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An inequality of rearrangements
on the unit circle

by

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AN INEQUALITY OF REARRANGEMENTS ON THE UNIT CIRCLE

CRISTINA DRAGHICI

Abstract. We prove that the integral of the product of two functions over a symmetric set in $S^1 \times S^1$, defined as $E = \{(x, y) \in S^1 \times S^1 : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\}$, where $\sigma_1$, $\sigma_2$ are diffeomorphisms of $S^1$ with certain properties and $d$ is the geodesic distance on $S^1$, increases when we pass to their symmetric decreasing rearrangement. We also give a characterization of these diffeomorphisms $\sigma_1$, $\sigma_2$ for which the rearrangement inequality holds. As a consequence, we obtain the result for the integral of the function $\Psi(f(x), g(y))$ ($\Psi$ a supermodular function) with a kernel given as $k[d(\sigma_1(x), \sigma_2(y))]$, with $k$ decreasing.

1. Introduction

On a measure space $(X, \mu)$, the Hardy-Littlewood inequality asserts [4]:
\[
\int_X f(x)g(x) \, d\mu(x) \leq \int_0^{\mu(X)} f^*(t)g^*(t) \, dt,
\]
where $f^*$ and $g^*$ are the decreasing rearrangements of $f$ and $g$, respectively. In what follows, $X = S^1$, or $X = [-\pi, \pi]$, and the above inequality can be written as:
\[
\int_{-\pi}^\pi f(x)g(x) \, dx \leq \int_{-\pi}^\pi f^\sharp(x)g^\sharp(x) \, dx,
\]
with $f^\sharp$, $g^\sharp$ the symmetric decreasing rearrangements of $f$ and $g$, given by $f^\sharp(x) = f^*(2|x|)$ and $g^\sharp(x) = g^*(2|x|)$.

These inequalities can be proved using the layer-cake formula [10]: Every measurable function $f : X \rightarrow \mathbb{R}_+$ can be written as an integral of the characteristic function of its level sets:
\[
f(x) = \int_0^\infty \chi_{\{f > t\}}(x) \, dt.
\]

A more general rearrangement inequality on $X = \mathbb{R}^n$ is the Riesz-Sobolev inequality:
\[
\int_{\mathbb{R}^{2n}} f(x)g(y)h(x - y) \, dx \, dy \leq \int_{\mathbb{R}^{2n}} f^\sharp(x)g^\sharp(x)h^\sharp(x - y) \, dx \, dy,
\]
where $f$, $g$, $h$ are non-negative functions which vanish at infinity in a weak sense. The case $n = 1$ is due to Riesz in 1930 (see [12]), and the case $n > 1$ is due to Sobolev in 1938 (see [13]). The proof can be found in the book by Hardy, Littlewood, Pólya [9] which sets the beginning of the systematic study of rearrangement inequalities.

A more general version of this inequality in $\mathbb{R}^n$, involving $n$ functions can be found in [5].
The equivalent of (1.3) for three non-negative functions on the unit circle was proved by Baernstein [1]:

\begin{equation}
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(e^{i\phi})g(e^{i\phi})h(e^{i(\phi-\theta)}) \, d\theta d\phi \leq \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f^\sharp(e^{i\phi})g^\sharp(e^{i\phi})h^\sharp(e^{i(\phi-\theta)}) \, d\theta d\phi.
\end{equation}

The proof of this inequality uses a variational principle applied to the convolution of characteristic functions of sets which does not seem to generalize in higher dimensions.

The Riesz-Sobolev inequality (1.3) is equivalent to the Brunn-Minkowski inequality from convex geometry [8, 11, 7] which states that if \( K \) and \( L \) are measurable sets in \( \mathbb{R}^n \), then their Minkowski (pointwise) sum \( K + L \) is related to the measure of the sets \( K \) and \( L \) by

\[ V(K + L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \]

where \( V \) denotes the \( n \)-dimensional volume. An analog of this inequality for \( \mathbb{S}^n \) is not known, and, since the proof of rearrangement inequalities in \( \mathbb{R}^n \) require it, an analog of the Riesz-Sobolev inequality (1.3) is not known in \( \mathbb{S}^n \), for \( n > 1 \).

However, a partial result in \( \mathbb{S}^n \) was proved by Baernstein and Taylor in [2]. They considered a version of the Riesz-Sobolev inequality where one of the functions is symmetric decreasing. They showed that, if \( h = K \) is already symmetric decreasing then

\[ \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} f(x)g(y)K(x \cdot y) \, d\sigma(x)d\sigma(y) \leq \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} f^\sharp(x)g^\sharp(y)K(x \cdot y) \, d\sigma(x)d\sigma(y), \]

where \( d\sigma \) is the surface measure on the unit sphere \( \mathbb{S}^n \) in \( \mathbb{R}^{n+1} \), \( x \cdot y \) is the usual inner product and \( K(t) \) is an increasing function on \([-1, 1]\). Since \( x \cdot y = \cos \alpha \), where \( \alpha \) is the angle between the vectors \( x \) and \( y \), we can write \( K(x \cdot y) = k(d(x, y)) \), with \( k \) decreasing. Here \( d(x, y) \) is the great circle (geodesic) distance between \( x \) and \( y \). Their proof is based on the polarization technique. They showed first that the inequality holds for the polarizations of \( f \) and \( g \) in any hyperplane and then they passed to the limit for the general case. They were led to this version of the Riesz-Sobolev inequality while trying to generalize a 2-dimensional result stating that \( u \) is subharmonic implies its star function is also subharmonic.

In this paper we are interested in the case \( n = 1 \) of this inequality with \( K \) replaced by the characteristic function of a symmetric set which does not depend on the distance between two points, but rather on the distance between their images under two diffeomorphisms \( \sigma_1, \sigma_2 \) of \( \mathbb{S}^1 \). We will also obtain a characterization of these diffeomorphisms for which the inequality holds. With the set \( E \) defined as

\[ E = \{(x, y) : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\}, \]

we will show that

\begin{equation}
\int_E f(x)g(y) \, dxdy \leq \int_E f^\sharp(x)g^\sharp(y) \, dxdy,
\end{equation}

for every \( \alpha > 0 \). This result implies the main result of this paper, Theorem 3.6:

\[ \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f(x), g(y))k[d(\sigma_1(x), \sigma_2(y))] \, dxdy \leq \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f^\sharp(x), g^\sharp(y))k[d(\sigma_1(x), \sigma_2(y))] \, dxdy, \]
with $k$ decreasing and $\Psi$ the distribution function of a measure $\mu$.

The paper is organized as follows: We will first prove (1.5) for $f$ and $g$ replaced by characteristic functions $\chi_A$, $\chi_B$, and $\sigma_2$ the identity. Then we will deduce the result (1.5) mentioned above, and we will show that we can replace the product $f(x)g(y)$ by a function $\Psi(f(x), g(y))$ and that we can replace $\chi_E$ by a decreasing function of the distance between $\sigma_1(x)$ and $\sigma_2(y)$, yielding Theorem 3.6.

2. Preliminaries

Recall that a function $f : I \to \mathbb{R}$, defined on an interval $I \subset \mathbb{R}$, is called convex if, for every $0 < \lambda < 1$ and every $a, b \in I$, the following inequality holds:

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b).$$

A convex function is differentiable almost everywhere on $I$ and its derivative is increasing.

We denote by $\mathbb{S}^1$ the unit circle in $\mathbb{R}^2$, i.e., $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$, and by $\mathbb{S}^1_+$ the upper half unit circle,

$$\mathbb{S}^1_+ = \{e^{i\theta} : 0 \leq \theta \leq \pi\}.$$  

**Definition 2.1.** A function $\sigma : \mathbb{S}^1_+ \to \mathbb{S}^1_+$ is called convex if the function $\sigma_1 : [0, \pi] \to [0, \pi]$, defined as:

$$\sigma(e^{i\theta}) = e^{i\sigma_1(\theta)} , \quad 0 \leq \theta \leq \pi,$$

is convex on $[0, \pi]$.

Let $f : \mathbb{S}^1 \to \mathbb{R}_+$ be a non-negative measurable function. We define its distribution function:

$$\lambda_f(t) = |\{f > t\}|, \quad t \in [0, \infty),$$

where $\{f > t\} := \{z \in \mathbb{S}^1 : f(z) > t\}$ denote the level sets of $f$, and $|A|$ is the linear measure on $\mathbb{S}^3$ of $A$. Functions which have the same distribution function are called equimeasurable.

We define the symmetric decreasing rearrangement of $f$ to be the function $f^\sharp : \mathbb{S}^1 \to \mathbb{R}_+$, given by:

$$f^\sharp(z) = \inf\{t : \lambda_f(t) \leq 2d(1, z)\},$$

where $d(1, z)$ is the geodesic distance on $\mathbb{S}^1$ between $z$ and $1$.

It is clear that $f^\sharp(z) = f^\sharp(\bar{z})$ and that $f^\sharp$ decreases as $d(1, z)$ increases. Also, $f$ and $f^\sharp$ are equimeasurable.

If we write $z = e^{i\theta}$, $-\pi \leq \theta < \pi$, then $d(1, z) = d(1, e^{i\theta}) = |\theta|$, and we can think of $f$ as a function of $\theta$ via the relation

$$\tilde{f}(\theta) = f(e^{i\theta}).$$

For $\tilde{f} : [-\pi, \pi] \to \mathbb{R}_+$, one defines its symmetric decreasing rearrangement as:

$$\tilde{f}^\sharp(\theta) = \inf\{t : \lambda_{\tilde{f}}(t) \leq 2|\theta|\},$$

where, as before, $\lambda_{\tilde{f}}(t) = |\{\tilde{f} > t\}|$, and thus, there is a one-to-one correspondence between $f^\sharp$ and $\tilde{f}^\sharp$, given by

$$\tilde{f}^\sharp(\theta) = f^\sharp(e^{i\theta}).$$

Whenever necessary, we will think of a function $f$ defined on $\mathbb{S}^1$ as a function on $[-\pi, \pi]$. If $f = \chi_A$ is the characteristic function of a measurable set $A \subset \mathbb{S}^1$, then
\[ f^\sharp = \chi_{A^\sharp}, \text{ where } A^\sharp \text{ is the open interval on the unit circle centered at } 1, \text{ having the same linear measure as } A. \]

Next, we introduce the Hardy-Littlewood-Pólya preorder relation \(<\) for non-negative functions defined on the interval \([-\pi, \pi]\). We say that (see [3, 4]):

\[ f < F \iff \int_{-t}^{t} f^\sharp(s) \, ds \leq \int_{-t}^{t} F^\sharp(s) \, ds, \text{ for all } 0 \leq t \leq \pi. \]

This is equivalent to

\[ \int_{-\pi}^{\pi} f^\sharp(s) h^\sharp(s) \, ds \leq \int_{-\pi}^{\pi} F^\sharp(s) h^\sharp(s) \, ds, \]

for every positive symmetric decreasing function \(h^\sharp\) defined on \([-\pi, \pi]\). To see this, write \(h^\sharp(s) = \int_{0}^{\infty} \chi_{\{h^\sharp > t\}}(s) \, dt\) (this is the layer cake formula (1.2)), and, using Fubini’s formula and the fact that \(\{h^\sharp > t\} = (-l(t), l(t))\) is a symmetric interval,

\[ \int_{-\pi}^{\pi} f^\sharp(s) h^\sharp(s) \, ds = \int_{0}^{\infty} \left[ \int_{-l(t)}^{l(t)} f^\sharp(s) \, ds \right] \, dt \leq \int_{0}^{\infty} \left[ \int_{-l(t)}^{l(t)} F^\sharp(s) \, ds \right] \, dt = \int_{-\pi}^{\pi} F^\sharp(s) h^\sharp(s) \, ds. \]

Yet another equivalent characterization is:

\[ f < F \iff \int_{E} f(s) \, ds \leq \int_{E^\sharp} F(s) \, ds, \text{ for every } E \subset [-\pi, \pi]. \]

The next result is well-known and it follows from the proof of the equality case in the Hardy-Littlewood inequality, presented by Lieb and Loss in [10, pp.82]. We will include a proof here for consistency.

**Lemma 2.2.** Let \(f : [-\pi, \pi] \to \mathbb{R}_+\) be a measurable function such that

\[ \int_{-t}^{t} f(x) \, dx \geq \int_{-t}^{t} f^\sharp(x) \, dx, \text{ for every } 0 \leq t \leq \pi. \]

Then \(f = f^\sharp\) a.e. on \([-\pi, \pi]\).

**Proof.** From (1.1) applied to \(\chi_{(-t,t)}\) and \(f\), it follows that we must have equality in (2.1), i.e.,

\[ \int_{-t}^{t} f(x) \, dx = \int_{-t}^{t} f^\sharp(x) \, dx. \]

We will use the layer-cake formula to write \(f(x) = \int_{0}^{\infty} \chi_{\{f > s\}}(x) \, ds\), and similarly for \(f^\sharp(x)\).

Using (1.1), we obtain:

\[ \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx \leq \int_{-t}^{t} \chi_{\{f^\sharp > s\}}(x) \, dx, \text{ for every } s \geq 0. \]

Fubini’s theorem and (2.2) imply that:

\[ \int_{-t}^{t} f(x) \, dx = \int_{0}^{\infty} \left[ \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx \right] \, ds = \int_{0}^{\infty} \left[ \int_{-t}^{t} \chi_{\{f^\sharp > s\}}(x) \, dx \right] \, ds = \int_{-t}^{t} f^\sharp(x) \, dx. \]
From this equality and (2.3) it follows that, for a fixed $t$, there exists a set of measure zero $S_t$, such that
\[ \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx = \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx, \quad \text{for every } s \in (0, \infty) \setminus S_t. \]

Next, we choose $T_N$, a countable dense set in $[0, \pi]$ and we denote by $S_{TN} = \cup_{t \in T_N} S_t$.

Then:
\[ (2.4) \quad \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx = \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx, \quad \text{for every } t \in T_N \text{ and } s \in (0, \infty) \setminus S_{TN}. \]

Since for every fixed $s$, $t \mapsto \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx$ is a continuous function of $t$, in fact (2.4) holds for every $0 \leq t \leq \pi$. Thus,
\[ \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx = \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx, \quad \text{for all } 0 \leq t \leq \pi \text{ and a.e. } s \in (0, \infty). \]

Now, let $t$ be such that $\{f^2 > s\} = (-t, t)$. Then, it follows that $\{f > s\} = (-t, t) = \{f^2 > s\}$ a.e., and thus, $f = f^\sharp$ by the layer cake formula.

The following result shows that $\int_{-t}^{t} f^\sharp(x) \, dx$ is attained as a supremum. A proof can be found in [4, Theorem 7.5, pp.82].

**Theorem 2.3.** (J. V. Ryff) For every measurable function $f$ as in Lemma 2.2, there exists a measure preserving transformation $T$ such that $f = f^\sharp \circ T$. This guarantees, for every $t$, the existence of a set $A \subset [-\pi, \pi]$ of measure $2t$ such that $\int_A f(x) \, dx = \int_{-t}^{t} f^\sharp(x) \, dx$.

3. Main results: inequalities on the circle

**Notation.** As before, $d$ is the geodesic distance, also called the arclength, on the unit circle $S^1$. We have:
\[ (3.1) \quad d(u, v) = d(w\bar{v}, 1), \quad \text{for all } u, v \in S^1, \]

where $\bar{v}$ denotes the complex conjugate of $v$.

We define, for $\alpha > 0$, the function:
\[ \chi_\alpha(u, v) = \begin{cases} 1, & \text{if } d(u, v) \leq \alpha, \\ 0, & \text{otherwise} \end{cases} \]

and we observe that $\chi_\alpha(u, v) = \chi_\alpha(w\bar{v}, 1)$, by (3.1).

We introduce a new function, which we call again $\chi_\alpha : S^1 \to \mathbb{R}_+$, given by $\chi_\alpha(z) := \chi_\alpha(z, 1)$, which is the characteristic function of the closed interval on $S^1$ of linear length $2\alpha$, centered at 1.

We will make use, in what follows, of the relation:
\[ (3.2) \quad \chi_\alpha(w\bar{v}) = \chi_\alpha(u, v), \quad \text{for all } u, v \in S^1. \]

Given two positive measurable functions $f, g : S^1 \to \mathbb{R}_+$, their convolution, $f * g$, is defined to be the function:
\[ (f * g)(z_0) = \int_{S^1} f(z_0z)g(z) \, dz = \int_{-\pi}^{\pi} f(e^{i(\theta_0 - \theta)})g(e^{i\theta}) \, d\theta, \]
with $z_0 = e^{i\theta_0}$ and $dz$ represents the arclength element on $S^1$, usually denoted by $|dz|$. Given three positive functions $f, g, h$ defined on $S^1$, we can write

$$I_n = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) g(e^{i\theta}) h(e^{i\theta}) \, dt \, d\theta = (f * g * h^*)(1),$$

where $h^*(z) = h(\bar{z})$, i.e., $h^*(e^{i\theta}) = h(e^{-i\theta})$.

**Theorem 3.1.** Let $\sigma : S^1 \to S^1$ be a $C^1$ diffeomorphism such that $\sigma(1) = 1$ and $\sigma(-1) = -1$. Additionally, we assume that $\sigma(S^1_+) \subseteq S^1_-$ and $\sigma(S^1_-) \subseteq S^1_+$. Let $d$ be the geodesic distance on the unit circle, $\alpha$ be a positive real number, and we define the set $E = \{(x, y) \in S^1 \times S^1 : d(\sigma(x), y) \leq \alpha\}$. For $A, B \subseteq S^1$ measurable sets, let

$$I_\alpha(A, B) = \int_{S^1 \times S^1} \chi_A(x) \chi_B(y) \chi_E(x, y) \, dx \, dy.$$

Then, for any $A, B$ measurable subsets of $S^1$, and $\alpha > 0$,

$$(3.4) \quad I_\alpha(A, B) \leq I_\alpha(A^1, B^2),$$

if and only if, $\sigma$ is symmetric (i.e. $\sigma(z) = \sigma(\bar{z})$, for every $z \in S^1$) and convex on $S^1_+$.

**Proof.** Sufficiency. We define $\sigma_1 : [-\pi, \pi) \to [-\pi, \pi)$ by $e^{i\sigma_1(\theta)} := \sigma(e^{i\theta})$ and we assume that $\sigma_1$ is convex on $(0, \pi)$. Using change of variables, $(\sigma(x), y) = (u, v)$, the integral $I_\alpha$ becomes:

$$I_\alpha(A, B) = \int_{S^1 \times S^1} \chi_{\sigma(A)}(u) \chi_B(v) \chi_{\sigma}^{-1}(u) \psi(u) \, dudv.$$

With $\chi_{\alpha}(u, v) = \chi_{\alpha}(uv)$, as in (3.2), the above expression becomes:

$$(3.5) \quad I_\alpha(A, B) = \int_{S^1 \times S^1} \chi_{\sigma(A)}(u) \chi_B(v) \chi_{\sigma}^{-1}(uv) \psi(u) \, dudv,$$

where $\psi(e^{i\theta}) = \tau_1(\theta)$ and $\tau_1$ is defined by $\sigma^{-1}(e^{i\theta}) = e^{i\tau_1(\theta)}$, and is the inverse of $\sigma_1$.

Thus, we can write using convolution and (3.3):

$$I_\alpha(A, B) = [(\chi_{\sigma(A)} \ast \psi) \ast \chi_{\alpha} \ast \chi_B](1),$$

where we used the fact that $\chi_{\alpha}$ is a symmetric function.

It was proved in [1] (see also (1.4)) by Baernstein that, for any three positive measurable functions $f, g, h$ on $S^1$, the following inequality holds:

$$(f * g * h^*)(1) \leq (f^2 * g^2 * h^2)(1).$$

One can replace $h^*$ in the inequality above by $h$ since they are equimeasurable functions. Thus, based on (3.6) and the fact that $\chi_{\alpha}$ is symmetric decreasing, we conclude that:

$$(3.7) \quad I_\alpha(A, B) \leq [(\chi_{\sigma(A)} \ast \psi)^\dagger \ast \chi_{\alpha} \ast \chi_B](1).$$

**Fact:** If $F$ is a positive symmetric decreasing function and if $f \prec F$ in the sense of Hardy-Littlewood-Pólya (i.e. $\sup_{|G| = 2\theta} \int_G f \leq \int_{-\theta}^{\theta} F$), then $f^2$ in inequality (3.6) can be replaced by $F$. Indeed, $f \prec F$ is equivalent to $\int_{S^1} f^2(z)g^2(z) \, dz \leq \int_{S^1} F(z)g^2(z) \, dz,$
for all positive symmetric decreasing functions \( g^\sharp \). Now, since \( g^\sharp * h^\sharp \) is symmetric decreasing and since the convolution \((f^\sharp * g^\sharp * h^\sharp)(1)\) can be written as the integral of the product \( f^\sharp(z)(g^\sharp * h^\sharp)(z)\), we conclude that:

\[
(f^\sharp * g^\sharp * h^\sharp)(1) \leq (F * g^\sharp * h^\sharp)(1).
\]

Therefore, using (3.7) and the Fact, we can prove (3.4) if we show that \( \chi_{\sigma(A^\sharp)} \psi \prec \chi_{\sigma(A^\sharp)} \psi \), i.e.

\[
\int_{E} \chi_{\sigma(A)} \psi \leq \int_{E} \chi_{\sigma(A^\sharp)} \psi.
\]

Let \( E' = \sigma^{-1}(E) \), and \( E'' = \sigma^{-1}(E^\sharp) \). With these notations, inequality (3.8) becomes:

\[
\int_{A \cap E'} dx \leq \int_{A \cap E''} dx,
\]

or equivalently, \(|A \cap E'| \leq |A^\sharp \cap E''|\), which is true if \(|E'| \leq |E''|\), since \( E'' \) is symmetric. Since \( \psi \) is symmetric decreasing, we have that \( \int_{E} \psi(u) \, du \leq \int_{E^\sharp} \psi(u) \, du \), which is equivalent to \( \int_{\sigma^{-1}(E)} \psi(x) \, dx \leq \int_{\sigma^{-1}(E^\sharp)} \psi(x) \, dx \), using change of variables. The latter inequality simply states that \(|E'| \leq |E''|\), and the proof of the sufficiency is now complete.

**Necessity.** Dividing (3.5) by \( 2\alpha \), and letting \( \alpha \) tend to zero, we obtain:

\[
I_0(A, B) = \int_{S^1} \chi_{\sigma(A)}(u) \chi_B(u) \psi(u) \, du,
\]

and inequality (3.4) implies that:

\[
I_0(A, B) \leq I_0(A^\sharp, B^\sharp).
\]

With the notation \( \tau = \sigma^{-1} \), the Jacobian of \( \tau \), and \( x = \tau(u) \), \( I_0 \) becomes:

\[
I_0(A, B) = \int_{S^1} \chi_A(x) \chi_{\tau(B)}(x) \, dx = |A \cap \tau(B)|.
\]

First, we will show that the symmetry condition is necessary. Suppose \( \tau \) is not symmetric. Then, there exists a point \( x = e^{i\theta} \) in \( S^1 \), such that \( \tau(\bar{x}) \neq \tau(x) \). If we consider \( A = \tau(\{e^{it} : |t| < \theta\}) \) and \( B = \{e^{it} : |t| < \theta\} \), then we have: \(|A \cap \tau(B)| = |\tau(B)| > |A^\sharp \cap \tau(B^\sharp)|\), since \( \tau(B^\sharp) \) is not symmetric and \( |A| = |\tau(B)| \). But this contradicts (3.9) and therefore (3.4).

Suppose now that \( \tau_1 \) is symmetric, but not concave (or, equivalently, \( \sigma_1 \) is symmetric, but \( \sigma_1 \) is not convex on \((0, \pi))\). Then, there exist \( e^{ib}, e^{ic} \in S^1_+ \) with \( b, c \in (0, \pi) \) such that:

\[
\frac{\tau_1(b) + \tau_1(c)}{2} > \tau_1\left(\frac{b + c}{2}\right).
\]

Without loss of generality we can assume that \( b > c \) and let us denote by \( a = \frac{b + c}{2} \). Letting \( B = \{e^{it} : -c < t < b\} \), it follows that \( B^\sharp = \{e^{it} : -a < t < a\} \). We calculate \(|\tau(B)| = \tau_1(b) - \tau_1(-c) = \tau_1(b) + \tau_1(c) \) and \(|\tau(B^\sharp)| = 2\tau_1(a) \).

From (3.11) we obtain that \(|\tau(B)| > |\tau(B^\sharp)|\) which shows that \( I_0(S^1, B) > I_0(S^1, B^\sharp) \) and contradicts (3.4). Therefore, \( \tau \) must also be concave. \( \square \)
**Theorem 3.2.** Suppose we have two functions $\sigma_1, \sigma_2$ satisfying the conditions of $\sigma$ in Theorem 3.1 and define $E = \{(x, y) \in S^1 \times S^1 : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\}$, for $\alpha \in \mathbb{R}_+$. Let

$$I_\alpha(A, B) = \int_{S^1 \times S^1} \chi_A(x) \chi_B(y) \chi_E(x, y) dx dy.$$  

Then, for any $A, B$ subsets of $S^1$ and $\alpha > 0$,

$$(3.12) \quad I_\alpha(A, B) \leq I_\alpha(A^\dagger, B^\dagger),$$

if and only if $\sigma_1, \sigma_2$ are symmetric and convex on $S^1_+$. 

**Proof.** Sufficiency. Very similar to Theorem 3.1. Using change of variables, $(\sigma_1(x), \sigma_2(y)) = (u, v)$, the integral becomes:

$$I_\alpha(A, B) = \int_{S^1 \times S^1} \chi_{\sigma_1(A)}(u) \chi_{\sigma_2(B)}(v) \chi_{\alpha(uv)} \psi_1(u) \psi_2(v) du dv,$$

where $\psi_1, \psi_2$ are defined similarly to $\psi$ in Theorem 3.1 (see (3.5)). Using convolution, this integral can be written as:

$$I_\alpha(A, B) = \left[(\chi_{\sigma_1(A)} \cdot \psi_1) \ast \chi_{\alpha} \ast (\chi_{\sigma_2(B)} \cdot \psi_2)^-(1)\right].$$

We have already proven that $\chi_{\sigma_1(A)} \psi_1 \prec \chi_{\sigma_1(A)} \psi_1$ and $\chi_{\sigma_2(B)} \psi_2 \prec \chi_{\sigma_2(B)} \psi_2$, from which it follows that $I_\alpha(A, B) \leq I_\alpha(A^\dagger, B^\dagger).$

Necessity. Using change of variable $v = \sigma_2(y)$, $I_\alpha$ becomes:

$$I_\alpha(A, B) = \int_{S^1 \times S^1} \chi_A(x) \chi_{\chi_{\sigma_2(B)}(\sigma_1(x))} \psi_2(\sigma_1(x)) dx dy.$$

Dividing by $\alpha$ and letting $\alpha \to 0$, we obtain:

$$I_0(A, B) = \int_{S^1} \chi_A(x) \chi_{\sigma_2(B)}(\sigma_1(x)) \psi_2(\sigma_1(x)) dx.$$  

Inequality (3.12) of the theorem implies the following inequality:

$$(3.13) \quad I_0(A, B) \leq I_0(A^\dagger, B^\dagger),$$

for all subsets $A$ and $B$ of $S^1$.

Now let $B = S^1$ in the above identity. Then:

$$I_0(A, S^1) = \int_{S^1} \chi_A(x) \psi_2(\sigma_1(x)) dx \leq \int_{S^1} \chi_{A^\dagger}(x) \psi_2(\sigma_1(x)) dx,$$

or equivalently,

$$\int_A \psi_2(\sigma_1(x)) dx \leq \int_{A^\dagger} \psi_2(\sigma_1(x)) dx,$$

for every measurable set $A \subset S^1$. Since the inequality is true for every measurable set $A$, we conclude by Lemma 2.2 and Theorem 2.3 that $\psi_2 \circ \sigma_1$ is symmetric (i.e., $\psi_2(\sigma_1(z)) = \psi_2(\sigma_1(\tilde{z}))$) and decreasing, which implies that $\psi_2$ is decreasing on $S^1_+$. Likewise, $\psi_1 \circ \sigma_2$ is symmetric and decreasing on $S^1_+$, implying that $\psi_1$ is decreasing on $S^1_+$. Thus, $\sigma_1^{-1}$ and $\sigma_2^{-1}$ are concave on $S^1_+$ and therefore, $\sigma_1$ and $\sigma_2$ are convex on $S^1_+$. 

Next, we denote by \( \tau = \sigma^{-1}_1 \circ \sigma_2 \). With this notation, \( I_0 \) becomes:

\[
I_0(A, B) = \int_{S^1} \chi_A(x)\chi_{\sigma_2(B)}(\sigma_1(x))[\psi_2 \circ \sigma_1](x) dx
\]

\[
= \int_{S^1} \chi_A(x)\chi_{\tau(B)}(x)[\psi_2 \circ \sigma_1](x) dx = \int_{A \cap \tau(B)} [\psi_2 \circ \sigma_1](x) dx.
\]

We will show that \( \tau \) is symmetric, i.e., \( \tau(\bar{x}) = \bar{\tau(x)} \), for every \( x \in S^1 \). Suppose this is not the case. Then there exists \( x = e^{i\theta} \), with \( \theta \in (0, \pi) \), such that \( \tau(\bar{x}) \neq \tau(x) \).

Let \( B = \{ e^{it} : |t| < \theta \} = B^2 \) and \( A = \tau(B) \neq A^2 \). Then, we have that \( A^2 \cap \tau(B^2) \subset A \cap \tau(B) = A \) and \( |A \cap \tau(B)| > |A^2 \cap \tau(B^2)| \). Since \( \psi_2 \circ \sigma_1 \) is positive, it follows that \( I_0(A, B) > I_0(A^2, B^2) \), which contradicts (3.13). Thus, \( \sigma_1^{-1} \circ \sigma_2 \) is symmetric.

We have shown before that \( \psi_1 \circ \sigma_2 \) is also symmetric.

**Claim:** \( \sigma_1^{-1} \circ \sigma_2 \) and \( \psi_1 \circ \sigma_2 \) symmetric imply \( \sigma_2 \) is symmetric.

**Proof of claim:** We define \( f_2 \) on the interval \([-\pi, \pi]\) as follows:

\[
\sigma_2(e^{i\theta}) = e^{if_2(\theta)}.
\]

Since \( \psi_1 \circ \sigma_2 \) is symmetric and \([\psi_1 \circ \sigma_2](e^{i\theta}) = \psi_1(e^{i\gamma_2(\theta)}) = \tau_1(f_2(\theta))\), as in (3.5), it follows that \( \tau_1' \circ f_2 \) is even.

Since \([\sigma_1^{-1} \circ \sigma_2](e^{i\theta}) = e^{i\tau_1(f_2(\theta))}\) is symmetric, it follows that \( \tau_1 \circ f_2 \) is odd.

Now, \((\tau_1 \circ f_2)' = (\tau_1' \circ f_2) \cdot f_2'\) is even and \( \tau_1' \circ f_2 \) is also even (as we have previously shown) and nonzero, so that \( f_2' \) is even and thus \( f_2 \) is odd. Therefore \( \sigma_2 \) is symmetric and the proof of the claim is now complete.

Following exactly the same steps, we can show that \( \sigma_1 \) is symmetric. We have shown that \( \sigma_1, \sigma_2 \) are symmetric and convex on \( S^1_+ \).

**Corollary 3.3.** With \( \sigma, \alpha \) and \( E = \{(x, y) \in S^1 : d(\sigma(x), y) \leq \alpha \} \), as in Theorem 3.1, we have the following result: For every \( f, g : S^1 \to \mathbb{R}_+ \) positive measurable functions, and every \( \alpha > 0 \),

\[
(3.14) \quad \int_E f(x)g(y) \, dx \, dy \leq \int_E f^\alpha(x)g^\alpha(y) \, dx \, dy,
\]

if and only if, \( \sigma \) is symmetric, and convex on \( S^1_+ \).

To sketch the proof, we write \( f \) and \( g \) as the integrals of their level sets, using the layer-cake representation formula (1.2):

\[
f(x) = \int_0^\infty \chi_{\{f > t\}}(x) \, dt \quad \text{and} \quad g(y) = \int_0^\infty \chi_{\{g > t\}}(y) \, dt,
\]

and we notice that \( \{f > t\}^t = \{f^t > t\} \) and \( \{g > t\}^t = \{g^t > t\} \) so that inequality (3.14) reduces to the case where \( f \) and \( g \) are characteristic functions, and thus, Theorem 3.1 applies.

**Corollary 3.4.** Let \( \sigma_1, \sigma_2 \) and \( E = \{(x, y) \in S^1 \times S^1 : d(\sigma_1(x), \sigma_2(y)) \leq \alpha \} \) be as in Theorem 3.2. For every \( f, g : S^1 \to \mathbb{R}_+ \) positive measurable functions, and every \( \alpha > 0 \),

\[
(3.15) \quad \int_E f(x)g(y) \, dx \, dy \leq \int_E f^\alpha(x)g^\alpha(y) \, dx \, dy,
\]

if and only if, \( \sigma_1 \) and \( \sigma_2 \) are symmetric, and convex on \( S^1_+ \).
The proof of Corollary 3.4 is indeed very similar to the proof of Corollary 3.3, in which one represents \( f \) and \( g \) as integrals of the characteristic functions of their level sets.

The next theorem is a generalization of the previous results, where one replaces the product by a function \( \Psi \) defined as follows:

\[
\Psi : \mathbb{R}^2_+ \to \mathbb{R} \text{ vanishes on the boundary of } \mathbb{R}^2_+ \text{, i.e., } \Psi|_{\{x_1=0\}} = \Psi|_{\{x_2=0\}} = 0, \text{ and } \Psi(x_1, x_2) + \Psi(y_1, y_2) \leq \Psi(x_1 \land x_2, y_1 \land y_2) + \Psi(x_1 \lor x_2, y_1 \lor y_2).
\]

If \( \Psi \) is twice continuously differentiable, then the above inequality is equivalent to

\[
\partial_{\{0\}} \Psi \geq 0.
\]

Crowe, Zweibel and Rosenbloom [6] noticed that a continuous such \( \Psi \) is the distribution function of a Borel measure \( \mu \) on \( \mathbb{R}^2_+ \), i.e.,

\[
\Psi(s, t) = \mu((0, s) \times (0, t)),
\]

and using Fubini’s theorem:

\[
\int \Psi(f(x), g(y)) \, dx \, dy = \int_{\mathbb{R}^2_+} \left[ \int \chi(f > s)(x) \chi(g > t)(y) \, dx \, dy \right] \, d\mu(s, t).
\]

We are now ready to state our next result.

**Theorem 3.5.** With \( \sigma_1, \sigma_2 \) and \( E = \{(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\} \) as in Theorem 3.2, and \( \Psi \) the distribution function of a Borel measure \( \mu \) on \( \mathbb{R}^2_+ \) as in (3.16), the following inequality holds for every \( \alpha > 0 \):

\[
\int_E \Psi(f(x), g(y)) \, dx \, dy \leq \int_E \Psi(f^2(x), g^2(y)) \, dx \, dy,
\]

if and only if, \( \sigma_1 \) and \( \sigma_2 \) are symmetric on \( \mathbb{S}^1 \), and convex on \( \mathbb{S}^1_1 \).

Again, we can reduce \( \Psi(f(x), g(y)) \) to a product of characteristic functions, using (3.17), and the result follows from Theorem 3.2.

The next theorem shows that we can replace the characteristic function of the set \( E \) by a decreasing function of the distance between \( \sigma_1(x) \) and \( \sigma_2(y) \), call it \( k[d(\sigma_1(x), \sigma_2(y))] \).

**Theorem 3.6.** Let \( \sigma_1, \sigma_2 \) be as in Theorem 3.2 and let \( k : [0, \infty) \to [0, \infty) \) be a decreasing function, and \( \Psi \) the distribution function of a Borel measure \( \mu \) on \( \mathbb{R}^2_+ \) as in (3.16). Then, the following inequality holds for every decreasing function \( k \),

\[
\int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f(x), g(y))k[d(\sigma_1(x), \sigma_2(y))] \, dx \, dy \leq \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \Psi(f^2(x), g^2(y))k[d(\sigma_1(x), \sigma_2(y))] \, dx \, dy,
\]

if and only if, \( \sigma_1 \) and \( \sigma_2 \) are symmetric on \( \mathbb{S}^1 \), and convex on \( \mathbb{S}^1_1 \).

**Proof.** Using (1.2), we can write:

\[
k(\tau) = \int_0^\infty \chi(k > t)(\tau) \, dt = \int_0^\infty \chi(0, l_t)(\tau) \, dt,
\]

and substituting \( d(\sigma_1(x), \sigma_2(y)) \) for \( \tau \) in the above formula, we have

\[
k[d(\sigma_1(x), \sigma_2(y))] = \int_0^\infty \chi(0, l_t)[d(\sigma_1(x), \sigma_2(y))] \, dt.
\]
We define the set $E_{l(t)}$ as follows:

$$E_{l(t)} = \{(x, y) : d(\sigma_1(x), \sigma_2(y)) \leq l(t)\}.$$

Then

$$\chi_{[0, l(t)]}(d(\sigma_1(x), \sigma_2(y)) = 1 \iff (x, y) \in E_{l(t)}.$$ 

Using this fact, (3.18), Fubini’s theorem and Theorem 3.5 we obtain the conclusion of Theorem 3.6 by:

$$\int_{S^1} \int_{S^1} \Psi(f(x), g(y)) k[d(\sigma_1(x), \sigma_2(y))] \, dxdy \leq \int_0^\infty \int_{S^1} \int_{S^1} \Psi(f^\sharp(x), g^\sharp(y)) \chi_{E_{l(t)}}(x, y) \, dxdy \, dt \leq \int_0^\infty \int_{S^1} \int_{S^1} \Psi(f^\sharp(x), g^\sharp(y)) \chi_{E_{l(t)}}(x, y) \, dxdy \, dt,$$

$$= \int_{S^1} \int_{S^1} \Psi(f^\sharp(x), g^\sharp(y)) k[d(\sigma_1(x), \sigma_2(y))] \, dxdy.$$ 

□

REFERENCES


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