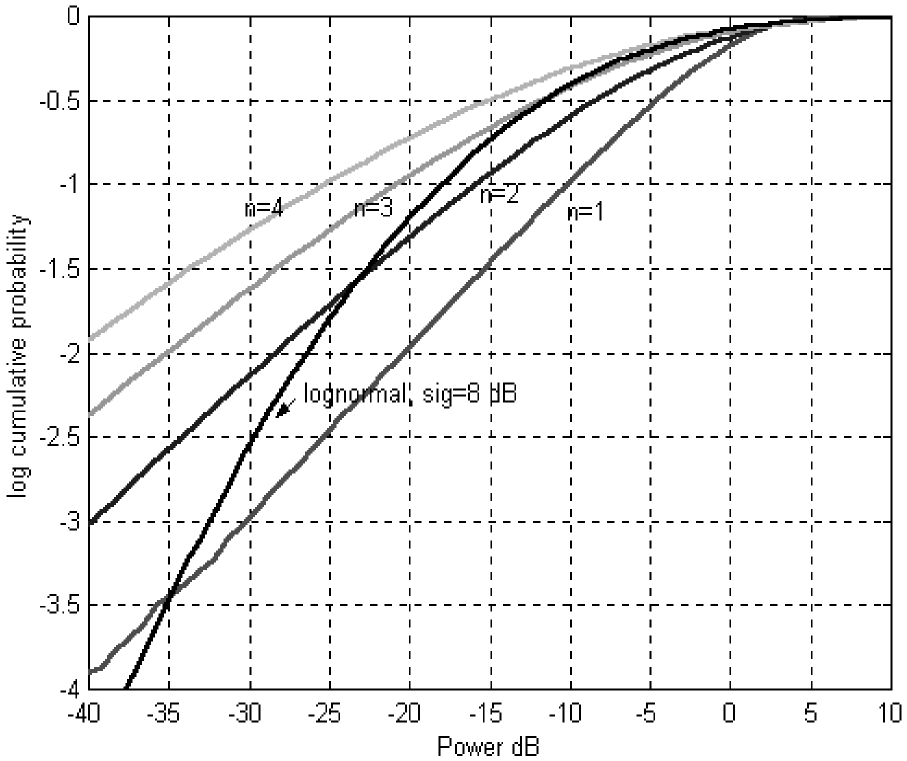


*Figure 2. The log cumulative probability curve for a product of n Rayleigh distributions and a lognormal distribution for comparison.*

It should be noted that in [7] it is shown that a random distribution of building heights in it self does not lead to a lognormal distribution, but that additional features may be added from the construction of the buildings to give a lognormal distribution over the 5%-95% range. It is interesting that Lin [9] in a study of rain attenuation has introduced the loglognormal distribution, where the dBs are lognormally distributed and found good agreement with observations. It seems unlikely as an explanation that many dBs should be multiplied to give the total attenuation. Thus, the lognormal distribution cannot be easily justified from a propagation point of view for an urban environment, and that it merely is a practical solution to the distribution of local means used for planning purposes

*3. Physically motivated distributions*

The model assumed in this paper is essentially one consisting of two (or more) sheaths as indicated in Figure 3. Each sheath consists of a number of scatterers. The bottom sheath is illuminated by a source (imagining *B* is the transmitter), and each scatterer in the sheath scatters some energy to each scatterer in the top sheath, which finally scatters to the receiver *A*. It is clear that the model may be extended to a multiplicity of sheaths, but two is sufficient for the purpose of this paper. Only forward scattering is assumed. A general description of the channel is given by the following equation (7), where the range dependencies have been suppressed, since it is assume that the phase variation is the important one.

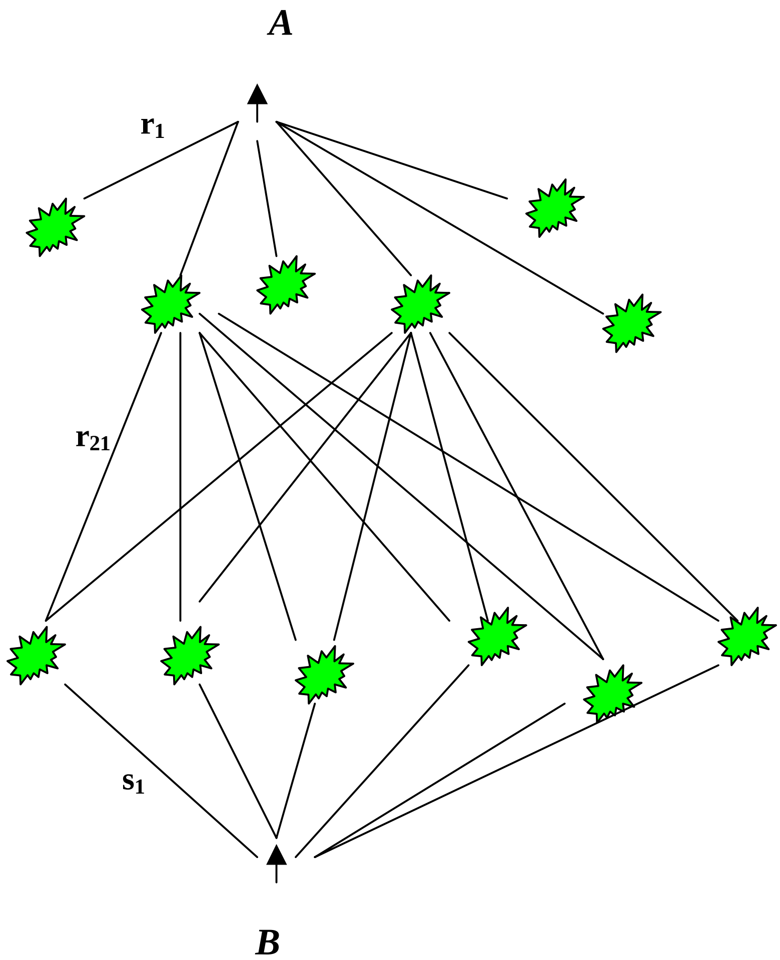


*Figure 3 Model of multiple scattering where the stars represent scatterers. There are two groups of scatterers, and there is a transmission path from each scatterer in one group to each scatterer in the other group. Not all ray paths are shown.*

$$
\begin{aligned}
H = & e^{-jkr_1} A_1 \left[ e^{-jkr_{11}} B_1 e^{-jks_1} + e^{-jkr_{12}} B_2 e^{-jks_2} + \ldots \right] \\
& + e^{-jkr_2} A_2 \left[ e^{-jkr_{21}} B_1 e^{-jks_1} + e^{-jkr_{22}} B_2 e^{-jks_2} + \ldots \right] \\
& + e^{-jkr_3} A_3 \left[ e^{-jkr_{31}} B_1 e^{-jks_1} + e^{-jkr_{32}} B_2 e^{-jks_2} + \ldots \right] \\
& + \ldots
\end{aligned} \tag{7}
$$

Here $r_i$ is the distance from the transmitter to the scatterer $A_i$ in the top sheath, $r_{ij}$ the distance from scatterer $A_i$ to scatterer $B_{j.}$ in the bottom sheath, and $s_j$ the distance from scatterer $B_j$ to the receiver. $A_i$ and $B_{j.}$ are the complex scattering cross sections of the scatterers. As it stands, equation (7) gives $H$ as a sum of a large number of complex numbers with zero mean, and $H$ will be complex Gaussian distributed, Rayleigh in magnitude. Let us now assume the two sheaths are some what removed from each other, so that $r_{ij}$ may be expanded around a common distance d

$$e^{-jkr_{ij}} = e^{-jk(d+\Delta_{ij})} = e^{-jkd}(1 - jk\Delta_{ij} + ..) \tag{8}$$

where the differences $\Delta_{ij}$ will be small when the two sheaths are far from each other. Introducing this into (7) gives

$$\begin{aligned}
H = {} & e^{-jkr_1} A_1 e^{-jkd}\left[B_1 e^{-jks_1} + B_2 e^{-jks_2} + \ldots\right] \\
& + e^{-jkr_2} A_2 e^{-jkd}\left[B_1 e^{-jks_1} + B_2 e^{-jks_2} + \ldots\right] \\
& + e^{-jkr_3} A_3 e^{-jkd}\left[B_1 e^{-jks_1} + B_2 e^{-jks_2} + \ldots\right] \\
& + e^{-jkr_1} A_1 e^{-jkd}(-jk)\left[\Delta_{11} B_1 e^{-jks_1} + \Delta_{12} B_2 e^{-jks_2} + \ldots\right] \\
& + e^{-jkr_2} A_2 e^{-jkd}(-jk)\left[\Delta_{21} B_1 e^{-jks_1} + \Delta_{22} B_2 e^{-jks_2} + \ldots\right] \\
& + e^{-jkr_3} A_3 e^{-jkd}(-jk)\left[\Delta_{31} B_1 e^{-jks_1} + \Delta_{32} B_2 e^{-jks_2} + \ldots\right] \\
& + \ldots
\end{aligned} \tag{9}$$

or

$$\begin{aligned}
H = {} & e^{-jkd}\left[A_1 e^{-jkr_1} + A_2 e^{-jkr_2} + A_3 e^{-jkr_3}\right]\left[B_1 e^{-jks_1} + B_2 e^{-jks_2} + \ldots\right] \\
& + e^{-jkr_1} A_1 e^{-jkd}(-jk)\left[\Delta_{11} B_1 e^{-jks_1} + \Delta_{12} B_2 e^{-jks_2} + \ldots\right] \\
& + e^{-jkr_2} A_2 e^{-jkd}(-jk)\left[\Delta_{21} B_1 e^{-jks_1} + \Delta_{22} B_2 e^{-jks_2} + \ldots\right] \\
& + e^{-jkr_3} A_3 e^{-jkd}(-jk)\left[\Delta_{31} B_1 e^{-jks_1} + \Delta_{32} B_2 e^{-jks_2} + \ldots\right] \\
& + \ldots
\end{aligned} \tag{10}$$

It is apparent that the transfer function now falls in two distinct parts, the first line consisting of a product of two complex Gaussians (called a double Rayleigh in the following), and the remainder consisting of a single complex Gaussian (a single Rayleigh).
The description above is sufficient for finding the distribution of the power ($|H|^2$) by simulation by assuming complex Gaussians. This will be done later, but first it must be understood why both the fast and the slow fading are involved. Assume that antenna *B* is fixed, all the scatterers are fixed, and antenna *A* moving. The only parameters changing are the $r_i$, phase mixing the contributions from the *A* scatterers, leading to a Rayleigh distribution. After some distance (depending on the correlation length of the scattering system) new scatterers appear and the previous ones disappear. If we assume a stationary environment this in itself will not change the statistics. However, if we assume that gradually the *B* scatterers also start to change, then the double Rayleigh will appear, and the statistics change. This latter change will of course take place over a different scale of length than the fast fading, where the scale of length is a fraction of a wavelength. Finally, it is easy to see that the double Rayleigh will also exist if both antennas are moving.

The model may be easily generalized to a multiplicity of sheaths leading to triple Rayleighs et cetera.

In general, the following model results, where a constant term (a Ricean term *K*) has been added

$$H = K + H_1 + \alpha H_2 H_3 + \beta H_4 H_5 H_6 + .... \tag{11}$$

This is the new model to be applied in the following where it is assumed that $H_i$ are complex, independent Gaussian fading signals. The normalization is such that each product of complex Gaussians has a mean of one. The cumulative distributions of single, double, triple, and quadruple product of Rayleigh fading paths are shown in Figure 2, and it is seen that the deep fading increases significantly. The double fading has been described earlier by Erceg et al [10] as cascaded Rayleigh fading, and it was shown that the probability density for the power was given by

$$p_z(z) = 2K_0(2\sqrt{z}) \tag{12}$$

where $K_0$ is the modified Bessel function of the second kind and zero order. The cumulative distribution function is then given by

$$P(Power \le z) = 1 - 2\sqrt{z}K_1(2\sqrt{z}) \approx 2z\ln(2\sqrt{z}) - z \tag{13}$$

where the approximation is valid for z<<1.

In the following the model will be limited to the four terms of eq.11, and constants related to the various powers are introduced. The basic Rayleigh term $H_1$ has a mean power of 1, $H_2H_3$ has also a mean power of 1, and so on.

The distribution of the magnitude of *H* or magnitude squared is not tractable analytically in the general case, so simulations of a few cases have been performed.

## *4. Comparison with the Suzuki distribution*

The model of eq. 11 is the basis for the simulations. The Suzuki distribution is a product of Rayleigh and a lognormal with standard deviation σ. The model parameters *K*, $\alpha$, $\beta$ from eq. 11 are found by minimizing the mean square error between the logarithm of the cumulative distribution functions. Figure 3 shows two examples for σ=4 and 6 dB. The minimized errors are 0.002 and 0.006 respectively. It is evident that the multiple scattering distributions may be made to equal the Suzuki distribution (eqs. 1,2,3), which involved the lognormal distribution as a running mean of the Rayleigh distribution. Comparing two models is not significant, except that the Suzuki distribution through its use of the lognormal distribution is known to agree with many experimental results. It is the comparison with experiment that matters.

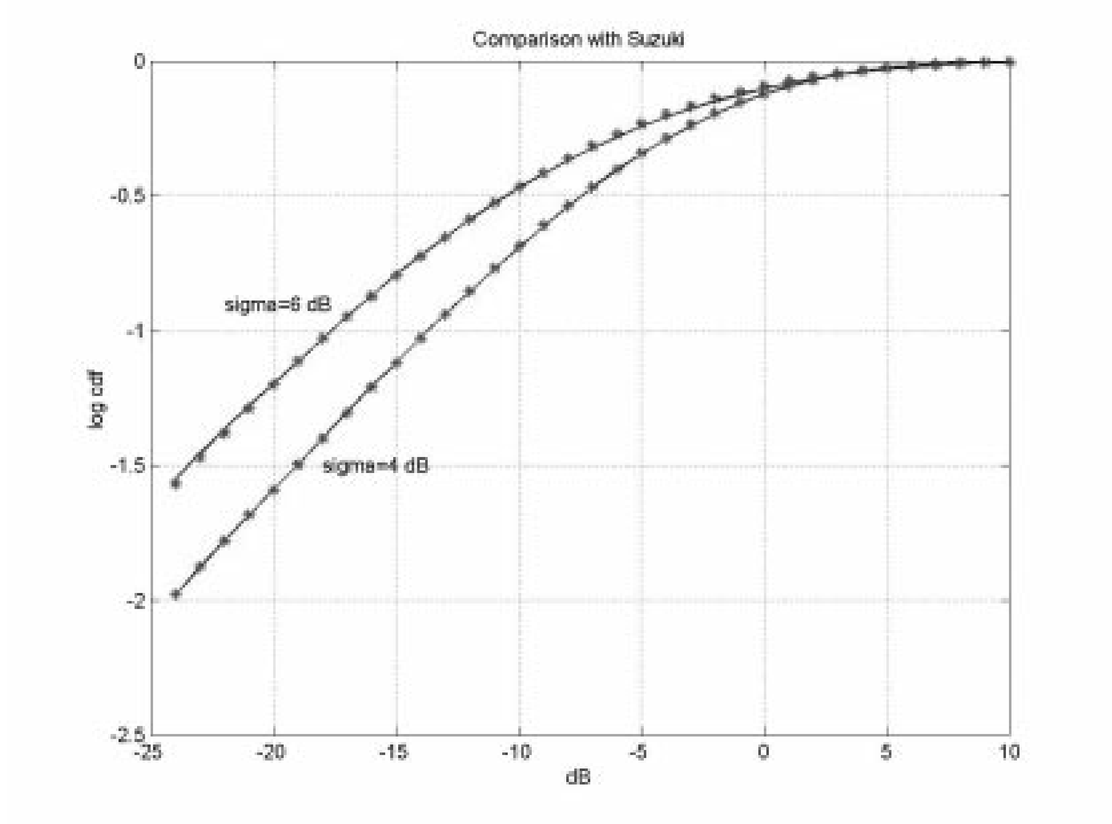


*Figure 3. Comparison between Suzuki distributions (-) with σ= 4 and 6 dB, and multiple Rayleighs (*). The parameters are found by minimizing the mean square error.*

## *5. Comparison with experimental results*

### *5a. Measurement in a forest.*

A forest is an environment where multiple scattering should be present. Figure 4 shows some examples at 1800 MHz for a mobile-to-mobile link. The best fit multiple-Rayleigh model gives the parameters in Table 1.

| d | α | β | K |
|---|---|---|---|
| 150 m | 1.05 | 0 | 0 |
| 330 m | 14.5 | 0 | 0 |

*Table 1 Best fit parameters for forest propagation (Fig. 4)*

The trend of increasing α with distance is in good agreement with the model. We can assume that a group of trees near the antennas form the effective set of scatterers. At close distance they tend to overlap and single Rayleigh is dominating. For large distances the two groups separate, and the double Rayleigh dominates, even if there are trees in the intervening area. There are also cases, not shown, where β is larger than zero.

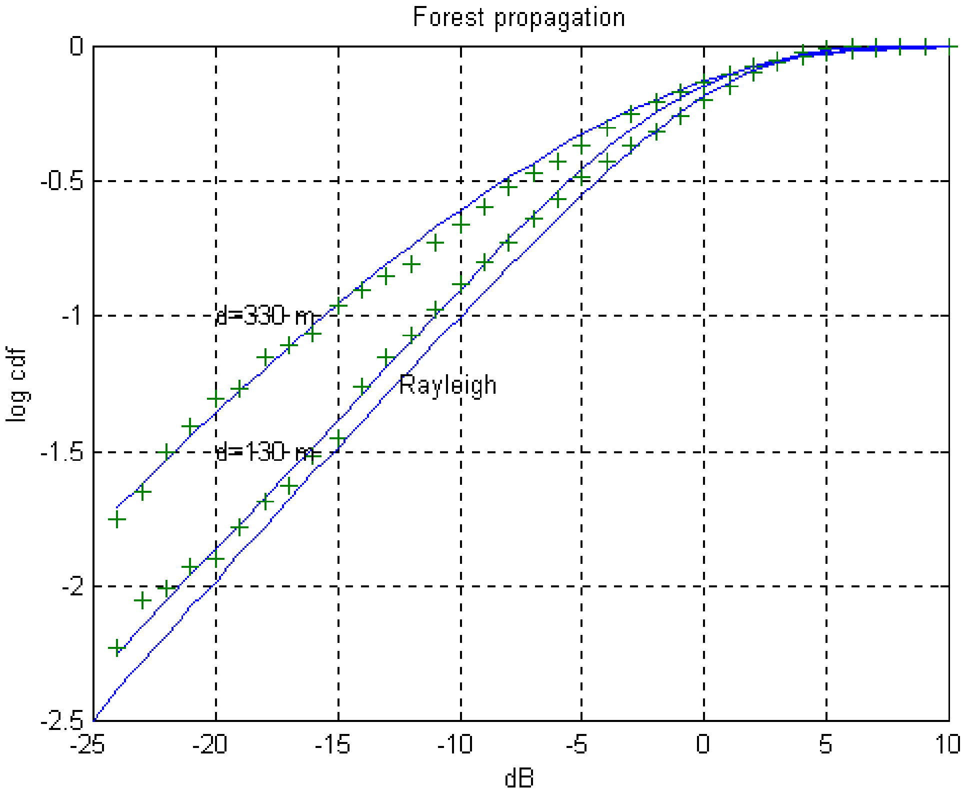


*Figure 4 Measurements (+) in a forest at two different distances compared with multiple Rayleigh model (-). Single Rayleigh shown for reference.*

*5b Measurement in an urban environment*

Experiments performed in Aarhus, Denmark are used for comparing with the simulated distributions. The frequency was 1800 MHz and the measurements were performed with the base station antenna 12 meters above rooftops (called high) and at rooftop level (called low). The measurements are described in greater detail in reference [3]. The total of 4000 samples were used in each case and compared with the theoretical distributions with a minimum error criterion.

| Base antenna | $\alpha$ | $\beta$ | K |
|---|---|---|---|
| High | 1.0 | 2.2 | 0 |
| Low | 0.75 | 0 | 0 |

*Table 2 Best fit parameters for urban environment (Fig. 5)*

Figure 5 shows street 3, low, the same case as indicated in Figure 1. The same street with a higher antenna position is also shown in figure 5. The two lowest values have been discarded as outliers. The street is running radially away from the base station, so in this case it makes sense to have more wave-guiding, which would increase the power in multiply reflected paths. In both cases a lognormal times Rayleigh distribution, i.e. a Suzuki, would give a good agreement, but without the physical interpretation of the results. For the high antenna the error is 0.0075 for the multi-

Rayleigh model, and 0.016 for the Suzuki, so the multi-Rayleigh is better. The difference is probably too small, however, to be statistically significant.

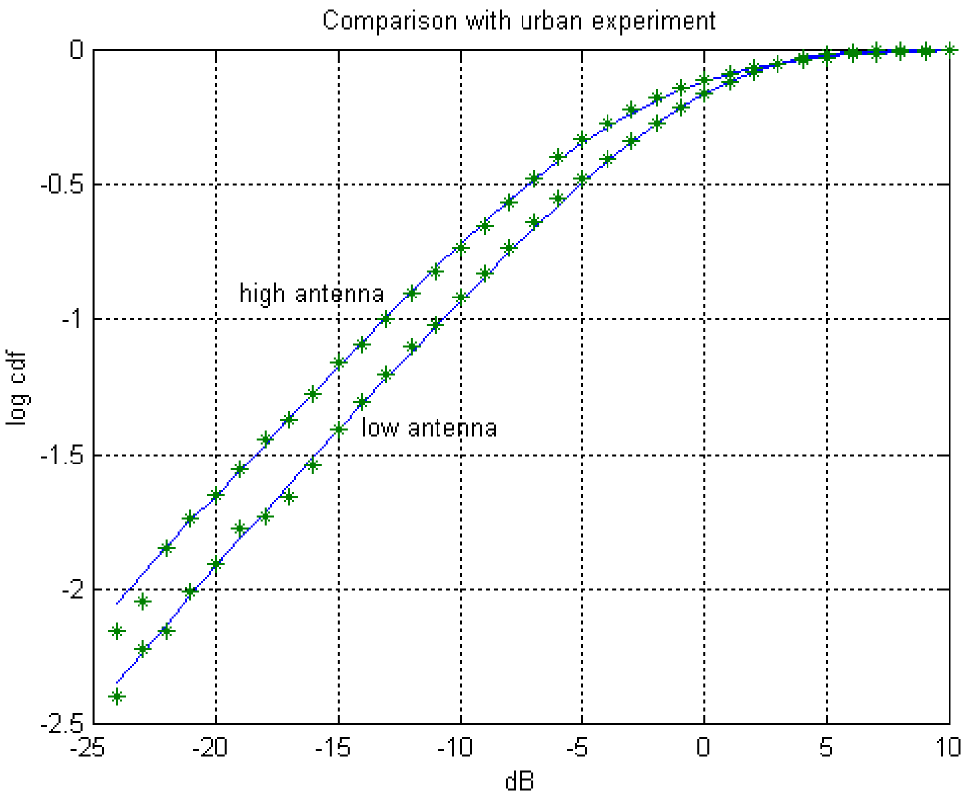


*Figure 5 Comparison between urban measurements (*) and a multi-Rayleigh model (-) for an urban situation with two different antenna heights.*

*Conclusion*

The traditional way of separating the spatial power variations in a short term and a long-term fading is satisfactory for many purposes. However, it is not related to the physical propagation mechanisms, and the standard lognormal distribution used for the slow fading does not have a simple interpretation. In stead we have chosen not to separate the fading in several parts, but rather study the total fading. The physical basis is a model of forward scattering between scatterers introducing multiply scattered waves defining a new transfer function. This function consists of a sum of a small number of terms, where each term is a multiple product of complex Gaussians. Under certain conditions multi-Rayleigh distributions, like the double Rayleigh, will dominate, but in general there will be a mix of single, double, triple Rayleighs, which form the complete picture. The advantage of the new distribution is the insight it gives into the origin of the slow fading, its disadvantage is the lack of a simple analytical function except in special cases.

The new model is similar in shape to the single reflected times lognormal (Suzuki), but it has a different interpretation. The lognormal is usually interpreted as a shadowing function, which influences the local mean value. The shadowing is supposed to be dependent on the local environment. The multi-Rayleigh distribution has a constant mean power for the single scattering for the whole environment, and the variation of the mean of the total power stems from the slowly varying scattering between the scatterers as the antenna moves. Thus there is no need for a shadowing argument to explain the slow fading. The resulting parameters from the fitting of the distributions may be interpreted as revealing the propagation mechanisms. One important conclusion from this study is that the slow fading is unpredictable, since it originates from the same random elements as the fast fading.

Two environments have been used for comparison, a forest and an urban environment. In both cases the agreement with the model has been excellent, in fact even better than the Suzuki model, which involves the lognormal distribution. Thus it seems that the lognormal distribution is just a practical tool, without any explaining power.

Shadowing will still occur and may give major predictable changes in the power, e.g. at street crossings and general terrain changes, but this will be additional mechanisms on top of those discussed here.

*References*


[1] W.C. Jakes, Jr., *Microwave Mobile Communications*, Wiley Interscience, 1974

[2] W.C.Y.Lee, *Mobile Communications Design Fundamentals*, Wiley, 1993

[3] K.I. Pedersen, P.E. Mogensen, B. Fleury, "A Stochastic Model of the Temporal and Azimuthal Dispersion Seen at the Base Station in Outdoor Propagation Environments", *IEEE Trans on Vehicular Technology*, vol. 49, No. 2, pp. 437-447, March 2000.

[4] H. Suzuki, “A Statistical Model of Mobile Radio Reception”, *IEEE Trans. Communications*, vol COM-25, July 1977, pp 673-680

[5] J. Goldhirsch, and W.J.Vogel, “Mobile Satellite System Fade Statistics for Shadowing and Multipath from Roadside Trees at UHF and L-band”, *IEEE Transactions on Antennas and Propagation*, vol 37, No 4, April, 1989, pp. 489-498

[6] C. Loo, “A Statistical Model for a Land Mobile Satellite Link”, *IEEE Transactions on Vehicular Technology*, vol. VT-34, No 3, August 1985, pp. 122-127

[7] C. Chrysanthou, H. Bertoni, ‘Variability of Sector Averaged Signals for UHF Propagation ijn Cities,’ *IEEE Transactions on Vehicular Technology*,vol. 39, No4, November 1990, pp 352-358

[8] A.J.Coulson, A.G.Williamson, R.G.Vaughan, “A Statistical Basis for Lognormal Shadowing Effects in Multipath Fading Channels”, *IEEE Transactions on Communications*,vol 46, No 4, April 1998, pp 494-502

[9] S. H. Lin, ‘Statistical behavior of rain attenuation, ’*Bell System Technical Journal*, vol. 52, no 4, April 1973, pp. 557-581

[10] V. Erceg, S.J.Fortune, J.Ling, A.J.Rustako, R.A.Valenzuela, ‘Comparisons of a Computer-Based Propagation Prediction Tool with Experimental Data Collected in Urban Microcellular Environments’, *IEEE Journal of Selected Areas in Communications*, vol.15, No 4, May 1997, pp 677-684