CHAPTER 14

LIFETIME RELIABILITY ESTIMATE AND EXTREME PERMANENT DEFORMATIONS OF RANDOMLY EXITED ELASTO-PLASTIC STRUCTURES

S.R.K. Nielsen, J. Dalsgård Sørensen & P. Thoft-Christensen
Aalborg University, Denmark

Abstract

A method is presented for lifetime reliability estimates of randomly excited yielding systems, assuming the structure to be safe, when the plastic deformations are confined below certain limits. The accumulated plastic deformations during any single significant loading history is considered to be the outcome of identically distributed, independent stochastic variables, for which a model is suggested. Further assuming the interarrival times of the elementary loading histories to be specified by a Poisson process, and the duration of these to be small compared to the designed life-time, the accumulated plastic deformation during several loadings can be modelled as a filtered Poisson process. Using the Markov property of this quantity the considered first-passage problem as well as the related extreme distribution problems are then solved numerically, and the results are compared to simulation studies.

1. INTRODUCTION

It is usually too expensive to design a structure in such a way that it can, sustain severe but infrequent loading without permanent deformations. It is therefore generally accepted in structural design practice to allow for some plastic deformations in structures due to, for example, strong motion earthquakes. Safe performance of the structure is only then ensured if the plastic deformations remain below certain limits specified by the ductility of the material.

This problem can be further explained with reference to the two-storey frame shown in Figure 1, where random loadings $F_1(t)$ and $F_2(t)$ take place at any of the random instants of time, $t_1, t_2, ...$, in Figure 2. Figure 2 shows a number of so-called elementary loadings initiated at time $t_1, t_2, ...$. During each of these elementary loadings

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the frame remains in the elastic range for most of the time. However, at some moment, the yield moment in a critical section may be exceeded, resulting in an increment to the mutual rotation, $\psi(t)$, of the joined sections (Figure 1). The frame may survive a number of such plastic deformations, but at some instant of time a section may fail, because the accumulated mutual rotation exceeds the rotational capacity. If the material is non-work hardening then critical combinations of load frequency and intensity may be such that yielding will not cease at a certain section and will therefore result in instantaneous collapse of the structure.

Figure 3 shows two realisations of the mutual rotation in a hinge obtained from numerical simulation of a load process similar to the load process indicated in Figure 2. In Figure 3(a) the plastic deformations have stabilised at each upcrossing in the plastic range but in Fig. 3(b) instantaneous collapse has occurred. The horizontal parts of the curves represent time intervals where the structure remains elastic.

In this paper a model for reliability estimates of elasto-plastic structures is presented. In this model the increment of plastic deformations, accumulated during an elementary loading, is modelled by a stochastic variable. As indicated above the distribution of this stochastic variable has a concentrated probability mass at zero, where no plastic deformations take place, and likewise a concentrated mass at infinity representing the possible instantaneous collapse of the structure.
In the model only loadings of engineering significance are considered, that is loadings, which are likely to induce plastic deformations in the structure. The counting process specifying the interarrival times of these elementary loadings is assumed to be of the Poisson type. If the increments of plastic deformation originating from different elementary loadings can be represented by identically distributed, mutually independent stochastic variables, then the accumulated plastic deformation becomes a filtered Poisson process, see Parzen [1].

The probability of failure of this process in the designed life-time, $T$, of the structure as well as the distribution of extreme plastic deformations can then be calculated by a numerical procedure using the Markov property of the considered process. The model is presented for two cases, namely when no repair of the structure is performed between the loadings, and when the structure is reconditioned if even the slightest damage occurs.

Mathematical models dealing with the life-time reliability of elasto-plastic structures have not yet been intensively studied. Vanmarcke et al. [2] and Veneziano & Vanmarcke [3] suggested a filtered Poisson process model of the accumulated plastic deformation during a single elementary loading history of bilinear one-degree-of-freedom systems. Dolinski [4] studied lifetime reliability estimates of multi-degree-of-freedom rigid plastic systems, assuming almost all realisations of the plastic deformation to be monotonically increasing functions of time. Practically the same problem as presented here was considered by Casciati et al. [5] where reliability estimates were performed from the first order distribution of the plastic deformation. This, however, is only correct if the realisations are monotonic functions with a probability of 1.

2. LIFETIME RELIABILITY OF ELASTIO-PLASTIC STRUCTURES

The loading is modelled by assuming that elementary loadings of engineering significance arrive at random instants of time $t_1, t_2, \ldots$ in the time interval $[t_0, t_0 + T]$ specified by a Poisson counting process {$N(t); t \in [t_0, t_0 + T]$} with intensity $\nu(t)$ (see, for example, Parzen [1]). Further, it is assumed that the duration of the elementary loadings are small compared with the design life-time, $T$, of the structure.

The magnitude of the plastic deformation between the arrival of the nth and the (n+1)th loading is represented by the stochastic variable $\Psi_n$. The accumulated plastic deformation can then be modelled by the filtered Poisson process $\{ \Psi(t); t \in [t_0, t_0 + T] \}$ where:

$$\Psi(t) = \sum_{n=0}^{N} \Psi_n (H(t-t_n) - H(t-t_{n-1})) \tag{1}$$

and where $H(\cdot)$ is Heaviside's step function. Note that permanent (plastic) deformations at time $t_0$ are specified by the stochastic variable $\Psi_0$ with a known distribution, i.e. the
initial conditions of the process need not be deterministic. A realisation $\psi(t)$ of the process $\{\Psi(t)\}$ is shown in Figure 4 where the safe area $S=[\psi_L, \psi_U]$. No re-establishment of the initial configuration is assumed to take place between the loadings in the realisation in Figure 4.

Let $P_f(t_0)$ denote the probability of instantaneous failure, i.e. failure at time $t_0$. When the safe area is $S=[\psi_L, \psi_U]$, then

$$P_f(t_0) = P(\Psi_0 \in CS)$$

(2)

Next, let $f_0$ be the first passage density function of $\psi(t)$. The probability of failure of the process, $\{\Psi(t)\}$, in the interval $[t_0, t_0+T]$ with the safe area, $S$, can then be written:

$$P_f(t_0, T) = P_f(t_0) + (1 - P_f(t_0)) \int_{t_0}^{t_0+T} f_0(t) dt$$

(3)

where, by definition,

$$f_0(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P(\text{crossing into } CS \text{ for the first time in } [t, t+\Delta t] | \Psi_0 \in S)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \sum_{n=0}^{\infty} P(\text{arrival of } (n+1)\text{th loading in } [t, t+\Delta t] \land \Psi_1 \in S \land \ldots \land \Psi_n \in S \land \Psi_{n+1} \in CS | \Psi_0 \in S)$$

(4)

Equation (4) can be written in a more convenient form. The probability that the $(n+1)$th loading arrives in the interval $[t, t+\Delta t]$ is $P(N(t) = n)$, where the first factor indicates the probability that a loading arrives in the considered interval, and the second factor is the probability that exactly $n$ loadings have arrived in the preceding interval $[t_0, t]$. In the expressions above $\Delta t$ should be considered small compared to $T$ but large relative to the duration of elementary loadings. The arrival of loadings is independent of the plastic deformations. Therefore eqn. (4) can be written:

$$f_0(t) = \nu(t) \sum_{n=0}^{\infty} a_n P(N(t) = n)$$

(5)

where:

$$a_n = P(\Psi_{n+1} \in CS \land \Psi_n \in S \land \ldots \land \Psi_1 \in S | \Psi_0 \in S)$$

(6)

and

$$P(N(t) = n) = \frac{1}{n!} \exp \left( - \int_{t_0}^{t} \nu(\tau) d\tau \right) \left( \int_{t_0}^{t} \nu(\tau) d\tau \right)^n$$

(7)

3. MODELLING THE PLASTIC DEFORMATIONS AS RANDOM WALK CHAINS

Generally the stochastic variables $\Psi_n$, $n = 0, 1, 2, \ldots$ will be arbitrarily related. However, if it is assumed that the plastic deformations between the loadings can be modelled as a continuously valued random walk chain [1] then $a_n$ can be simplified. Let:

$$\Psi_n = \Psi_{n-1} + \Delta \Psi_n \quad (n = 1, 2, \ldots)$$

(8)
where \( \{ \Delta \Psi_n, n = 1, 2, \ldots \} \) is a sequence of mutually independent stochastic variables and independent of \( \Psi_0 \), all identically distributed as a stochastic variable \( \Delta \Psi \). This implies that \( \{ \Psi_n, n = 1, 2, \ldots \} \) becomes a continuously valued Markov chain \([1]\) and from eqn. (8) it follows that

\[
\Psi_n = \Psi_0 + \sum_{l=1}^{n} \Delta \Psi_l \quad (n \in 1, 2, \ldots)
\]

(9)

\[
\Psi_n = \Psi_m + \sum_{l=1}^{m-n} \Delta \Psi_{m+l} \quad (0 \leq m < n)
\]

(10)

Clearly the stochastic variables on the right hand side of eqn. (10) are mutually independent. The density function \( p_{\Psi_{n},\Psi_{m}} : R^2 \to R \) of the 2-dimensional stochastic variable \((\Psi_{n},\Psi_{m})\) can therefore be written:

\[
p_{\Psi_{n},\Psi_{m}}(x_1, x_2) = p_{\Psi_{n}}(x_1) p_{\Psi_{m}-\Psi_{n}}(x_2 - x_1) \quad (0 \leq m < n)
\]

(11)

where \( p_{\Psi_{n}} : R \to R \) and \( p_{\Psi_{m}-\Psi_{n}} : R \to R \) are the density functions for the stochastic variables \( \Psi_{m} \) and \( \Psi_{n} - \Psi_{m} \).

The coefficients \( a_n \) given by eqn. (6) can now be rewritten as:

\[
a_n = P(\Psi_{n+1} \in \text{CS} | \Psi_n \in S \wedge \ldots \wedge \Psi_{i} \in S \wedge \Psi_{0} \in S) P(\Psi_{n} \in S \wedge \ldots \wedge \Psi_{i} \in S | \Psi_{0} \in S)
\]

\[
= (1 - \lambda_n) P(\Psi_{n} \in S \wedge \ldots \wedge \Psi_{i} \in S | \Psi_{0} \in S)
\]

(12)

where:

\[
\lambda_n = P(\Psi_{n+1} \in \text{CS} | \Psi_n \in S)
\]

(13)

and where the Markov property of the sequence \( \{ \Psi_n \} \) has been used. By successive applications of this property

\[
a_n = (1 - \lambda_n) \prod_{i=0}^{n-1} \lambda_i
\]

(14)

4. MODELLING THE DENSITY FUNCTION FOR THE INCREMENTS IN PLASTIC DEFORMATIONS

The conditional probability, \( \lambda_n \), defined by eqn. (13), can be calculated when the density function of \((\Psi_n, \Psi_{n+1})\) is known. It follows from eqn. (11) that this is the case when the density functions of \( \Psi_n \) and \( \Delta \Psi \) are known. Initially it is assumed that \( \Psi_L < 0 \) and \( \Psi_U > 0 \). To obtain the indicated density functions it is further assumed that:

\[
P(\Psi_0 = 0) = 1
\]

(15)

which implies that no plastic deformations with a probability of 1 occur at time \( t_0 \). From eqn. (2) it follows then that: \( P_f(t_0) = 0 \).

The probability that no plastic deformations take place during an elementary loading, i.e. the structure remains elastic throughout the load history, is called \( \alpha \). Further, the probabilities of infinitely positive and negative plastic deformations during an elementary loading due to instantaneous collapse, as shown in Figure 3(b), are called \( \beta \) and \( \gamma \), respectively. This can be formulated by
\[ P(\Delta \Psi = 0) = \alpha \]
\[ \forall k_1 \in R_+ : P(\Delta \Psi > k_1) = \beta \quad (0 \leq \alpha + \beta + \gamma \leq 1) \]  \hspace{1cm} (16)
\[ \forall k_2 \in R_- : P(\Delta \Psi < k_2) = \gamma \]

The remainder, 
\[ (1 - \alpha - \beta - \gamma) , \] of the probability mass in the distribution of \( \Delta \Psi \) is assumed to be distributed as a continuously valued stochastic variable \( \Delta \psi \), with density function \( \psi \Delta \psi \) and distribution function \( F_{\Delta \psi} \). The density function of \( \Delta \psi \) is shown in Figure 5.

5. FAILURE PROBABILITY WHEN NO REPAIR TAKES PLACE

In this section it is assumed that no repair takes place between the loadings. It follows then from eqns. (11) and (13) that the conditional probability, \( \lambda_n \), can be written:

\[
\lambda_n = \frac{\int_{\psi_l}^{\psi_u} \int_{\psi_l}^{\psi_u} p_{\psi_n \psi_{\psi}}(x_1, x_2)dx_1dx_2}{\int_{\psi_l}^{\psi_u} p_{\psi_n}(x)dx} = \frac{\int_{\psi_l}^{\psi_u} g(x)p_{\psi_n}(x)dx}{\int_{\psi_l}^{\psi_u} p_{\psi_n}(x)dx} \]  \hspace{1cm} (17)

where

\[
g(x) = \int_{\psi_l}^{\psi_u} p_{\Delta \psi}(y-x)dt \]  \hspace{1cm} (18)

The denominator and counter in eqn. (17) can be calculated using eqns. (9), (15) and (16). It is shown in the Appendix that:

\[
\lambda_n = \frac{\sum_{i=1}^{n} \alpha^i (1-\alpha-\beta-\gamma)^{n-i} r_{n-i}}{\sum_{i=1}^{n} \alpha^i (1-\alpha-\beta-\gamma)^{n-i} s_{n-i}} \]  \hspace{1cm} (19)

where, for \( n = 1, 2, \ldots \):

\[
r_n = \int_{\psi_l}^{\psi_u} g(x)p_{\psi_n}(x)dx \]  \hspace{1cm} (20)

and

\[
s_n = \int_{\psi_l}^{\psi_u} p_{\psi_n}(x)dx \]  \hspace{1cm} (21)

In eqns. (20) and (21) \( p_{\psi_n} \) is the density function of the stochastic variable.
\[ \Psi^*_n = \sum_{l=1}^{n} \Delta \Psi^*_l \]  

(22)

Further, \( r_0 = g(0) \) and \( s_0 = 1 \).

The stochastic variables \( \Delta \Psi^*_n \) are mutually independent and identically distributed, as \( \Delta \Psi^* \) introduced in Section 4.

When the distribution of \( \Delta \Psi^*_n \) and the quantities \( \alpha, \beta, \gamma \) are known, the probability of failure, \( P_f(t_0, T) \), can be calculated numerically from eqns. (3), (5), (14), and (19). This is shown in Section 8 for normally distributed and for gamma distributed \( \Delta \Psi^* \).

Extension of the above theory to other combinations of \( \psi_L \) and \( \psi_U \) is straightforward, and is presented in the Appendix.

6. FAILURE PROBABILITY WHEN THE SYSTEM IS PERFECTLY RE-ESTABLISHED

In Section 5 it is assumed that no repair of the system takes place between the loadings. As the other extreme to be considered, it is now assumed in this section that the initial distribution of plastic deformations are re-established after each elementary loading whenever plastic deformations take place. The restriction to the initial distribution of plastic deformations as specified by eqn. (15) is repeated here.

Clearly, the assumption of perfect re-establishing of the system can be, stated as:

\[ \lambda_n = \lambda_0 \quad (n = 1, 2, 3, \ldots) \]  

(23)

where \( \lambda_0 \) is given by eqn. (17) for \( n = 0 \).

It follows from eqns. (3), (5), (7), and (14) that the probability of failure in this case is given by

\[ P_f(t_0, T) = 1 - \exp\left(-\lambda_0 \int_0^{t_0 + T} v(t) \, dt \right) \]  

(24)

7. LIFETIME EXTREMES OF PLASTIC DEFORMATIONS

Let the maximum of plastic deformation in the time interval \([t_0, t_0 + T]\) be represented by a stochastic variable \( \Psi_{\max} \) and let the safe interval for \{\( \Psi(t) \)\} be \( S = [-\infty, x] \). \( \Psi_{\max} \) is then less than \( x \) when no failure of \{\( \Psi(t) \)\} takes place and the distribution function of \( \Psi_{\max} \) is determined by:

\[ F_{\Psi_{\max}}(x) = P(\Psi_{\max} \leq x) = 1 - P_f(t_0, T; x) \]  

(25)

where \( P_f(t_0, T, x) \) is the probability of failure in the time interval \([t_0, t_0 + T]\) and with the safe interval \( S = [-\infty, x] \). \( P_f(t_0, T, x) \) is calculated from eqn. (3).

Likewise, the minimum of plastic deformation is represented by a stochastic variable \( \Psi_{\min} \), which will be less than or equal to \( x \), when at least one failure of \{\( \Psi(t) \)\}
in \([t_0, t_0 + T]\) takes place with the safe area defined as \(S = [x, \infty]\). Therefore, the distribution function of \(\Psi_{\text{min}}\) is given by

\[
F_{\Psi_{\text{min}}} (x) = P(\Psi_{\text{min}} \leq x) = P_f (t_0, T; x)
\]  

Again \(P_f (t_0, T; x)\) can be calculated from eqn. (3), but now on the basis of \(S = [x, \infty]\).

8. NUMERICAL EXAMPLE 1

In order to investigate the influence of the distribution of \(\Delta \Psi^*\) the following two cases are considered:

1. \(\Delta \Psi^*\) normally distributed \((\mu, \sigma)\) and
2. \(\Delta \Psi^*\) gamma distributed \((\alpha^*, \beta^*) = (\mu^2 / \sigma^2, \sigma^2 / \mu)\).

\[
\mu \text{ and } \sigma \text{ indicate the mean value and standard deviation of } \Delta \Psi^*. \ (\alpha^*, \beta^*) \text{ are the parameters of the gamma distribution, not to be confused with } \alpha \text{ and } \beta \text{ defined earlier. Case (1) represents a symmetrical distribution, where negative increments may occur. The gamma distribution is asymmetrical and rules out the possibility of any negative increment.}
\]

It follows from eqn. (22) that \(\Psi_n^*\) then becomes normally distributed \((n\mu, \sqrt{n}\sigma)\) and gamma distributed \((\alpha^*, \beta^*) = (n(\mu^2 / \sigma^2), \sigma^2 / \mu)\) respectively.

The following parameters are chosen for this numerical example:

\[
\begin{align*}
\nu(t) & = 0.1 \text{ year}^{-1} & \sigma & = 0.01 \\
\bar{\psi}_L & = \bar{\psi}_U = 0.07 & \alpha & = 0.40 \\
P(\Psi_0 = 0) & = 1 & \beta & = 0.03 \\
\mu & = 0.007 & \gamma & = 0
\end{align*}
\]

It is assumed that no repair takes place between the elementary loadings.
Numerical calculation of the quadratures (18), (20) and (21) is performed by means of a Chebyshev expansion with 50 terms, and the integration of the first-passage density in eqn. (3) is carried out with the trapezoidal formula with a step length of 0.5 years.
The first-passage densities obtained are shown in Figure 6. As seen, \( f_0 \) has a pronounced maximum. This may be explained from the accumulated plastic deformations which, on average, will approach the upper \( \psi_U \) after a number of loadings, when \( \mu > 0 \), resulting in an increased rate of failure. The different ordinates \( f_0(0) = v(1- \lambda_0) \) originate solely from the different distributions of \( \Delta \Psi^* \). If the system is perfectly re-established between the loadings \( f_0(T) \) will decrease exponentially from \( v(1-\lambda_0) \).

The impact on the probability of failure due to independent variations of \( \mu, \sigma, \alpha \), and \( \beta \) from the indicated referential values is illustrated in Figures 7-10. The same nomenclature as used in Figure 6 is used again in these figures. From these results it may be concluded that the distribution of \( \Delta \Psi^* \) only influences the lifetime reliability significantly when \( \sigma > 0.01 \). This is explained from the parameters \( \lambda_n \) (depending on the distribution of \( \Psi_n^* \), which approaches a normal distribution as \( n \) goes to infinity due to the central limit theorem, independently of the actual distribution of \( \Delta \Psi^* \)). Otherwise, when \( n \) is small, and \( \sigma \) is simultaneously small compared to \( \psi_U - \psi_L \), it follows from eqn. (16) that \( \lambda_n \sim \lambda_0 \sim 1-\beta-\gamma \). In the latter case it may then be assumed that \( \lambda_n \) and hence the first-passage density, is independent of the distribution of \( \Delta \Psi^* \).

![Fig. 11. Distribution of maximum plastic deformation.](image)

The distribution functions of the extreme plastic deformations \( \Psi_{\text{max}} \) and \( \Psi_{\text{min}} \) are shown in Figures 11 and 12. When \( \Delta \Psi^* \) is gamma distributed, the accumulated plastic deformations will be positive with a probability of 1, whereas negative deformations may occur in the case of normally distributed increments. This explains the quantitative difference between the distribution functions of \( \Psi_{\text{min}} \) in the two cases as shown in Figure 12. The horizontal asymptotics of the distribution functions different from 1 originate from the probability mass \( \beta \) located at infinity of the incremental plastic deformations.
9. NUMERICAL EXAMPLE 2

The results of the model have been compared with simulation estimates of the life-time reliability of the frame shown in Figure 1, where only the mechanism shown was observed. Initially the number of elementary loadings during the life-time $T$ is simulated, assuming a stationary Poisson counting process model of the interarrival times. The elementary loading processes $(F_1(t), F_2(t))$ are assumed to be Gaussian with fully correlated coordinate processes, i.e. $\rho_{F_1F_2}(t, t) = 1$. Mean values and autocovariance function are functions of a number of parameters, which change from loading to loading, and are considered as random variables with known distribution. With known outcomes of these variables and a random duration of the elementary loading a realisation of the vector process $(F_1(t), F_2(t))$ may be simulated. From a dynamic analysis of the frame with known initial permanent set $\psi(t)$ at the start of the $i$th elementary loading, equal to the permanent rotation at the end of the $(i - 1)$th loading, a realisation of the accumulated mutual rotation in the designed lifetime can be simulated.

Reiterating this process $N$ times, a certain number, $\Delta N$, will lead to failure, either because of instantaneous failure in an elementary loading, or due to excessively accumulated finite plastic increments. The life-time probability of failure can then be estimated from

$$P_f(t_0, T) \approx \frac{\Delta N}{N} \quad (27)$$

The results obtained, as well as the number of life-time simulations used, are shown in Table 1.

<table>
<thead>
<tr>
<th>$T$ (year)</th>
<th>$N$</th>
<th>$P_{A(t_0, T)}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>88</td>
<td>14.8</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>33.0</td>
</tr>
<tr>
<td>150</td>
<td>99</td>
<td>40.4</td>
</tr>
</tbody>
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* $v = 0.1 \text{ year}^{-1}$, $\psi_0 = -\psi_L = 0.07$, $P(\psi_0 = 0) = 1$. 

Fig. 12. Distribution of minimum plastic deformation.
The parameters $\mu$, $\sigma$, $\alpha$, $\beta$, $\gamma$ are estimated from 300 realisations; of the elementary loading process, each simulated as indicated above. The frame is re-established with the initial distribution of plastic deformations, $P(\Psi_0 = 0) = 1$, before each loading. The following results were obtained:

$$\begin{align*}
\mu &= 0.00861 \\
\beta &= 0.030 \\
\sigma &= 0.0130 \\
\gamma &= 0 \\
\alpha &= 0.889
\end{align*}$$

The results of the model with these parameters and the simulation results from Table 1 are shown in Figure 13. The discrepancy between the results is below the estimation error inherent in the simulation results.

10. CONCLUSIONS

A mathematical model has been suggested, from which life-time reliability estimates of the accumulated permanent deformations of elasto-plastic structures can be estimated.

The model assumes the interarrival times of loadings to be specified by a Poisson counting process with known intensity. Besides, the parameters $\alpha$, $\beta$, and $\gamma$ and the distribution function of continuous plastic increments $\Delta \Psi^r$ must be known, either from analysis or numerical simulation studies. It is shown that the actual distribution of $\Delta \Psi^r$, besides the expectation $\mu$ and standard deviation $\sigma$, is of minor importance.

The model has been tested with simulation studies with reasonable concordance between the corresponding results.

REFERENCES

APPENDIX

The characteristic function of $\Delta \Psi$ becomes

$$
\varphi_{\Delta \Psi}(t) = E[\exp(it\Delta \Psi)]
= \int_{-\infty}^{\infty} \exp(itx) \left( \alpha \delta(x) + \beta \lim_{x_0 \to -\infty} \delta(x-x_0) + \gamma \lim_{x_1 \to -\infty} \delta(x-x_1) + (1-\alpha-\beta-\gamma)p_{\Delta \Psi_0}(x) \right) dx
= \alpha + \beta \lim_{x_0 \to -\infty} \exp(itx_0) + \gamma \lim_{x_1 \to -\infty} \exp(itx_1) + (1-\alpha-\beta-\gamma)\varphi_{\Delta \Psi_0}(t)
$$

(28)

Initially it is assumed that $\gamma = 0$. Using eqns. (9) and (15) the characteristic function of $\Psi_n$ can then be written as

$$
\varphi_{\Psi_n}(t) = \varphi_{\Delta \Psi_n}(t) = \left( \alpha + \beta \lim_{x_0 \to -\infty} \exp(itx_0) + (1-\alpha-\beta)\varphi_{\Delta \Psi_n}(t) \right)^n
= \lim_{x_0 \to -\infty} \sum_{l=0}^{n} \sum_{k=0}^{l} \binom{n}{l} \left( \frac{1}{k} \right) \alpha^{-k} \beta^k (1-\alpha-\beta)^{n-1} \times \exp(itkx_0) \times \varphi_{\Delta \Psi_0}^{-k}(t)
$$

(29)

The frequency function of $\Psi_n$ is obtained from an inverse Fourier transform of eqn. (29). The result is:

$$
p_{\Psi_n} = \lim_{x_0 \to -\infty} \sum_{l=0}^{n} \sum_{k=0}^{l} \binom{n}{l} \left( \frac{1}{k} \right) \alpha^{-k} \beta^k (1-\alpha-\beta)^{n-1} p_{\Psi_{n-1}}(x-kx_0)
$$

(30)

where $\Psi_n$ is defined by eqn. (22).

$g(x)$ given by eqn. (18) can, when $0 < \psi_U < x_0$ and $\psi_L < 0$, be written as

$$
g(x) = \alpha + (1-\alpha-\beta) \int_{\psi_L}^{\psi_U} p_{\Delta \Psi_0}(t-x) dt
$$

(31)

When $\psi_U \geq x_0$ it is seen that

$$
g(x) = \alpha + \beta + (1-\alpha-\beta) \int_{\psi_L}^{\psi_U} p_{\Delta \Psi_0}(t-x) dt
$$

(32)

Consider the case where $-\infty < \psi_L < 0$ and $0 < \psi_U < \infty$ ($\psi_U < x_0$). For any function $g(x)$ integratable in the interval $[\psi_L, \psi_U]$ it follows, for $n = 1, 2, ..., n:

$$
limit_{x_0 \to -\infty} \int_{\psi_L}^{\psi_U} g(x)p_{\Psi_n}(x-kx_0)dx = \begin{cases} 0 & k \neq 0 \\ r_n & k = 0 \end{cases}
$$

(33)

with $r_n$ as given by eqn. (20).
From eqns. (30) and (33) it follows that
\[
\int g(x)p_{\psi_L}(x)dx = \sum_{l=0}^{n} \binom{n}{l} \alpha^l (1-\alpha-\beta)^{n-l} r_{n-l}
\]  
(34)

The denominator in eqn. (17) can be calculated from the above expressions, with \( g(x) \equiv 1 \). Equation (19) is then obtained in the case \( \gamma = 0 \). A similar, somewhat more tedious, derivation with \( \gamma \neq 0 \) will finally reveal the more general result of eqn. (19).

Next we consider the case where \( \psi_L = -\infty \) \( (\gamma = 0) \). When \( \psi_U < 0, s_0 = r_0 = 0 \). For \( \psi_U \to -\infty \) it is seen that \( \lambda_n = 0 \). For \( \psi_U \to \infty (\psi_U < x_0) \) it follows, from eqn.(19), that \( \lambda_n = 1-\beta, n = 1, 2,\ldots \) .

Finally, when \( \psi_U = \infty (x_0 \leq \psi_U) \) and \( \psi_L < 0 \), it follows that:
\[
I_n = \lim_{x_0 \to \infty} \int_{\psi_L}^{\psi_U} g(x)p_{\psi_a}(x-kx_0)dx = \begin{cases} 1 & k \neq 0 \\ r_n & k = 0 \end{cases}
\]  
(35)

with, \( r_n \) given by eqn. (20). Instead of eqn. (34) we now get
\[
\int g(x)p_{\psi_a}(x)dx = \sum_{l=0}^{n} \binom{n}{l} \alpha^l \beta^k (1-\alpha-\beta)^{n-l} I_{n-l}
\]  
(36)

Again the denominator in eqn. (17) can be calculated from eqn. (36) with \( g(x) \equiv 1 \). When \( \psi_L > 0, s_0 = r_0 = 0 \). For \( \psi_L \to -\infty \) it is seen that \( \lambda_n = 1, n = 0, 1, 2,\ldots \) . For \( \psi_L \to \infty (\psi_L < x_0) \lambda_n = \beta \) and \( \lambda_n = 1, n = 1, 2,\ldots \) .

From these results it follows that the distribution function of \( \Psi_{\text{max}} \) and \( \Psi_{\text{min}} \) as given by eqns. (25) and (26), will have the following limits \( (v(t) = v \) and \( \gamma = 0) \):

\[
(1-P_f(t_0))\exp(-Tv) < F_{\Psi_{\text{max}}}(t_0, T) \leq (1-P_f(t_0)) \int_{t_0}^{T} \left( 1 + \sum_{n=0}^{\infty} a_n \exp(-v(t-t_0)) (v(t-t_0))^n \right) dt \]  
(37)

where \( a_0 = \beta \) and \( a_n = (1-\beta)\beta^n, n = 1, 2,\ldots \). 

\[
0 < F_{\Psi_{\text{max}}}(t_0, T) < P_f(t_0) + (1-P_f(t_0))(1-\beta)(1-\exp(-Tv))
\]  
(38)