CHAPTER 11

BOUNDS ON THE PROBABILITY OF FAILURE IN RANDOM VIBRATION

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Abstract

A new method to construct upper and lower bounds for the failure probability of a structure exposed to random excitations is presented. The method is based on an extension of the classical inclusion-exclusion series of Rice where the main idea is to take into consideration only realizations, which are in the safe area at a number of previous instants of time. By a numerical example it is demonstrated that this method may lead to rather sharp bounds.

1. INTRODUCTION

During its lifetime a structure will be exposed to a variety of static and dynamic loads with variability in space and time, which cannot be foreseen by the designers. Such loadings may suitably be modelled as realizations from stochastic processes. In this paper only structures exposed to dynamic random loadings will be treated, which are defined as external loadings, which with a high probability will induce substantial inertial forces in the structure. It is then of great interest to estimate the probability that the response processes of such structures will withstand the prescribed design loading process during the specified lifetime. This probability is called the reliability probability and is equal to one minus the failure probability.

If the considered response process, which may typically represent a stress or a deflection at a critical point of the structure, and its derivative process of any order, can be modelled as a Markov vector process the problem stated above can at least in principle be solved. The solution fulfils the Kolmogorov backward equation with absorbing boundaries, which can at present only be solved in the one-dimensional case.

Therefore, it is natural to look at the possibility of constructing upper and lower bounds of the failure probability. These bounds must be sufficiently close to be of any use in estimating the reliability of the structure. In connection with structural design

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upper bounds are of course the most interesting because designs based on these bounds are conservative.

For stationary or non-stationary response processes which are differentiable in mean square, upper bounds for the numerical value of the corresponding extreme in a given time interval can be constructed on the basis of the continuous generalization of Chebyshev’s inequality, [1], [2]. As usual when bounds are based on Chebyshev’s inequality one gets rather crude bounds. Sharper upper bounds for the same quantity can be found by reducing the class of structures in consideration. If only linear time-invariant structures under stationary excitation are considered, Drenick’s inequality (31 in connection with Chebyshev’s and Markov’s inequalities will in this way generally lead to better upper bounds, [4], [5].

Shinozuka [6] has constructed an upper-bound solution based on the expected number of passages per unit time of a given limit state. An improvement of this result may be obtained as stated by Shinozuka and Yang [7], by considering only those limit state passages, which are in the safe area at a discrete number of preselected moments prior to the considered time.

It can be shown that the upper-bound solution by Shinozuka [6] corresponds to the first term in the classical inclusion-exclusion series by Rice [8]. Successive partial sums of this series result in upper and lower bounds to the first-passage density function see Slepian [9] and Roberts [10]. Upper bounds can be constructed by retaining an odd number of terms and lower bounds by retaining an even number of terms. This provides obviously a general method in which still closer upper and lower bounds may be constructed. In [10] the first three terms are calculated for a Gaussian process and for small excitation times.

The present paper presents a new method to derive upper and lower bounds, which can be considered a generalization of the method, based on inclusion-exclusion series. The generalization is obtained by only considering realizations, which are in the safe area at a number of previous instants of time. It is demonstrated that any lower or upper bound constructed from the series will be improved and converge monotonically against the exact failure probability, as the number of pre-scribed intermediate instants of time is increased. Further, for a specified number of intermediate instants of time it is shown how the bounds may be optimized.

A numerical example is presented, which indicates that the upper and lower bound from the inclusion-exclusion series with a limited number of intermediate instants of time may lead to rather sharp bounds even at low barrier heights. The first upper bound is finally used to obtain some approximate results of the distribution of extremes of a Gaussian response process. The results indicate that the extreme value distribution to some extent depends on the damping of the structure, at least when the excitation time is moderate.

2. DEFINITIONS AND ASSUMPTIONS

The response process is assumed to be a vector-valued stochastic process \( \{X(t), t \in [0; \infty]\} \), \( X(t) : \Omega \rightarrow R^n \) with index values in the time interval \([0; \infty] \). The safe area at time \( t \) is defined as an open, simply connected subset \( S_t \subset R^n \). The complement of \( S_t \) is the unsafe area. For a fixed instant of time \( t \) the limit state \( \partial S_t \) is defined by the function \( L : [0; \infty] \times R^n \rightarrow R \), where
\[ \partial S_t = \{ \mathbf{r} \in \mathbb{R}^n | L(t, \mathbf{r}) = 0 \} \quad (1) \]

Notice, that \( \partial S_t \) is in the unsafe area.

For fixed \( \omega \in \Omega \), \( X(t) \) represents an ordinary vector-valued function of time, called a realization of the response process. The event that a realization leaves the safe area is termed a limit state passage. It is assumed that the set of \( \omega \in \Omega \) for which the realizations are sufficiently smooth to ensure that only a finite number of limit state passages take place in a finite time interval has the probability measure one.

The investigations will be limited to the set of realizations \( \Gamma_0 \), which at time \( t = 0 \) is in the safe area \( S_0 \). During the time interval \( ]0; t] \) a subset \( \Gamma \subseteq \Gamma_0 \) of the realizations will leave the safe area at least once. The probability measure of \( \Gamma \) is
\[ P[\Gamma] = Q(0,t) \quad (2) \]
where \( Q(0,t) \) represents the probability of failure in the time interval \( ]0; t] \).

3. LIMIT STATE PASSAGE DENSITIES

A subset \( \Delta \Gamma \) of \( \Gamma \)-realizations will leave the safe area during the interval \( ]\tau; \tau + \Delta \tau[ \). It is assumed that the probability of this set can be written
\[ P(\Delta \Gamma) = f_i(\tau)\Delta \tau + o(\Delta \tau) \quad (3) \]
where the function \( f_i: ]0; \infty[ \rightarrow \mathbb{R}_+ \) will be called the limit state passage density function. Because of the assumed smoothness of the realizations the probability that more than one limit state passage will occur in the interval \( \Delta \tau \) is negligible. Therefore, the function \( f_i(\tau) \) can be interpreted as the expected number of limit state passages per unit time at time \( \tau \). Note that the condition imposed on the considered realizations at time \( t = 0 \) implies that \( f_i \) will be time-dependent, even when the unconditioned process \( \{ X(t) \} \) is stationary and the limit state is time-invariant.

Next we consider the subset \( \Delta' \Gamma \) of \( \Gamma \)-realizations, which leaves the safe area at least once in each of the non-overlapping intervals \( ]\tau_i; \tau_i + \Delta \tau_i[ \), \( ]\tau_i + \Delta \tau_i; \tau_{i+1} + \Delta \tau_{i+1}[ \), \( \ldots \), \( ]\tau_l; \tau_l + \Delta \tau_l[ \), where \( 0 < \tau_1 < \tau_2 < \ldots < \tau_l < \tau + \Delta \tau \leq t \). It is assumed that the probability of this set may be written
\[ P(\Delta' \Gamma) = f_i(\tau_1, \ldots, \tau_l)\Delta \tau_1 \ldots \Delta \tau_l + o((\Delta \tau)^l) \quad (4) \]
where \( \Delta \tau = \max(\Delta \tau_1, \ldots, \Delta \tau_l) \). The function \( f_i: ]0; \infty[ \rightarrow \mathbb{R}_+ \) will be called the joint limit state passage density function of order \( l \).

The function \( f_i \) can be calculated from the following expression, which is a generalization of Rice's formula to \( n \)-dimensional vector processes ([11], [12])
\[ f_i(\tau) = \int_{S_0} \left( \int_{\mathbb{R}_+} (v - \bar{b}) p_{XV}(r, \tau | S_0) dV \right) dr \quad (5) \]
\[ V(\tau) N = \sum_{i=1}^n \hat{X}_i(\tau)n_i \quad (6) \]
\[ b(\tau, r) = \sum_{i=1}^{n} \frac{\partial L(\tau, r)}{\partial r_i} n_i \]  

(7)

where \( \dot{X}(\tau) \) and \( n_i \) are the \( i \)th components of the derivative vector process \( X(\tau) \) and the unit normal vector of the limit state in the outward direction at position \( r \) and time \( \tau \), respectively. Therefore, \( V \) and \( b \) represent the normal component of velocity and the gradient of the limit state. \( \dot{b} \) is the total time derivative of \( b \).

In equation (5) \( p_{XV} \) is the joint probability density of \( X \) and \( V \) on the condition that \( X_0 = X(0) \) is in the safe area \( S_0 \). Therefore, \( p_{XV} \) can be calculated from the following equation

\[
p_{XV}(x, V, \tau|S_0) = \frac{\int_{S_0} p_{XVX}(x, V, x_0) dx_0}{\int_{S_0} p_{X_0}(x_0) dx_0}
\]

(8)

where \( p_{XVX} \) is the unconditioned joint probability density function of \( X(\tau), X_0, \) and \( V(\tau) \), and where \( p_{X_0} \) is the corresponding marginal density function of \( X_0 \).

4. CONDITIONED LIMIT STATE PASSAGE DENSITIES

In section 3 realizations were considered which leave the safe area at least once in the interval \( [0; t] \). In this section a subset \( \Gamma_N \subseteq \Gamma \) of \( \Gamma \)-realizations, which at a finite number of intermediate times \( t_1, t_2, \ldots, t_N \), where \( 0 < t_1 < \ldots < t_N < t \) are in the safe area, will be considered. When these conditions are imposed the limit state passage densities defined in section 3 will be called **conditioned limit state passage densities**. The functions \( f_i \) will then be functions of \( \tau_1, \ldots, \tau_i \) and also of \( t_1, \ldots, t_N \). It is obvious that the value of \( f_i \) will not increase when the number of intermediate instants of time is increased. This can be formulated by

\[
f_i(\tau_1, \ldots, \tau_j; t_1, \ldots, t_j, t_{i+1}, \ldots, t_N) \geq f_i(\tau_1, \ldots, \tau_j; t_1, \ldots, t_{N+1}, t_{i+1}, \ldots, t_N)
\]

(9)

Both sides represent the conditioned joint limit state passage density of order \( N \) with limit state passage at \( \tau_1, \ldots, \tau_i \) and in the safe area at time \( t_1, \ldots, t_i, t_{i+1}, \ldots, t_N \). However, the right-hand side has also a restriction at time \( t_{N+1} \) where \( t_i < t_{N+1} < t_{i+1} \).

The function \( f_i \) can still be calculated by (5) but \( p_{XV} \) is now given for \( \tau \in (t_N; t) \) by

\[
p_{XV}(x, V, \tau, t_1, \ldots, t_N|S_0) = \frac{\int_{S_0} \int_{S_0} \ldots \int_{S_0} p_{XVX_1 \ldots X_N}(x, V, x_0, x_1, \ldots, x_N) dx_0 dx_1 \ldots dx_N}{\int_{S_0} p_{X_0}(x_0) dx_0}
\]

(10)

where \( X_i = X(i) \) and \( p_{XVX_1 \ldots X_N} \) are the joint probability density functions of the stochastic vectors \( X, X_0, X_1, \ldots, X_N \) and \( V \).
As a straightforward generalization of (5) the functions \( f_l \), \( l \geq 1 \) are given by

\[
f_l(\tau_1, \ldots, \tau_l) = \int \cdots \int_{\mathcal{S}_1 \cdots \mathcal{S}_l} (V_i - b_i) \ldots (V_1 - V_l) \times p_{X_{i_1}, \ldots, X_{i_{\mu_l}}} \left( r_i, V_{i_1}, \ldots, V_{i_{\mu_l}} | S_0 \right) dV_1 \ldots dV_l dr_1 \ldots dr_l , \quad l \geq 1
\]

where \( X_i = X(\tau_i), V_i = V(\tau_i), b_i = \frac{d}{d\tau} b(\tau, r(\tau)) \big|_{\tau = \tau_i} \).

The probability density in (11) can be determined from a conditioned probability in a way analogous to (8). Further, when intermediate conditions are prescribed, the probability density has to be calculated from an integral similar to (10).

5. FIRST-PASSAGE PROBABILITY DENSITIES

In the last two sections limit state passage probabilities have been treated without taking into consideration whether a given passage is a first-passage or not. In estimating the failure probability of a structure exposed to random excitations, first-passage probability densities play an important role.

A subset \( \Gamma_0 \) of \( \Gamma \)-realizations will leave the safe area for the first time during the interval \( [\tau ; \tau + \Delta \tau] \). Note that \( \Gamma_0 \subseteq \Gamma \subseteq \Gamma \). It is assumed that the probability of the set \( \Delta \Gamma_0 \) can be written

\[
P(\Delta \Gamma_0) = Q(\tau, \tau + \Delta \tau) = f_0(\tau) \Delta \tau + o(\Delta \tau)
\]

where the function \( f_0 : ]0; \infty[ \rightarrow \mathbb{R}_+ \) will be called the first passage probability density function. Obviously,

\[
Q(0, t) = \int_0^t f_0(\tau) d\tau
\]

It is the ultimate object of this paper to show how upper and lower bounds for \( f_0 \) can be constructed. Before doing this some important limits can be proved. It is easy seen that the following limits hold

\[
f_1(\tau ; t_1, \ldots, t_N) \rightarrow f_0(\tau) \quad \text{for} \quad \delta_N \rightarrow 0
\]

\[
f_l(\tau_1, \ldots, \tau_l ; t_1, \ldots, t_N) \rightarrow 0 \quad \text{for} \quad \delta_N \rightarrow 0
\]

where \( 0 < \tau_1 < \ldots < \tau_l \leq t, \quad 0 < t_1 < \ldots < t_N < \tau_i \) and where

\[
\delta_N = \max \left( t_{i+1} - t_i \right) , \quad t_0 = 0 , \quad t_{N+1} = t
\]

The limits (14) and (15) apply, because as \( \delta_N \rightarrow 0 \), the assumed smoothness of the realizations ensure that almost all realizations at time \( \tau \) are first passages. Further the convergence of the sequences (14) and (15) will be monotonic according to (9).

Another consequence of the smoothness of the realizations is that almost none of them will leave the safe area in a differential time. Therefore, the following limits apply
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\[ f_i(\tau_1, \ldots, \tau_i; t_1, \ldots, t_{i-1}, t_i, t_{i+1}, t_N) \rightarrow \]
\[ f_i(\tau_1, \ldots, \tau_i; t_1, \ldots, t_{i-1}, t_i, t_{i+1}, t_N) \quad \text{for} \quad t_i \to t_{i+1} \quad \text{or} \quad t_i \to t_{i-1} \quad \text{(17)} \]

The limit state passages become independent events if the structure has extreme damping or if the failure probability is small. Then the number of limit state passages is a Poisson counting process with intensity \( f_i(\tau) \). \( f_0(\tau) \) is then given by (see [1])

\[ f_0(\tau) = f_i(\tau) \exp\left( -\int_0^\tau f_i(u)du \right) \quad \text{(18)} \]

Finally assume that the response process is stationary and the safe area \( S_i \) is time-invariant. Then a random variable \( Y \), representing the length of time intervals between succeeding in- and out crossings of a certain realization, can be defined. If the probability of tangency of the limit state at any time is 0, the distribution functions \( F_Y \) of \( Y \) are related to the first passage probability density function by the following relation, obtained independently by Rice [13] and Cook [14]

\[ f_0(\tau) = f_i(0)(1 - F_Y(\tau)) \quad \text{(19)} \]

It follows from (19) that \( f_0 \) is a non-increasing function under these assumptions.

6. INCLUSION - EXCLUSION SERIES

In this section a generalization of the inclusion-exclusion series by Rice [8] will be derived. The derivation will be based on \( \Gamma_N \)-realizations, that is realizations, which are in the safe area at a finite number of intermediate times \( t_1, t_2, \ldots, t_N \). Then in the next section it will be shown how upper and lower bounds for \( f_0 \) can be derived on the basis of this inclusion-exclusion series.

Let \( A(\tau; t_1, \ldots, t_N | \tau_i) \) be the limit state density at time \( \tau \) of \( \Gamma_N \)-realizations on the condition that a first passage has occurred at a previous time \( \tau_i \). Similarly \( B(\tau, \tau_i; t_1, \ldots, t_N | \tau_2) \) is the joint limit state passage at times \( \tau \) and \( \tau_i \) on condition that a first passage has occurred at a previous time \( \tau_2 < \tau_i < \tau \).

The following fundamental identity can then be formulated

\[ f_0(\tau) = f_i(\tau; t_1, \ldots, t_N) - \int_0^\tau A(\tau; t_1, \ldots, t_N | \tau_i) f_0(\tau_i) d\tau_i \quad \text{(20)} \]

The purpose of the last term in (20) is to subtract the probability density of the set of \( \Gamma_N \)-realizations with limit state passages at time \( \tau \), which have already had their first passage at times prior to \( \tau \). The kernel in this integral equation (20) can be expressed in the following way

\[ A(\tau; t_1, \ldots, t_N | \tau_i) f_0(\tau_i) = \]
\[ f_2(\tau, \tau_i; t_1, \ldots, t_N) - \int_0^\tau B(\tau, \tau_i; t_1, \ldots, t_N | \tau_2) f_0(\tau_2) d\tau_2 \quad \text{(21)} \]
where the last term subtracts the probability of the set of $\Gamma_N$-realizations with joint limit state passages at times $\tau$ and $\tau_1$ which have had their first passages prior to $\tau_1 < \tau$.

Inserting (21) into (20) gives

$$f_0(\tau) = f_1(\tau; t_1, \ldots, t_N) - \int_0^{\tau} f_2(\tau, \tau_1; t_1, \ldots, t_N) d\tau_1$$

$$+ \int_0^{\tau_1} \int_0^{\tau_2} B(\tau, \tau_1; t_1, \ldots, t_N | \tau_2) f_0(\tau_2) d\tau_1 d\tau_2$$

where $B(\tau, \tau_1; t_1, \ldots, t_N | \tau_2)$ is defined above. The outlined process leading from (20) to (22) can be continued, so that $f_0(\tau)$ can be expressed by a series of the form

$$f_0(\tau) = \sum_{i=0}^{\infty} (-1)^i Q_i(\tau; t_1, \ldots, t_N)$$

where

$$Q_i(\tau; t_1, \ldots, t_N) = \begin{cases} f_1(\tau; t_1, \ldots, t_N) & \text{for } i = 0 \\ \int_0^{\tau_1} \cdots \int_0^{\tau_{i-1}} f_i(\tau; \tau_1, \ldots, \tau_{i-1}; t_1, \ldots, t_N) d\tau_1 \cdots d\tau_{i-1} & \text{for } i > 1 \end{cases}$$

The series (23) is a generalization of Rice's method of inclusion-exclusion [8], which essentially corresponds to the case $N = 0$.

**7. CONSTRUCTION OF BOUNDS**

In this section it is demonstrated how upper and lower bounds for the first-passage probability density function $f_0$ can be constructed by truncating the series (23). If, for example, the last term in (20) is omitted, an upper bound for $f_0$ will be obtained because the integral is positive. In the same way omission of the last term in (22) will result in a lower bound. In general it is seen that the series (23) provides upper bounds when an odd number of terms remain and lower bounds when the number of terms are even. This can be formulated by the following inequality

$$R_k(\tau; t_1, \ldots, t_N) \leq f_0(\tau) \leq S_k(\tau; t_1, \ldots, t_N)$$

where $k = 1, 2, \ldots$ and

$$R_k(\tau; t_1, \ldots, t_N) = \sum_{i=0}^{2k} (-1)^i Q_i(\tau; t_1, \ldots, t_N)$$

$$S_k(\tau; t_1, \ldots, t_N) = \sum_{i=1}^{2k-1} (-1)^i Q_i(\tau; t_1, \ldots, t_N)$$

$R_k$ and $S_k$ will be referred to as the $k$th lower and upper bound, respectively.

The first upper bound $S_1(\tau; t_1, \ldots, t_N) = f_1(\tau; t_1, \ldots, t_N)$ is essentially the upper bound indicated by Shinozuka and Yang [7].
Next, it will be shown that for a fixed \( k \) the sequence of lower bounds is non-decreasing when the number of intermediate times \( N \) is increased and that the corresponding sequence of upper bounds is non-increasing. Further, the two sequences mentioned will converge to the same limit under specific circumstances. All this is formulated in the following theorem.

**Theorem**

For any \( k \geq 1 \) the lower bounds form a non-decreasing sequence \( \{ R_k(t_1, t_i, \ldots, t_N) \} \) and the upper bounds a non-increasing sequence \( \{ S_k(t_1, t_i, \ldots, t_N) \} \) as the number \( N \) of intermediate times is increased. When the maximum interval length \( \delta_N \) between any succeeding intermediate times goes to zero, both sequences converge to \( f_0(\tau) \).

**Proof**

Consider the first lower bound. From equation (22), where the lower bound is equal to the first two terms on the right hand side, one gets

\[
R_1(\tau; t_1, t_i, t_{i+1}, \ldots, t_N) = f_0(\tau) - \int_0^{\frac{\tau}{N}} \int_0^{\frac{\tau}{N}} B(\tau, \tau_i; t_1, t_i, t_{i+1}, t_N) \, d\tau_i \, d\tau_2
\]

and

\[
R_1(\tau; t_1, t_i, t_{i+1}, \ldots, t_N) = f_0(\tau) - \int_0^{\frac{\tau}{N}} \int_0^{\frac{\tau}{N}} B(\tau, \tau_i; t_1, t_i, t_{i+1}, t_N) \, d\tau_i \, d\tau_2
\]

By a similar argument as used in derivation of (9) it is seen that

\[
B(\tau, \tau_1; t_1, t_i, t_{i+1}, t_N) \leq B(\tau, \tau_1; t_1, t_i, t_{i+1}, t_N) \]

It then follows from (28) and (29) that

\[
R_1(\tau; t_1, t_i, t_{i+1}, t_N) \leq R_1(\tau; t_1, t_i, t_{i+1}, t_N)
\]

This proves that the first lower bound forms a non-decreasing sequence, when the number of intermediate times is increased in this way. By repeating this argument, the same result is seen to hold for any lower bound \( R_k, k \geq 1 \). By a similar argument it is easy to see that any upper bound \( S_k, k \geq 1 \), forms a non-increasing sequence as the number of intermediate times are increased. The sequences \( \{ R_k \} \) and \( \{ S_k \} \) are both bounded by \( f_0(\tau) \) so they converge to a limit when \( N \to \infty \) in the prescribed sense. It follows from (10) and (11) that this limit is \( f_0(\tau) \) when the intervals between succeeding intermediate times go to zero. This completes the proof of the theorem.

Consider the lower and upper bounds in (25). For a given \( k \) and a fixed number of intermediate times \( N \) a certain position of these intermediate times \( t_1, \ldots, t_N \) gives the best \( k \)th lower and upper bound, defined by

\[
R_k^*(\tau, N) = \max_{0 < t_i < \ldots < t_N < \tau} R_k(\tau; t_1, \ldots, t_N)
\]
and

\[ S_i^*(\tau, N) = \min_{0 < t_1 < \ldots < t_N < \tau} S_i(\tau; t_1, \ldots, t_N) \]  (33)

If the first passage density function is a non-decreasing function of time, as in the case of stationary processes and time-invariant limit state, cf. (19), the previous bounds may be sharpened in the following way

\[ R^*_i(\tau, N) = \max_{\tau \leq t_1 < \infty} R^*_i(\tau, N) \]  (34)

\[ S^*_i(\tau, N) = \min_{0 \leq t < t_1} S^*_i(\tau, N) \]  (35)

8. NUMERICAL RESULTS

The upper and lower bounds derived in section 7 will in this section be evaluated on the basis of a simple example. To simplify the numerical calculations the response process \( \{X(t), t \in [0; \infty]\} \) is assumed to be a stationary one-dimensional Gaussian process. Without any loss of generality the process can be assumed to be normalized to zero mean and unit standard deviation. Such a process is completely defined by its autocorrelation coefficient \( \rho \). Note that due to the stationarity \( \rho \) only depends on the index values \( t_1 \) and \( t_2 \) through the difference \( \Delta t = t_2 - t_1 \).

Further the considered system is assumed to be a linear time-invariant one-degree-of-freedom system under stationary excitation of white noise. Then (see [17])

\[ \rho(\Delta t) = \exp(-\zeta \omega_0 |\Delta t|)(\cos \omega_d \Delta t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d |\Delta t|) \]  (36)

where \( \omega_d = \omega_0 \sqrt{1 - \zeta^2} \) and where \( \zeta \) and \( \omega_0 \) are the critical damping ratio and cyclical eigenfrequency of the system, respectively. The time will be measured in units of the fundamental period \( T_0 = 2\pi / \omega_0 \) where \( \omega_0 = \sigma_X \) is the standard deviation of the derivative process.

The safe area is assumed to be time-invariant and defined by the semi-interval \( S_0 = (-\infty; b] \). As shown in the appendix the function \( f_i(\tau; t_1, \ldots, t_N) \) can then be calculated from

\[ f_i(\tau; t_1, \ldots, t_N) = \frac{\exp(-\frac{1}{2} b^2)}{\Phi(b)} \int_0^\infty \exp(-\frac{1}{2} y^2) F(b, y) dy \]  (37)
where $F$ is given by the formulas (A.27) - (A.34) in the appendix, and where $\Phi$ is the distribution function for the standardized normal distribution. Note that $F$ is a function also of $\tau; t_1, \ldots, t_N$.

The numerical calculations are restricted to the case $N = 0$, $N = 1$, and $N = 2$ all with $\xi = 0.01$, and $b = 2.00$. All numerical quadratures have been solved by means of a Chebyshev expansion with 35 terms.

**Upper bounds.** In the figures 1, 2 and 3 upper bounds are shown according to (33) and (35) for the three cases $N = 0$, $N = 1$ and $N = 2$. Simulation results obtained from equation (19) are also shown for comparison.

On figure 1 the minimum $f_1(\tau) = 0.045050$ occurs when $\tau_1 = 0.92478$. This value 0.045050 is the global minimum of $f_1(\tau)$. Therefore, the upper bound (35) attains this value at all succeeding times.

As shown in figure 2, the optimal position of the intermediate time $t_1$ making $f_1(\tau_1; t_1)$ a local minimum, occurs when $\tau_1 - t_1 \approx 0.90$-0.92. The local minima, which are indicated in table 1, will also be upper bounds at succeeding times according to (35) as long as these are less than the upper bound after (33).

![Figure 2. First upper bound, $N = 1$, $b = 2$, $\xi = 0.01$](image1)

<table>
<thead>
<tr>
<th>$t_1$</th>
<th>$r_1$</th>
<th>$f_1(\tau_1, t_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.061</td>
<td>0.961</td>
<td>0.04050</td>
</tr>
<tr>
<td>1.019</td>
<td>1.941</td>
<td>0.03723</td>
</tr>
<tr>
<td>2.016</td>
<td>2.939</td>
<td>0.03712</td>
</tr>
</tbody>
</table>

Table 1. Local minima of first upper bound, $N=1$.

![Figure 3. First upper bound, $N = 2$, $b = 2$, $\xi = 0.01$](image2)

<table>
<thead>
<tr>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$r_1$</th>
<th>$f_1(\tau_1, t_1, t_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.046</td>
<td>0.091</td>
<td>0.978</td>
<td>0.03943</td>
</tr>
<tr>
<td>0.975</td>
<td>1.039</td>
<td>1.937</td>
<td>0.03415</td>
</tr>
<tr>
<td>1.019</td>
<td>2.028</td>
<td>2.950</td>
<td>0.03375</td>
</tr>
<tr>
<td>2.002</td>
<td>3.021</td>
<td>3.942</td>
<td>0.03320</td>
</tr>
</tbody>
</table>

Table 2. Local minima of first upper bound, $N=2$. 
The upper bound in figure 3 is constructed solely on the basis of the local minima of $f_i(\tau_i, t_1, t_2)$. However, it is reasonable in analogy with figure 2 to expect that the optimal upper bound (35) is only slightly sharper. The local minima of $f_i$ are shown in table 2. Note that at the first two local minima $t_1$ and $t_2$ are very close together, namely almost one period before the minimum value $\tau_1$. At the succeeding minima the three times $t_1$, $t_2$ and $\tau_1$ are separated by almost one period. However, this pattern may change with different damping ratios and barrier levels.

**Lower bounds.** The first lower bounds according to (32) and (34) are shown in figures 4 and 5, respectively.

![Figure 4. First lower bound, $N = 0, b = 2, \xi = 0.01$](image1)

![Figure 5. First lower bound, $N = 1, b = 2, \xi = 0.01$](image2)

<table>
<thead>
<tr>
<th>$\tau_1$</th>
<th>$R_1(\tau_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.500</td>
<td>0.03872</td>
</tr>
<tr>
<td>2.888</td>
<td>0.003742</td>
</tr>
<tr>
<td>3.892</td>
<td>-0.03132</td>
</tr>
<tr>
<td>4.893</td>
<td>-0.07014</td>
</tr>
<tr>
<td>5.893</td>
<td>-0.1110</td>
</tr>
</tbody>
</table>

Table 3. Local maxima of first lower bound, $N = 0$.

<table>
<thead>
<tr>
<th>$t_1$</th>
<th>$\tau_1$</th>
<th>$R_1(t_1, \tau_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.50</td>
<td>1.50</td>
<td>0.0387</td>
</tr>
<tr>
<td>2.50</td>
<td>1.58</td>
<td>0.0278</td>
</tr>
<tr>
<td>3.89</td>
<td>2.95</td>
<td>0.0202</td>
</tr>
<tr>
<td>4.89</td>
<td>3.96</td>
<td>0.0132</td>
</tr>
<tr>
<td>5.89</td>
<td>4.95</td>
<td>0.00530</td>
</tr>
</tbody>
</table>

Table 4. Local maxima of first lower bound, $N = 1$. 

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The local maxima of \( R_i(\tau_i) \) (see figure 4) are shown in table 3. A local minimum, almost invisible at figure 4, occurs at \( \tau_i = 1.000 \), where \( R_i(1.000) = 0.03848 \).

The local maxima of \( R_i(\tau_i; t_i) \) (see figure 5) are shown in table 4. Within the accuracy of the numerical calculations the ordinate of the first horizontal part of the curve has not been increased, which indicates that the obtained value is close to the first passage density. Combining the results from table 2 and table 4 we have \( 0.03872 < f(1.5) < 0.039431 \). The local minimum in the vicinity of \( \tau_i = 1.000 \), mentioned above, has not been registered in the calculation of \( R_i'(\tau_i; 1) \).

9. DISTRIBUTION OF MAXIMUM RESPONSE

In this section it is shown how the theory presented in the previous sections can be used to construct an approximate distribution for the maximum response \( X_{\text{max}} \) in a time interval \((0; t]\). The distribution function is related to the probability of failure \( Q(t, x) \) of the process \( \{X(\tau), \tau \in (0; t]\} \) by the formula

\[
F_{X_{\text{max}}}(x) = 1 - Q(t, x)
\]  

where it is assumed that the safe area is \( S_0 = (-\infty; x] \).

If \( F_{X_{\text{max}}}(x) = 0 \) for \( x < 0 \) the moments of \( X_{\text{max}} \) can be computed from

\[
E[X_{\text{max}}^n] = \int_0^\infty x^{n-1}(1 - Q(t, x))dx
\]  

In the literature a Poisson type of approximation for \( f_0(\tau) \) is often used

\[
f_0(\tau) = f_1(0) \exp(-f_1(0)\tau) \Rightarrow Q(t, x) = 1 - \exp(-f_1(0)\tau)
\]  

where \( f_1(0) \) is the limit state passage density at \( \tau = 0 \), calculated from (5).

In this section we will use a different approximation for \( f_0(\tau) \) based on the upper bound \( f_1(\tau) \), namely

\[
f_0(\tau) = \begin{cases} f_1(\tau) & \text{for } \tau \in (0; \tau_i] \\ f_1(\tau_i) \exp[-\alpha f_1(0)(\tau - \tau_i)] & \text{for } \tau \in [\tau_i; \infty) \end{cases}
\]  

where

\[
\alpha = \frac{f_1(\tau_i)}{f_1(0)}(1 - \int_0^{\tau_i} f_1(u)du)^{-1}
\]  

and where \( f_1(\tau_i) \) is the minimum value of \( f_1 \). In this way the area under the corresponding curve becomes equal to one. This curve (41) is shown in figure 1 as curve (4). It is an improvement if \( \alpha < 0 \), which will be the case for narrow-banded random processes, unless the safety levels are extremely low. Actually \( \alpha f_1(0) \) is an estimate of the so-called limiting decay rate, assuming an asymptotic exponential relation of the first-passage probability density curve [15, 18]. Simulation estimates indicate that the approximation (41) leads to conservative results, merely because \( \alpha f_1(0) \) is greater than the actual decay rate.
The variation of $\alpha$ with the barrier level $b$ is shown in figure 6 for four damping values. From this figure it is seen that the approximation (41) is most favourable, when the damping is low and the barrier level is moderate.

![Figure 6. Variation of the decay rate coefficient $\alpha$ vs. threshold level and damping ratio.](image)

The so-called *peak factor*, that is, the expected value of $X_{\text{max}}$, is shown in figure 7 for four values of the damping ratio as a function of the excitation time in units of the fundamental period $T_0$. The calculation is based on equation (39), where the minimum resulting from the approximation (40) and (41) has been used for $Q(\tau, x)$ As a standard of reference the following well-known approximation is also shown [16]

$$E[X_{\text{max}}] = \sqrt{2\ln \tau} + \frac{0.577216}{\sqrt{2\ln \tau}}$$

(43)

The approximation (43) is based essentially on the approximation (40). Figure 7 indicates that the peak factors may be reduced somewhat in narrow-banded vibration compared to the common approach to the subject.

The approximation (41) can also be applied to the upper bound $f_i(\tau; t_1, \ldots, t_N)$ when $f_i(\tau_1)$ is interpreted as the global minimum of $f_i(\tau; t_1, \ldots, t_N)$.

10. CONCLUSIONS

It is shown how upper and lower bounds for the first passage density function can be constructed for stationary and non-stationary processes. A further improvement is possible when the first passage density function is non-increasing. This is for example the case when the process is stationary and the limit state time-invariant.

In the method a number of intermediate instants of time $N$ are introduced. When $N \to \infty$ the gap between the upper and lower bounds will decrease and can in principle be made so small as one wishes with a proper selection of intermediate times.

The method presented has been tested on a low barrier problem with a promising result. It is reasonable to expect that even closer bounds will appear for a more realistic problem with a higher barrier.
APPENDIX
Calculation of \( f_1(\tau) \) for Stationary Gaussian Vector Processes and Rectangular Safe Areas

Let \{\( X(t), t \in T \), \( T = [0, \infty) \} \) be a stationary Gaussian vector process where all coordinate processes are assumed to be normalized to mean value 0 and standard deviation 1. Moreover, it is assumed for all \((t_1, t_2)\in T^2\)

\[
E[X_i(t_1)X_j(t_2)] = \begin{cases} 0 & i \neq j \\ \rho_i(t_1 - t_2) & i = j \end{cases} \tag{A.1}
\]

where \( \rho_i \) is the autocorrelation coefficient of the coordinate process \{\( X_i(t) \)\}.

From (A.1) it follows

\[
E[\dot{X}_i(t_1)X_j(t_2)] = \begin{cases} 0 & i \neq j \\ \rho_i'(t_1 - t_2) & i = j \end{cases} \tag{A.2}
\]

\[
E[\dot{X}_i(t_1)\dot{X}_j(t_2)] = \begin{cases} 0 & i \neq j \\ \rho_i''(t_1 - t_2) & i = j \end{cases} \tag{A.3}
\]

where \{\( \dot{X}_i(t) \)\} is the derivative process corresponding to \{\( X_i(t) \)\}.

Defining

\[
V(t) = \sum_{i=1}^{n} \dot{X}_i(t)n_i \tag{A.4}
\]

it follows from (A.2) and (A.3) that

\[
E[V(t_1)X_j(t_2)] = n_i\rho_i'(t_1 - t_2) = \dot{\lambda}(t_1 - t_2) \tag{A.5}
\]

\[
E[V(t)V(t)] = \sigma_n^2 = \sum_{i=1}^{n} n_i^2\sigma_{\dot{X}_i}^2 \tag{A.6}
\]

\[
\sigma_{\dot{X}_i}^2 = -\rho_i''(0) \tag{A.7}
\]

where \( \sigma_{\dot{X}_i}^2 \) is the variance of \{\( X_i(t) \)\}. It is seen that \( V(t) \equiv n(0, \rho_n) \).

The safe area is assumed to be time-invariant and represented by an interval of the type

\[
S_0 = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n | a_i < x_i < b_i, \ldots, a_n < x_n < b_n \} \tag{A.8}
\]

The probability that a realization will be in the safe area at time \( t = 0 \) then becomes

\[
P(S_0) = \prod_{i=1}^{n} (\Phi(b_i) - \Phi(a_i)) \tag{A.9}
\]

The joint probability density of the stochastic vectors \( X = X(\tau), X_0 = X(0) \) and the stochastic variable \( V \) becomes
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\[
p_{x_v}(x, v, x_0) = \frac{1}{(2\pi)^{n+1}|C|} \exp(-\frac{1}{2} \begin{pmatrix} x^T \\ v \\ \lambda^T \\ v \\ x_0 \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & \sigma_n^2 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix})
\]  
(A.10)

\[
C = \begin{pmatrix} c_{11} & c_{12} \\ c_{12}^T & c_{22} \end{pmatrix}
\]  
(A.11)

\[
e_{11} = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & \sigma_n^2 \end{pmatrix}
\]  
(A.12)

\[
e_{12} = \begin{pmatrix} \rho_{01} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho_{0n} \\ \dot{\lambda}_1 & \cdots & \dot{\lambda}_n \end{pmatrix}
\]  
(A.13)

\[
e_{22} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}
\]  
(A.14)

where \( \rho_{0i} = \rho_i(\tau) \) and \( \dot{\lambda}_i = \dot{\lambda}_i(\tau) \).

Equation (A.10) may now be written on the form

\[
p_{x_v}(x, v, x_0) = \prod_{i=1}^n \varphi(x_i) \frac{\varphi(v)}{\sigma_n} \times \frac{1}{(2\pi)^{n+1/2} |m_{22}|} \exp(-\frac{1}{2} (x_0 - \mu)^T m_{22}^{-1} (x_0 - \mu))
\]  
(A.15)

where

\[
m_{22} = e_{22} - e_{12}^T e_{11}^{-1} e_{12}
\]  
(A.16)

\[
\sigma_{0i}^2 = 1 - \rho_{0i}^2 - (\dot{\lambda}_i^2/\sigma_n^2)
\]  
(A.17)
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\[ \mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} = \mathbf{e}_1^T \mathbf{c}_{11} \begin{pmatrix} x \\ v \end{pmatrix} = \\
\begin{pmatrix} \rho_1 x_1 + \frac{\lambda_1}{\sigma_n} v \\
\vdots \\
\rho_n x_n + \frac{\lambda_n}{\sigma_n} v \end{pmatrix} \]  
(A.18)

From (A.5) it follows that

\[ \lambda_i(x) \lambda_j(x) = n_i n_j \rho_i^2(x) \rho_j^2(x) \]  
(A.19)

On the surface \( \partial S \) we have \( n_i n_j = 0 \) when \( i \neq j \). It then follows that \( m_{22} \) is a diagonal matrix on \( \partial S \). On this surface the conditioned probability density may then easily be calculated from (8) and (A.15)

\[ p_{xy}(x, v, \tau | S) = \frac{1}{P(S)} \Phi \left( \frac{v}{\sigma_n} \right) \prod_{i=1}^{n} \phi(x_i) F_i(x_i, n_i) \frac{v}{\sigma_n} \]  
(A.20)

where

\[ F_i(x, y) = \int_{\sigma_0}^{\beta_{0i}} \phi(u) du = \Phi(\beta_{0i}) - \Phi(\alpha_{0i}) \]  
(A.21)

\[ \alpha_{0i} = \frac{a_i - \rho_{0i} - \rho_{0i}^2 y}{\sigma_{0i}} \]  
(A.22)

\[ \beta_{0i} = \frac{b_i - \rho_{0i} x - \rho_{0i}^2 y}{\sigma_{0i}} \]  
(A.23)

and \( \rho_{0i}^2 = \rho_i^2(x) \). When (A.20) is applied on surfaces normal to the \( x_i \)-axis, it follows from (A.6) that \( \sigma_i = \sigma_{x_i} \). This result should then be used in (A.17), (A.22) and (A.23).

(A.20) may now be inserted into equation (5) and \( f_1(x) \) be calculated. Each of the \( 2^n \) hyper planes of \( \partial S \) contributes to the integral. The result may be written on the form

\[ f_1(x) = \sum_{i=1}^{n} f_1^{(i)}(x) \]  
(A.24)

\( f_1^{(i)}(x) \) represents the contributions from the 2 surfaces normal to the \( x_i \)-axis, which turns out to be

\[ f_1^{(i)}(x) = \frac{\sigma_{x_i}}{P(S)} g_i(x) \int_{0}^{\pi} y \phi(y) (\phi(b_i) F_i(b_i, y) + \phi(a_i) F_i(a_i, -y)) dy \]  
(A.25)
\[ g_i(\tau) = \begin{cases} \prod_{k=1}^{n} \int_{a_k}^{b_k} \phi(x) F_k(x,0) \, dx & n > 1 \\ 1 & n = 1 \end{cases} \quad \text{(A.26)} \]

In this example the multidimensional integral involved in (12) thus reduces to products of n first order integrals.

When intermediate conditions are specified at times \( \tau_1, \ldots, \tau_N \) a similar deduction shows that the expressions (A.24), (A.25) and (A.26) remain valid, \( f_1 \) and \( f_1^{(i)} \) now being functions of all arguments \( \tau_1, \ldots, \tau_N \). However, \( F_i(x, y) \) are now given by the expression

\[
F_i(x, y) = \int_{a_i}^{b_i} \int_{a_i}^{b_i} \frac{1}{(2\pi)^{N/2} |D_i|} \exp \left\{ -\frac{1}{2} \begin{pmatrix} z_0 \\ \vdots \\ z_n \end{pmatrix}^T \begin{pmatrix} z_0 \\ \vdots \\ z_n \end{pmatrix} \right\} \, dz_0 \ldots dz_N \quad \text{(A.27)}
\]

\[
D_i = \begin{pmatrix} 1 & R_{0i} & \cdots & R_{Ni} \\ \vdots & 1 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \text{symm} & 1 & R_{N-1,N} & \cdots \end{pmatrix} \quad \text{(A.28)}
\]

\[
a_i - \rho_n x - \frac{\rho_n'}{\sigma_x} y \\
\alpha_n(x, y) = \frac{a}{\sigma_n} \quad r = 0, 1, \ldots, N \quad \text{(A.29)}
\]

\[
b_i - \rho_n x - \frac{\rho_n'}{\sigma_x} y \\
\beta_n(x, y) = \frac{b}{\sigma_n} \quad r = 0, 1, \ldots, N \quad \text{(A.30)}
\]

\[
R_{rs}^{\prime} (\tau, \tau, \tau) = \frac{\rho_r (\tau - \tau_r) - \rho_{rs} \rho_{s} - \rho_r' \rho_{s}'}{\sigma_n \sigma_x}, \quad 0 \leq r < s \leq N \quad \text{(A.31)}
\]

\[
\sigma_n^2 = 1 - \rho_n^2 \left( \frac{\rho_n'}{\sigma_x} \right)^2 \quad \text{(A.32)}
\]

\[
\rho_n = \rho_r (\tau - \tau_r) \quad \text{(A.33)}
\]

\[
\rho_n' = \rho_r' (\tau - \tau_r) \quad \text{(A.34)}
\]
In above expressions \( \tau_0 \) should be interpreted as 0. \( R_{\alpha}(\tau, \tau, \tau) \) represents the correlation coefficients of \( X_\alpha(\tau) \) and \( X_\alpha(\tau) \), and \( \sigma_\alpha \) the standard deviation of \( X_\alpha(\tau) \), on condition of prescribed values of \( X_\alpha(\tau) \) and \( V(\tau) \).

The integral in (A.27) of order \( N + 1 \) may easily be seduced to one of the order \( N \) by splitting up the involved quadratic form in a way similar to the formulas leading from (A.10) to (A.15).

REFERENCES

