Simulation of Floaters in Action
1 - Theory : Release: SOFIA v1.0
Nielsen, Morten Eggert; Ulriksen, Martin Dalgaard; Damkilde, Lars

Publication date:
2016

Document Version
Publisher’s PDF, also known as Version of record

Link to publication from Aalborg University

Citation for published version (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

Users may download and print one copy of any publication from the public portal for the purpose of private study or research.

? You may not further distribute the material or use it for any profit-making activity or commercial gain
?

Take down policy
If you believe that this document breaches copyright please contact us at vbn@aub.aau.dk providing details, and we will remove access to the work immediately and investigate your claim.
Simulation of Floaters in Action
1 - Theory

M. E. Nielsen
M. D. Ulriksen
L. Damkilde
Simulation of Floaters in Action
1 - Theory

by

M. E. Nielsen
M. D. Ulriksen
L. Damkilde

December 2016

© Aalborg University
Scientific Publications at the Department of Civil Engineering

**Technical Reports** are published for timely dissemination of research results and scientific work carried out at the Department of Civil Engineering (DCE) at Aalborg University. This medium allows publication of more detailed explanations and results than typically allowed in scientific journals.

**Technical Memoranda** are produced to enable the preliminary dissemination of scientific work by the personnel of the DCE where such release is deemed to be appropriate. Documents of this kind may be incomplete or temporary versions of papers—or part of continuing work. This should be kept in mind when references are given to publications of this kind.

**Contract Reports** are produced to report scientific work carried out under contract. Publications of this kind contain confidential matter and are reserved for the sponsors and the DCE. Therefore, Contract Reports are generally not available for public circulation.

**Lecture Notes** contain material produced by the lecturers at the DCE for educational purposes. This may be scientific notes, lecture books, example problems or manuals for laboratory work, or computer programs developed at the DCE.

**Theses** are monograms or collections of papers published to report the scientific work carried out at the DCE to obtain a degree as either PhD or Doctor of Technology. The thesis is publicly available after the defence of the degree.

**Latest News** is published to enable rapid communication of information about scientific work carried out at the DCE. This includes the status of research projects, developments in the laboratories, information about collaborative work and recent research results.
Simulation of Floaters in Action

1 - Theory

Release: SOFIA v1.0

M.E. Nielsen, M.D. Ulriksen and L. Damkilde
Aalborg University
Department of Civil Engineering
Division of Structures, Materials and Geotechnics
Preface

This report is the first in a series of three, which altogether documents:

1. **theory**
2. **numerical implementation**
3. **application**

for Simulation of Floaters in Action (SOFIA), which is a structural analysis tool for slender offshore structures, such as monopiles, jacket structures and floating space frame structures. The current report represents the theoretical basis, while the numerical implementation and application of SOFIA are documented in two individual reports.

In relation to other structural analysis tools, the present tool allows for geometrical nonlinearities, which may be exhibited by mooring lines and floating structures. Therefore, SOFIA provides an alternative to traditional analysis tools, which may only be able to handle bottom founded structures, exhibiting linear behavior.

The work has been conducted in the Department of Civil Engineering, Aalborg University, and as further developments are in progress, the reports are updated continuously.
## Contents

1 Introduction to floating offshore structures 1
   1.1 Numerical methods in offshore engineering 3
   1.2 Nonlinear behavior of floating offshore structures 6
   1.3 Outline 7

2 Structural mechanics 8
   2.1 Co-rotational beam formulation 9
      2.1.1 Local deformations 10
      2.1.2 Rotation matrix 11
      2.1.3 Tangent stiffness matrix 13
   2.2 Nonlinear Newmark time integration 14

3 Water wave mechanics 15
   3.1 Linear wave theory 15
      3.1.1 Linear regular waves 16
      3.1.2 Kinematic stretching 17
      3.1.3 Irregular waves 18
      3.1.4 Multi-directional waves 20
      3.1.5 Current 22
   3.2 Stream function wave theory 23

4 Hydrodynamic and gravitational forces 26
   4.1 Inertial forces 26
   4.2 Viscous forces 28
   4.3 Buoyancy and gravity 30
   4.4 Morison equation 31
1 Introduction to floating offshore structures

During the latest decades, offshore structures have been a major topic within structural engineering. Especially, floating offshore structures have been of interest to several industries, which find the floating alternatives beneficial to the respective areas. A number of industrial applications are highlighted below.

Wind Energy
The wind industry is interested in harvesting the wind, not only offshore, but also in deep waters. In figure 1 different offshore wind turbine concepts are illustrated.

Already around 50m water depths the conventional jacket structures become too expensive, which points toward moored floating platform concepts.

Oil & Gas
Semi-submersibles are used within the Oil & Gas industry, see figure 2. At great water depths ships have been used for drilling, as traditional jackup-rigs only operate at relative shallow depths.

Semi-submersibles, with an anchor mooring system, have the advantage that the ballasted pontoon floaters are placed below the free surface. In waves this introduces less motion, compared to a drilling ship, which makes drilling operations easier to carry out.
Wave Energy
As for wind energy, floating concepts have also been proposed within the wave energy industry. In figure 3 a prototype of the WEPTOS wave energy converter is shown.

![WEPTOS wave energy converter](image)

Figure 3: Moored wave energy converter. [3]

The floating wave energy converter allows for arbitrary wave directions, as the mooring system allows the structure to rotate. The mooring system also allows for installation on various water depths.

Infrastructure
Floating structures are not only related to the energy industry. At the western part of Norway, the fjords reach water depths of around 550m and spans up to several kilometers. This draws attention to alternative solutions to well-known bridge designs. One proposed solution is illustrated in figure 4.

![Submerged floating tunnels](image)

Figure 4: Submerged floating tunnels. [4]

In order to overcome the great water depth and spans, submerged floating tunnels have been proposed. The advantageous use of buoyancy allows for large underwater spans, while either mooring or pontoons are used for stabilization.

Indisputably, the application of floating offshore structures is of great interest; not only to the above-mentioned, but to many industries. The most commonly used numerical methods, for structural analysis of offshore structures, are briefly highlighted in the following section.
1.1 Numerical methods in offshore engineering

Because of rather complex fluid-structure interactions, numerous numerical methods have been proposed for a variety of offshore structure designs. Despite the significant number of contributions, no general approach seems to have been adopted yet. The most common and well-established methods are briefly introduced in the following.

Bottom founded offshore structures, such as monopiles, are widely used within the wind energy industry. A variety of space frame foundations also exist, including the jacket structures shown in figure 5. Jacket structures consist of multiple cylindrical members, often welded together.

Figure 5: Offshore wind turbines installed on jacket structures. [5]

Jacket structures are relatively transparent to waves, due to the slender cylindrical members. Hence, these members are assumed not to influence the incoming waves. Morison’s equation [6], which is based on fluid particle kinematics and dimensionless parameters, leads to fairly good approximations of in-line wave inertia and viscous drag forces. Assuming small displacements, the jacket is discretized by linear beam elements. Additionally, wave loads are applied as time-dependent in-line forces based on fixed element positions. Hence, time series of stresses are generated by solving the equation of motion in time domain.

In general, fluid viscosity introduces drag forces, which are divided into skin friction and form drag. The two concepts are illustrated in figure [7]. Skin friction is composed of shear stresses in the fluid-structure boundary layer, which for most offshore structures are negligible unless severe marine growth is present.
Form drag is due to fluid-structure flow separation, which may cause significant wave drag forces on slender members. However, for large structures, which experience small displacements, the drag forces are often negligible. The incoming fluid flow attaches to the body surface, leading to insignificant flow separation, and thereby allowing for the application of potential flow theory.

Ships are examples of offshore structures where no significant drag forces are introduced under operational conditions, due to the streamlined design. A ship consists of a hull with stiffeners at the inside. In motion response analysis it is often idealized as a rigid body with 6 degrees of freedom. Assuming the wave surface elevations negligible, compared to the ship size, only the mean wetted surface is considered to interact with the fluid flow. Hence, the equation of motion is linearized about the mean water level, allowing only small vertical changes in the free surface piercing body area. Because of their relative large dimensions, ships influence the incoming waves (see figure 7), introducing diffraction and radiation effects, which are often handled by boundary element methods.

Diffraction is introduced if the structure disturbs the fluid flow because no fluid flow is allowed through the body surface. In still waters, radiation is introduced by oscillating a body, developing radiating waves from the fluid-structure boundary. Thus, diffraction and radiation effects change the incoming fluid flow, and thereby the wave excitation forces. The wave excitation forces and moments are determined by direct pressure integration over the mean wetted surface.

Some offshore structures combine the application of slender members and bluff bodies, exemplified by the wave energy converter shown in figure 8. Bottom founded space frame structures with rigid body

Figure 7: Numerical simulation of forward moving vessel in waves. [8]

Figure 8: Bottom founded wave energy converter with slender coloumns and rigid body floaters. [9]
floaters necessitate a coupling between the previously mentioned methods, assuming the floaters may be linearized about the mean water level.

Nowadays, floating wave energy converters are being developed. The WEPTOS WEC [3], illustrated in figure 9, constitutes such a device. As the mooring allows large displacements/rotations, the WEPTOS WEC system may experience large global displacements. Additionally, the rotors may experience large local rotations as wave energy is absorbed. The energy absorption may lead to considerable influence on the incoming waves and thus the assumption of undisturbed incident waves is invalid, which may provide poor estimations of the wave excitation forces on the space frame members.

Considering nonlinearities, the presented linearizations of both beam- and potential theory may generally not be valid. Additionally, for moored floating offshore structures, such as semi-submersibles (figure 10), the rigid body formulation does not directly predict the stresses within the structural members.

Applying potential flow theory to some concepts, such as the spar buoy depicted in figure 10 important viscous effects of flow separation may be omitted at the bottom. By omitting the flow separation, significant overestimation of the heave motion may occur. Other medium sized structures may show the same tendencies, such as the tension leg platform in figure 10 depending on the specific design dimensions.

Application of viscous flow methods (Navier-Stokes equations) has been proposed, but until now the computational costs have been too high for structural analysis of floating offshore structures. A combination of the above-mentioned methods is often used in order to account for the important hydrodynamic effects.
1.2 Nonlinear behavior of floating offshore structures

Mooring systems
Floating offshore structures, as seen in figure 10, most often include a mooring system. Different mooring systems have been developed, but the overall purposes are station keeping and/or stabilization. An example of a moored floating structure is given in figure 11, which could be regarded as a simplified drawing of the WEPTOS prototype in figure 3.

![Figure 11: A moored floating structure in (a) initial configuration and (b) displaced by incoming wave load $F_w$, introducing nonlinearities of the mooring system.](image)

The moored floating structure, in figure 11, is able to rotate around the buoy and thereby pull the mooring buoy in arbitrary directions. Displacement of the buoy will increase or decrease tension forces in the mooring lines, but also change their configuration. This will influence the structural stiffness and thereby change the dynamic behavior of the floating structure. Ultimately, this nonlinear dynamic behavior may introduce significant contributions to the internal stresses in the mooring lines and structural members.

Fluid-structure interaction
Hydrodynamic loads are essentially pressure distributions integrated over the fluid-structure interface. Hence, the hydrodynamic loading does not only depend on the wave and current conditions, but also the instantaneous position and motion of the floating structure.

Floating structures may, as mentioned above, be exposed to relatively large displacements. In figure 12, the hydrodynamic loading changes as the floating structure displaces. Therefore, the hydrodynamic loads have to be found at the instantaneous configuration.

![Figure 12: A moored floating structure exposed to a steady current $\dot{u}_c$, which change the structural orientation from (a) the initial configuration to (b) the actual configuration.](image)

Motions of the floating structure influence the hydrodynamic loads too. Based on instantaneous velocities and accelerations of the floating structure, additional loading is introduced. Hence, if the structural motions are of reasonable size, the hydrodynamic loading is a nonlinear function of these.

The present nonlinearities are also interdependent, which demand a coupling of structural nonlinearities with the nonlinear hydrodynamic loading. This introduces a nonlinear behavior, which is too complex to describe in full detail, and therefore simplifications are introduced.
1.3 Outline

The aim of this report is to establish a theoretical basis for structural analysis of slender offshore structures. This section briefly outlines the physical problem and overall methodology. Further details are given in subsequent sections.

Problem statement

In general the structural equation of motion may be expressed by

\[ M(x)\ddot{x}(t) + C(x)\dot{x}(t) + K(x)x(t) = f(t) \]  

(1)

where \( M(x) \) is the mass, \( C(x) \) is the damping, \( K(x) \) is the stiffness, \( f(t) \) is the time dependent external force, \( x \) is the displacement, \( \dot{x} \) is the velocity and \( \ddot{x} \) is the acceleration. For structures experiencing large displacements, the system matrices depend on the instantaneous structural state. Therefore, these depend on the actual orientation and level of internal stresses, which is discussed in section 2. In the following, the independent variable definitions, such as \( (t) \) and \( (x) \), are intentionally left out.

For floating offshore structures the equation of motion in (1) must include added mass \( M_a \), viscous damping \( C_v \), radiation damping \( C_r \), hydrostatic stiffness \( K_h \), gravitational loading \( f_g \), inertial forces \( f_I \) and viscous drag forces \( f_D \). These terms will be explained more carefully in section 4. Expanding (1) for the description of floating structures leads to

\[ (M + M_a)\ddot{x} + (C + C_v + C_r)\dot{x} + (K + K_h)x = f_I + f_D + f_g \]  

(2)

due to the presence of waves, current, buoyancy and gravity.

If diameters \( D \) of the structural members may be considered small relative to the wave length \( \lambda \), often stated as \( D < 0.2\lambda \), the members are assumed to have no significant influence on the incoming waves. In this version of SOFIA the diffracted wave potential \( \phi_D \) and radiated wave potential \( \phi_R \) are neglected, so that

\[ \phi = \phi_I + \phi_D + \phi_R \approx \phi_I \]  

(3)

which states that the total velocity potential \( \phi \) is assumed equal to the incident wave potential \( \phi_I \). For large members, also known as bluff bodies, the incoming waves are diffracted and radiated considerably and therefore this assumption is only acceptable for slender members. Consequently, the equation of motion is expressed by

\[ M\ddot{x} + C\dot{x} + Kx = f_w + f_G \]  

(4)

where the gravitational forces \( f_G = -K_hx + f_g = f_b + f_g \), in which the gravity \( f_g \) and buoyancy force \( f_b \) are included. Hydrodynamic mass force, Froude-Krylov force, viscous drag and damping are included in \( f_w = f_{FK} + f_D - C_s\dot{x} - M_a\ddot{x} \) by relative Morison equations. All of these forces are explained more carefully in section 4.

Methodology

The structural domain is modeled by means of structural mechanics and discretized by using the Finite Element Method (FEM). The external forces are modeled from particle kinematics in an empirical manner. The particle kinematics are modeled by an appropriate water wave theory, where the governing Laplace equation is solved within the fluid domain. Based on the sea state conditions, the free surface boundary conditions of the fluid domain are either linear or nonlinear. An analytical solution is found for linear boundary conditions, whereas nonlinear boundary conditions necessitate a numerical method.

The structural equation of motion in (4) becomes nonlinear, due to possibly large structural displacements, stress-stiffening effects and nonlinear external loads. By using a co-rotational element formulation and a nonlinear time integration scheme, the equation of motion in (4) are solved in time domain. As mentioned previously, the structural feedback to the incident wave field is neglected in the current version and therefore the fluid domain is not influenced by the structural domain.
2 Structural mechanics

To deal with arbitrary motion in three-dimensional space, a nonlinear geometrical formulation is introduced. In contrast to linear analysis, when displacements and rotations become large, the structural stiffness becomes a function of these, which results in nonlinear behavior.

Consider the truss system illustrated in figure 13. The vertical translational degree of freedom \( u \), is governed by the axial stiffness of the truss, represented in the global domain as \( k_{eq} \).

\[
\theta \quad P \quad \uparrow \quad u \\
(a) \\
(\theta) \quad k_{eq} \\
(b)
\]

**Figure 13:** (a) Simply supported truss system, loaded by \( P \) in the vertical direction and (b) equivalent 1-dof system.

The equivalent spring stiffness \( k_{eq} \) changes due to orientation and deformation. In an updated Lagrangian formulation the equivalent stiffness is expressed by \( k_{eq} = k_e(\theta) + k_\sigma(u) \), where \( k_e(\theta) \) is the current elastic stiffness as a function of the truss orientation and \( k_\sigma(u) \) is the geometric stiffness as a function of deformation (internal force). A load-displacement relation is sketched in figure 14 based on the truss system in figure 13 for a linear and nonlinear stiffness formulation.

\[
\begin{align*}
& \text{Figure 14: Tangent stiffness definitions for a linear and nonlinear analysis of the truss system.} \\
& \text{Because the unknown displacement } u \text{ depends on the nonlinear stiffness } k_{eq}, \text{ the static equilibrium equation} \\
& \quad k_{eq}u = P \\
& \text{can not be solved directly and therefore equilibrium iterations are necessary. For this purpose a nonlinear Newmark time integration scheme is introduced in section 2.2. However, first a geometrical nonlinear formulation of beam structures are introduced, specifically the co-rotational concept in three-dimensional space [11].}
\end{align*}
\]
2.1 Co-rotational beam formulation

In order to deal with arbitrary motion and stress stiffening, a co-rotational framework is briefly introduced. The main idea is to formulate a tangent stiffness, based on a 'co-rotating' frame of reference. The co-rotating frame can be applied to various formulations. In the following, the main principles are presented. For more detailed description, see [11].

Virtual work

In figure 15, an initially straight beam element is decomposed into a body translation and nodal rotations.

The beam element in figure 15 has two nodes and in total 12 local degrees of freedom, defined by

\[
p_l = \begin{bmatrix} d_{l1} \\ \theta_{l1} \\ d_{l2} \\ \theta_{l2} \end{bmatrix}
\]

with the following nodal component definitions

\[
d_{l1} = \begin{bmatrix} u_{l1} \\ v_{l1} \\ w_{l1} \end{bmatrix}, \quad \theta_{l1} = \begin{bmatrix} \theta_{l1} \\ \theta_{l2} \\ \theta_{l3} \end{bmatrix}, \quad d_{l2} = \begin{bmatrix} u_{l2} \\ v_{l2} \\ w_{l2} \end{bmatrix}, \quad \theta_{l2} = \begin{bmatrix} \theta_{l4} \\ \theta_{l5} \\ \theta_{l6} \end{bmatrix}
\]

By introducing the local elastic stiffness matrix \( K_{el} \), the local internal forces are found by

\[
q_{li} = K_{el} p_l
\]

while the relation between global and local virtual displacements are defined by a transformation matrix \( F \) as

\[
\delta p_l = F \delta p
\]

The global and local internal virtual work must be equal for arbitrary virtual displacements

\[
V_i = \delta p_i^T q_i = \delta p_{el}^T q_{li} = \delta p_{el}^T F^T q_{li}
\]

leading to the global internal forces being expressed as

\[
q_i = F^T q_{li} = F^T K_{el} p_l
\]

From differentiation of \([11]\), the tangent stiffness is defined

\[
\delta q_i = F^T \delta q_{li} + \delta F^T q_{li} = F^T K_{el} F \delta p + K_{el} \delta p = K \delta p
\]

as described previously, the global tangent stiffness is constituted by two parts.
\[ K = K_e + K_\sigma \] (13)

where \( K_e \) is the global elastic part, and \( K_\sigma \) is the geometric part. Before generating the tangent stiffness, the internal forces have to be computed from the local deformations.

### 2.1.1 Local deformations

From the current nodal positions \( \mathbf{x} \) and translational displacements \( \mathbf{d} \), in figure 15, the local axis is updated by

\[ \mathbf{e}_1 = \frac{1}{\ell_n} (\mathbf{x}_{21} + \mathbf{d}_{21}) \] (14)

where the current element length \( \ell_n = \| \mathbf{x}_{21} + \mathbf{d}_{21} \| \), while \( \mathbf{x}_{21} = \mathbf{x}_2 - \mathbf{x}_1 \) and \( \mathbf{d}_{21} = \mathbf{d}_2 - \mathbf{d}_1 \).

Neglecting axial deformations, due to local rotational deformations, the axial deformation is found by

\[ u_l = \ell_n - \ell_0 = \ell_n - \| \mathbf{x}_{21} \| \] (15)

expressing the difference between the initial and current straight-line length between the element nodes.

The local rotational deformations \( \theta_l = [\theta_{l1}, \theta_{l2}, \theta_{l3}, \theta_{l4}, \theta_{l5}, \theta_{l6}]^T \) are expressed by

\[ \theta_{l1} = \sin \left( \frac{-t_1^T \mathbf{e}_2 + t_2^T \mathbf{e}_1}{2} \right), \quad \theta_{l2} = \sin \left( \frac{-t_2^T \mathbf{e}_1 + t_1^T \mathbf{e}_2}{2} \right), \quad \theta_{l3} = \sin \left( \frac{-t_3^T \mathbf{e}_1 + t_1^T \mathbf{e}_3}{2} \right), \quad \theta_{l4} = \sin \left( \frac{-u_1^T \mathbf{e}_2 + u_2^T \mathbf{e}_1}{2} \right), \quad \theta_{l5} = \sin \left( \frac{-u_2^T \mathbf{e}_1 + u_1^T \mathbf{e}_2}{2} \right), \quad \theta_{l6} = \sin \left( \frac{-u_3^T \mathbf{e}_1 + u_1^T \mathbf{e}_3}{2} \right) \] (16)

while the local axis \( \mathbf{e}_1 \) is updated by (14), and the last to base vectors are approximated by

\[ \mathbf{e}_2 = r_{av2} \cdot \frac{r_{av2} - r_{av1} \cdot (\mathbf{e}_1 + r_{av1})}{2} \] (19)

\[ \mathbf{e}_3 = r_{av3} \cdot \frac{r_{av3} - r_{av1} \cdot (\mathbf{e}_1 + r_{av1})}{2} \] (20)

which results in the current base triad as

\[ \mathbf{E}_n = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \] (21)

By equations (15) and (16), the local deformations can be written in the form

\[ \mathbf{p}_l = [0, 0, 0, 0, \theta_{l1}, \theta_{l2}, \theta_{l3}, u_l, 0, 0, \theta_{l4}, \theta_{l5}, \theta_{l6}]^T \] (22)

with respect to the co-rotating frame of reference. In the equations (17) - (20), a rotation matrix \( \mathbf{R} (\Delta \theta_i) \) and average rotation vectors \( \mathbf{r}_i \) were introduced. Spatial rotations will be introduced in the following for the purpose of defining the rotation matrix and an elemental mean of this.
2.1.2 Rotation matrix

Rotations in 3D are in general not commutative as they are in 2D. This is illustrated in figure 16 simply by rotating an object by $\frac{\pi}{2}$ two times, but in different sequences.

The non-commutativity of 3D rotations leads to the introduction of pseudo-vectors. The three nodal rotation components in figure 17 forms a pseudo-vector, expressed by

$$\mathbf{\theta} = \theta \mathbf{e} = [\theta_x, \theta_y, \theta_z]^T$$

where $\mathbf{e}$ is the axis of rotation, as illustrated in figure 18 while $\theta$ defines the rotation itself. The initial direction $\mathbf{t}_0$, in figure 18 is decomposed into a parallel and perpendicular component, with respect to the axis of rotation

$$\mathbf{t}_0 = \mathbf{t}_{01} + \mathbf{t}_{0\perp} = \mathbf{e}(\mathbf{e}^T \mathbf{t}_0) - \mathbf{e} \times (\mathbf{e} \times \mathbf{t}_0)$$

which shows that the rotation $\theta$ is only described by components perpendicular to the axis of rotation. From geometrical interpretation of figure 18 the current direction $\mathbf{t}_1$ is found to be

$$\mathbf{t}_1 = \mathbf{e}(\mathbf{e}^T \mathbf{t}_0) + \sin\theta(\mathbf{e} \times \mathbf{t}_0) - \cos\theta \mathbf{e} \times (\mathbf{e} \times \mathbf{t}_0)$$

By considering figure 18 simplifications are introduced as
and introducing the skew-symmetric matrix $S(e)$, so that

$$S(e)t_0 = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix} t_0 = e \times t_0$$  \hspace{1cm} (27)

allows for (25) to be rewritten as

$$t_1 = t_0 + \sin \theta S(e)t_0 + (1 - \cos \theta)S(e)S(e)t_0$$  \hspace{1cm} (28)

Factoring out the initial configuration $t_0$ leads to

$$t_1 = (I + \sin \theta S(e) + (1 - \cos \theta)S(e)S(e)) t_0$$  \hspace{1cm} (29)

from which the rotation matrix $R$ is finally isolated as

$$R(\theta) = I + \sin \theta S(e) + (1 - \cos \theta)S(e)S(e)$$  \hspace{1cm} (30)

which in pseudo-vector form, is given by

$$R(\theta) = I + \frac{\sin \theta}{\theta} S(\theta) + \frac{1 - \cos \theta}{\theta^2} S(\theta)S(\theta)$$  \hspace{1cm} (31)

which then is used to update the nodal triads in equations (17) and (18).

**Average rotation matrix**

In order to update the base triad, from equations (19) and (20), the average rotation is determined by

$$R_{av} = R\left(\frac{\gamma}{2}\right) = [r_{av1}, r_{av2}, r_{av3}]$$  \hspace{1cm} (32)

where $\gamma$ is the pseudo-vector describing the rotation from $T$ to $U$, expressed as

$$R(\gamma) = UT^T$$  \hspace{1cm} (33)

Assuming $0 < |\gamma| < \pi$, the antisymmetric part of (31) is given by

$$R_a = \frac{1}{2} \left( R(\gamma) - R(\gamma)^T \right) = \sin \gamma S(e) = \frac{\sin \gamma}{\gamma} S(\gamma)$$  \hspace{1cm} (34)

Recalling the skew-symmetric matrix form in (27), the expression in (34) is rewritten into

$$\sin \gamma e = \frac{\sin \gamma}{\gamma} = \frac{1}{2} \begin{bmatrix} R_{32}(\gamma) - R_{23}(\gamma) \\ R_{13}(\gamma) - R_{31}(\gamma) \\ R_{21}(\gamma) - R_{12}(\gamma) \end{bmatrix}$$  \hspace{1cm} (35)
from which the pseudo-vector $\gamma$ is obtained and finally used in (32) to determine the average rotation matrix.

Local internal forces and transformation to global domain
For the generating the tangent stiffness, the local internal forces are determined by

$$q_{li} = K_{el}p_l$$  \hspace{1cm} (36)

where $K_{el}$ is the local elastic stiffness matrix, known from e.g. Bernoulli-Euler beam theory. The local internal forces are used for establishing the geometric part of the tangent stiffness, explained in the following section. In order to solve the equation of motion, for which a method is introduced in section 2.2, the internal forces in the global domain is needed. The local internal forces are transformed into the global domain by

$$q_i = F^T q_{li} = F^T K_{el}p_l$$  \hspace{1cm} (37)

where $F$ is the transformation matrix, which was stated in (9), assumed as a known quantity. The transformation is generated on basis of the previous introduced variables, and the final form is a 12x12 matrix

$$F = [f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}]$$  \hspace{1cm} (38)

where $f_1 = f_2 = f_3 = f_8 = f_9 = 0$. Definitions of the non-zero components, may be found in [11], along with a detailed description of the derivation.

2.1.3 Tangent stiffness matrix
Recalling (13), the tangent stiffness matrix was considered to have two parts

$$K = \frac{\delta q_i}{\delta p} = K_e + K_\sigma$$  \hspace{1cm} (39)

where the local elastic stiffness is transformed into the current global elastic stiffness by

$$K_e = F^T K_{el}F$$  \hspace{1cm} (40)

where for Bernoulli-Euler beams in 3D, the local elastic stiffness matrix can be defined as

$$K_{el} = \begin{bmatrix} GJ & 0 & 0 & 0 & -GJ & 0 & 0 \\ 0 & 4EI_y & 0 & 0 & 0 & -2EI_y & 0 \\ 0 & 0 & 4EI_z & 0 & 0 & 0 & 2EI_z \\ -GJ & 0 & 0 & GJ & 0 & 0 & 0 \\ 0 & 2EI_y & 0 & 0 & 4EI_y & 0 & 0 \\ 0 & 0 & 2EI_z & 0 & 0 & 4EI_z \end{bmatrix}$$  \hspace{1cm} (41)

because the local frame of reference is co-rotated and local deformations are decoupled, as noticed in (22). Additionally, by omitting all zero terms in the local deformation vector and the transformation matrix, the local elastic stiffness matrix becomes a 7x7 matrix.

The geometric stiffness matrix $K_\sigma$ is generated, based on the previously found variables. As for the transformation matrix, the geometric stiffness will not be included in this. For further details, see [11].
2.2 Nonlinear Newmark time integration

Given initial displacements $x_0$ and velocities $\dot{x}_0$, the initial accelerations $\ddot{x}_0$ are estimated by

$$\ddot{x}_0 = M^{-1}(F^e_0 - F^i_0)$$  \hfill (42)

where $F^e_0$ is the external forces, based on the initial configuration, expressed by

$$F^e_0 = F^w_0 + F_0^i$$  \hfill (43)

Predictions of time step $(i + 1)$ are determined by

$$\ddot{x}_{i+1} = \ddot{x}_i$$  \hfill (44)

$$\dot{x}_{i+1} = \dot{x}_i + \ddot{x}_i dt$$  \hfill (45)

$$x_{i+1} = x_i + \dot{x}_i dt + \frac{1}{2} \ddot{x}_i dt^2$$  \hfill (46)

where $dt$ is the time step size.

In order to correct the predictions of time step $(i + 1)$, a residual is determined from the updated internal and external forces, based on the predicted configuration

$$r = F^e_{i+1} - M\ddot{x}_{i+1} - F^i_{i+1}$$  \hfill (47)

while the tangent stiffness is updated (described in section 2.1), based on the current configuration

$$K = \frac{\delta F^i_{i+1}}{\delta x_{i+1}}$$  \hfill (48)

and a modified tangent stiffness is formulated by

$$K_{mod} = K + \gamma \frac{\beta dt}{\delta} C + \frac{1}{\beta dt^2} M$$  \hfill (49)

where $\beta$ and $\gamma$ are integration constants.

Incremental displacements are determined by solving

$$\Delta x = K_{mod}^{-1} r$$  \hfill (50)

and the predicted motion is corrected by

$$x_{i+1} = x_{i+1} + \Delta x$$  \hfill (51)

$$\dot{x}_{i+1} = \dot{x}_{i+1} + \frac{\gamma}{\beta dt} \Delta x$$  \hfill (52)

$$\ddot{x}_{i+1} = \ddot{x}_{i+1} + \frac{1}{\beta dt^2} \Delta x$$  \hfill (53)

Based on the corrected values a new residual is determined from the updated internal and external forces by

$$r = F^e_{ext_{i+1}} - M\ddot{x}_{i+1} - F^i_{int_{i+1}}$$  \hfill (54)

and while $\|r\| > tolerance$, equilibrium iterations should be carried out by repeating (48-54). A new time step is solved by returning to (44), when the previous time step has converged. The procedure is repeated for all time steps. For further details, literature as [12] is suggested.
3 Water wave mechanics

The following sections introduce the theoretical basis for the modeling of linear and nonlinear water waves, which may appropriately model fatigue- and ultimate limit states, respectively. In order to model realistic water waves, an appropriate wave theory has to be chosen. Regions of validity for three different wave theories are shown in figure 19, namely linear waves, 5th order Stokes waves and Stream function waves.

![Diagram of wave theories](image)

*Figure 19: Regions of wave theory validity, where \( g \) is gravitational acceleration, \( T \) is wave period, \( H \) is wave height, \( d \) is water depth, \( L \) is wave length and \( \phi \) is a stream function wave of order \( n \). Modified version of the original [13].*

In order to handle all regions in figure 19 except wave breaking, a linear and nonlinear wave theory is introduced. Linear wave theory is often used for the modeling of operational conditions, which are irregular and may be directionally spreaded. These concepts are introduced in section 3.1. Stream function wave theory describes all regions in figure 19 except wave breaking. Thus, stream function wave theory may be chosen for the modeling of steep regular waves, which may be present in both steep and shallow waters. Stream function wave theory is introduced in section 3.2.

### 3.1 Linear wave theory

In the following linear wave theory is introduced, along with other topics, such as wave spectra, directional spreading and kinematic stretching.
3.1.1 Linear regular waves

Assuming waves can be modeled by incompressible, inviscid and irrotational fluid flow, potential flow theory is valid. For potential flows the Laplace equation \( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \) is the governing equation, while the boundary conditions are listed below.

1. Kinematic boundary condition at \( z = -d \), due to an impermeable seabed:
   \[
   \frac{\partial \phi}{\partial z} = 0
   \]  
   (55)

2. Kinematic boundary condition at \( z = \eta \), stating that particles at the surface stays at the surface:
   \[
   \frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} \frac{\partial \phi}{\partial x}
   \]  
   linearized \( \approx 0 \)
   (56)

3. Dynamic boundary condition at \( z = \eta \), stating that pressure on the surface must be constant (neglecting wind and assuming atmospheric pressure to be only a function of time):
   \[
   g\eta + \frac{1}{2} \left( \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right) + \frac{\partial \phi}{\partial t} = 0
   \]  
   linearized \( \approx 0 \)
   (57)

4. Periodicity condition, stating that the waves are periodic, progressive and of constant form:
   \[
   \eta(x, t) = \eta(x + L, t) = \eta(x, t + T) \quad \& \quad \phi(x, t) = \phi(x + L, t) = \phi(x, t + T)
   \]  
   (58)

characterizes the free surface waves. The mathematical problem is illustrated in figure 20.

![Figure 20: Illustration of the flow problem.](image)

The Laplace equation, with corresponding boundary conditions in figure 20, can not be solved analytically, because of the nonlinear boundary conditions at the free surface. In the case of linear waves, the wave amplitude \( a = \frac{H}{2} \) is considered small relative to the wave length \( L \). Thus, the free surface boundary conditions (2) and (3) can be linearized about the mean water level \( \eta \). Based on an exact solution of the Laplace equation \([14]\), with linearized boundary conditions, the velocity potential is found to be

\[
\phi = -\frac{ag}{\omega} \frac{\cosh(k(z + d))}{\cosh(kd)} \sin(\omega t - kx)
\]  
(59)
where $k = \frac{2\pi}{T}$ is the wave number, $\omega = \frac{2\pi}{T}$ is the angular wave frequency, $T$ is the wave period, $x$ and $z$ are the horizontal and vertical coordinates, respectively.

Linear regular waves are fully described by the four parameters $(a, L, T, d)$, as seen in equation (59). At finite water depth, the dispersion relation is

$$L = \frac{gT^2}{2\pi} \tanh \left( \frac{2\pi d}{L} \right)$$  (60)

relating wave length $L$ to the wave period $T$, which only necessitate the knowledge of $H$, $d$ and either $T$ or $L$, while the unknown is determined by iterating until it is converged.

The free surface elevation is described by

$$\eta = a\cos(\omega t - kx)$$  (61)

The water particle velocities are determined from the velocity potential in (59) by

$$\dot{u} = \frac{\partial \phi}{\partial t} = \frac{agk \cosh(k(z + d))}{\omega \cosh(kd)} \cos(\omega t - kx)$$  (62)

$$\dot{w} = \frac{\partial \phi}{\partial z} = -\frac{agk \sinh(k(z + d))}{\omega \cosh(kd)} \sin(\omega t - kx)$$  (63)

and the water particle accelerations are approximated by

$$\ddot{u} = \frac{d\dot{u}}{dt} \approx \frac{\partial \dot{u}}{\partial t} = -agk \frac{\cosh(k(z + d))}{\cosh(kd)} \sin(\omega t - kx)$$  (64)

$$\ddot{w} = \frac{d\dot{w}}{dt} \approx \frac{\partial \dot{w}}{\partial t} = -agk \frac{\sinh(k(z + d))}{\cosh(kd)} \cos(\omega t - kx)$$  (65)

where the higher order convective terms are neglected. An example of a linear regular wave is shown in figure 21.

Because the free surface boundary conditions are linearized, the water particle kinematics are only valid from the seabed to the mean water level. However, waves above the mean water level are still present and therefore contribute to the wave loading. Thus, kinematic stretching is introduced to adjust the linear theory by approximating particle kinematics up to the free surface elevation.

### 3.1.2 Kinematic stretching

Various stretching formulas have been proposed, such as the ones in [15]. In the following, we only consider a commonly used method, named Wheeler stretching [16].

By modification of the vertical coordinate $z$, the wave kinematics are stretched from the mean water level to the instantaneous free surface elevation by introduction of the modified vertical coordinate

$$z' = \frac{d(z - \eta)}{(d + \eta)} \quad \text{for} \quad -d \leq z \leq \eta$$  (66)

mapping the actual $z$ coordinate to the range between $-d$ and 0. In figure 21, the use of Wheeler stretching is exemplified for the horizontal particle velocities.

The Wheeler stretching was originally proposed for irregular waves [16]. Thus, the method is also applicable for linear irregular wave kinematics, introduced in the following section.
3.1.3 Irregular waves

Long-crested sea states can be modeled by random phased linear regular waves, which are distributed in accordance to a wave energy spectrum. The wave energy spectrum, shown in figure 22, relates the amount of energy a certain wave component introduces, during a given sea state. In this manner sea states are modeled, based on measured wave characteristics.

The Pierson-Moskowitz wave spectrum [17] is used for fully developed seas and is expressed by

\[
S_{\eta,PM}(f) = \frac{1}{3.2} \frac{H_s^2 f_p^4}{f^3} \exp \left( -\frac{5 f_p^4}{4 f^4} \right)
\]

where \( H_s \) is the significant wave height, \( f_p = \frac{1}{T_p} \) is the peak frequency, \( T_p \) is the peak period and \( f \) is the wave frequency.

At locations with limited fetch, the JONSWAP spectrum [17] is commonly used, which is expressed by

\[
S_{\eta,JS}(f) = C(\gamma) S_{\eta,PM}(f) \gamma^3
\]

where

\[
\beta = \exp \left( -\frac{(f - f_p)^2}{2\sigma^2 f_p^2} \right)
\]

\[
\sigma = \begin{cases} 
0.07 & \text{for } f \leq f_p \\
0.09 & \text{for } f > f_p
\end{cases}
\]

while the peak enhancement factor may be found from

\[
\gamma = \begin{cases} 
5 \exp \left( 5.75 - 1.15 \frac{T_p}{\sqrt{H_s}} \right) & \text{for } \frac{T_p}{\sqrt{H_s}} \leq 3.6 \\
1 & \text{for } 3.6 \leq \frac{T_p}{\sqrt{H_s}} \leq 5 \\
1 & \text{for } \frac{T_p}{\sqrt{H_s}} > 5
\end{cases}
\]

and the normalizing factor by

\[
C(\gamma) = \frac{\int_0^\infty S_{\eta,PM}(f) df}{\int_0^\infty S_{\eta,PM}(f) \gamma^3 df} \approx 1 - \ln(\gamma) 0.287
\]

Because the JONSWAP spectrum is a modification of the Pierson-Moskowitz spectrum, the Pierson-Moskowitz spectrum is recovered for \( \gamma = 1 \).
Figure 22: JONSWAP spectra with $H_s = 1\text{m}$ and $T_p = 12\text{s}$, for various peak enhancement factors.

Based on a wave spectrum, $n$ number of linear regular wave amplitudes are generated by

$$a_i = \sqrt{2S_\eta(f_i)\Delta f}$$  \hspace{1cm} (73)

where $a_i$ is wave amplitude for wave component $i$, $S_\eta(f_i)$ is the spectral energy density for wave frequency $f_i$ and $\Delta f$ is the frequency bandwidth, which depends on the number of wave components.

The $n$ wave amplitudes are superposed, which is acceptable due to linearity, and defines an irregular free surface by

$$\eta = \sum_{i=1}^{n} a_i \cos(\omega_i t - k_i x + \delta_i)$$  \hspace{1cm} (74)

and a corresponding velocity potential by

$$\phi = \sum_{i=1}^{n} -\frac{a_i g \cosh(k_i(z+d))}{\omega_i} \frac{\cosh(k_i d)}{\cosh(k_i d)} \sin(\omega_i t - k_i x + \delta_i)$$  \hspace{1cm} (75)

where $\delta_i$ is a random phase angle between 0 and $2\pi$ for wave component $i$. The water particle kinematics are also determined from the principle of superposition

$$\dot{u} = \sum_{i=1}^{n} \frac{a_i g k_i \cosh(k_i(z+d))}{\omega_i} \frac{\cosh(k_i d)}{\cosh(k_i d)} \cos(\omega_i t - k_i x + \delta_i)$$  \hspace{1cm} (76)

$$\dot{w} = \sum_{i=1}^{n} -\frac{a_i g k_i \sinh(k_i(z+d))}{\omega_i} \frac{\cosh(k_i d)}{\cosh(k_i d)} \cos(\omega_i t - k_i x + \delta_i)$$  \hspace{1cm} (77)

$$\ddot{u} \approx \sum_{i=1}^{n} -a_i g k_i \cosh(k_i(z+d)) \sin(\omega_i t - k_i x + \delta_i)$$  \hspace{1cm} (78)

$$\ddot{w} \approx \sum_{i=1}^{n} -a_i g k_i \frac{\sinh(k_i(z+d))}{\cosh(k_i d)} \cos(\omega_i t - k_i x + \delta_i)$$  \hspace{1cm} (79)

As an example, a linear irregular sea state is generated and the corresponding free surface elevations and particle velocities are shown in figure 23.

In reality, not all waves can be assumed long-crested due to directional variations. These directional variations cause waves to become short-crested, leading to multi-directional sea states. In the following
a spreading function is introduced, which distributes linear irregular waves in the two-dimensional plane, which represents a multi-directional sea state.

3.1.4 Multi-directional waves

Short-crested sea states are modeled by introducing directional variations to the wave spectrum, expressing a directional wave spectrum by

\[ S(f, \theta) = S_\eta(f)D(\theta) \]  \hspace{1cm} (80)

where \( S(f, \theta) \) is the directional wave spectrum, \( S_\eta(f) \) is the wave energy spectrum for wave frequency \( f \) and \( D(\theta) \) is the spreading function, which has to obey \( \int_{-\pi}^{\pi} D(\theta) d\theta = 1 \). Different spreading functions have been proposed [15, 18]. A commonly used spreading function is a cosine-power distribution expressed by

\[ D(\theta) = \frac{2^{2s-1} \Gamma^2(s+1)}{\pi \Gamma(2s+1)} \cos^{2s} \left( \frac{\theta - \theta_0}{2} \right) \quad \text{for} \quad -\pi \leq \theta \leq \pi \]  \hspace{1cm} (81)

where \( \theta_0 \) is the mean wave direction, \( s \) is the spreading index and the Gamma function \( \Gamma(n) = (n-1)! \) for \( n > 0 \). The spreading function in (81) distributes the irregular waves in the range \( \theta_0 \pm 180^\circ \). In figure 24 the influence of spreading index \( s \) is illustrated, indicating uni-directional waves when \( s \to \infty \). The spreading index \( s \) can be approximated in various ways, as discussed in [18].
As for linear irregular waves, the principle of superposition is valid. Thus, multi-directional linear irregular waves are generated in the same manner as previously described. The wave amplitude components are now found by

\[ a_{ij} = \sqrt{2S_n(f_i)D(\theta_j)\Delta f \Delta \theta} \quad (82) \]

while the free surface elevation is defined by

\[ \eta = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} \cos (\omega_i t - k_i (x \cos(\theta_j) + y \sin(\theta_j)) + \delta_{ij}) \quad (83) \]

where \( m \) is the number of spreading function divisions, based on the direction bandwidth \( \Delta \theta \). The corresponding particle velocities are determined by

\[ \dot{u} = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} g k_i \frac{\cos(\theta_j)}{\omega_i} \frac{\cosh(k_i(z+d))}{\cosh(k_i d)} \cos (\omega_i t - k_i (x \cos(\theta_j) + y \sin(\theta_j)) + \delta_{ij}) \quad (84) \]

\[ \dot{v} = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} g k_i \frac{\sin(\theta_j)}{\omega_i} \frac{\cosh(k_i(z+d))}{\cosh(k_i d)} \cos (\omega_i t - k_i (x \cos(\theta_j) + y \sin(\theta_j)) + \delta_{ij}) \quad (85) \]

\[ \dot{w} = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} g k_i \frac{\sinh(k_i(z+d))}{\cosh(k_i d)} \sin (\omega_i t - k_i (x \cos(\theta_j) + y \sin(\theta_j)) + \delta_{ij}) \quad (86) \]

whereas the particle accelerations are once again approximated, so that

\[ \ddot{u} \approx \sum_{i=1}^{n} \sum_{j=1}^{m} -a_{ij} g k_i \frac{\cos(\theta_j)}{\omega_i} \frac{\cosh(k_i(z+d))}{\cosh(k_i d)} \sin (\omega_i t - k_i (x \cos(\theta_j) + y \sin(\theta_j)) + \delta_{ij}) \quad (87) \]

\[ \ddot{v} \approx \sum_{i=1}^{n} \sum_{j=1}^{m} -a_{ij} g k_i \frac{\sin(\theta_j)}{\omega_i} \frac{\cosh(k_i(z+d))}{\cosh(k_i d)} \sin (\omega_i t - k_i (x \cos(\theta_j) + y \sin(\theta_j)) + \delta_{ij}) \quad (88) \]

\[ \ddot{w} \approx \sum_{i=1}^{n} \sum_{j=1}^{m} -a_{ij} g k_i \frac{\sinh(k_i(z+d))}{\cosh(k_i d)} \cos (\omega_i t - k_i (x \cos(\theta_j) + y \sin(\theta_j)) + \delta_{ij}) \quad (89) \]

As the multi-directional irregular waves not only depend on the wave frequency components, but also the wave directions, the spectral density is now defined in three-dimensional space. The directional wave spectrum of an arbitrary sea state is illustrated in figure 25.

**Figure 25:** Directional wave spectrum \( S(f, \theta) \) with \( H_s = 2m \), \( T_p = 12s \), \( \gamma = 3.3 \), \( \theta_0 = 0 \) and \( s = 1 \).
Expectedly, the spectral energy density $S(f, \theta)$ is maximum at $f_p = \frac{1}{T_p} = 0.0833$ Hz and $\theta = \theta_0 = 0^\circ$. In figure 26 the free surface of a short-crested sea state is shown. The free surface corresponds to multidirectional irregular waves, modeled by the directional wave spectrum in figure 25.

Figure 26: Free surface of a short-crested sea state, based on a directional wave spectrum with $H_s = 2m$, $T_p = 12s$, $\theta_0 = 0$, $s = 1$ at $t = 100s$.

The particle kinematics, corresponding to a 100s realization of the directional wave spectrum in figure 25, are shown in figure 27. Thus, short-crested sea states may be modeled by use of linear wave theory, along with a wave energy spectrum and spreading function.

Figure 27: Particle kinematics, based on a directional wave spectrum with $H_s = 2m$, $T_p = 12s$, $\theta_0 = 0$, $s = 1$, at $(x, y, z) = (0, 0, -2)$.

As mentioned previously, the short-crested sea states are mostly used for representing operational conditions. However, at shallow waters or for steep waves, the waves may become nonlinear. Thus, the nonlinear boundary conditions have to be fulfilled, which require the governing mathematical problem to be solved numerically. For this purpose, Stream function wave theory is introduced in section 3.2.

3.1.5 Current

Besides the wave induced fluid motion, additional flow may be introduced by ocean currents, which are assumed to be steady. Currents are generated by several factors, including wind, tides, water temperature,
salinity etc. For floating structures, surface currents are of great importance, as it causes the structures to drift, which may introduce significant stresses in the structure. Thus, the current velocities should be included in the sea state modeling.

\[ \eta \]

\[ \dot{u}(z)_{\text{const.}} \]

\[ \dot{u}(z)_{\text{lin.}} \]

\[ \dot{u}(z)_{\text{poly.}} \]

Figure 28: Different types of assumed velocity profiles

In figure 28 different current velocity profiles are shown, but others have been proposed [15]. In SOFIA the current velocities are superimposed on the wave induced particle velocities and in the present version the constant current profile is available.

3.2 Stream function wave theory

Different nonlinear wave theories have been proposed for the regions sketched in figure 19 where linear wave theory becomes invalid and thus provide inaccurate results. The term ‘nonlinear’ refers to the nonlinear nature of the free surface boundary conditions, expressed in (56) and (57), before linearization. An example of a nonlinear wave, computed by means of stream function wave theory, is illustrated in figure 29 which shows the shallow throughs and steep crests of a highly nonlinear water wave, which is near the breaking limit.

\[ \eta \]

\[ -250 \ -200 \ -150 \ -100 \ -50 \ 0 \ 50 \ 100 \ 150 \ 200 \ 250 \]

\[ x\text{-direction [m]} \]

\[ -150 \ -100 \ -50 \ 0 \ 50 \ 100 \ 150 \]

\[ \eta \text{ [m]} \]

Figure 29: A sharp-crested stream function wave with \( \frac{d}{gT^2} = 0.04 \) and \( \frac{H}{gT^2} = 0.02 \).

Stream function wave theory [19] constitute a nonlinear wave theory, which wish to find an approximate solution to the governing equation while fulfilling the exact free surface boundary conditions, (56) and (57), without neglecting the nonlinear terms. A significant advantage of stream function wave theory, compared to other nonlinear wave theories, is the range of applicability as illustrated in figure 19.
Stream function wave theory is applicable to all regions within the wave breaking limit, and hence no requirements are necessary for the steepness of the wave.

In the case of plane incompressible flow a scalar function, namely a stream function, can be used for the description of the flow field. The stream function guarantees fulfillment of the continuity equation (for incompressible flow: $\nabla \cdot \mathbf{u} = 0$), due to the velocity field definition

$$
\mathbf{u} = [u, w] = \left[ -\frac{\partial \psi}{\partial z}, \frac{\partial \psi}{\partial x} \right]
$$

which by substitution into the continuity equation for incompressible flows

$$
\nabla \cdot \mathbf{u} = -\frac{\partial^2 \psi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial z \partial x} = 0
$$

proves always to be fulfilled. Furthermore, for irrotational flow the vorticity is zero, i.e.

$$
\nabla \times \mathbf{u} = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = 0
$$

which expressed in terms of the stream function

$$
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = 0
$$

shows that the Laplace equation also will be fulfilled in the fluid domain.

A wave-like fluid domain is assumed and illustrated in figure 30. The $(x_r, z_r)$-system is fixed to a wave body moving with the velocity $c_r$ relative to a still water body (fluid domain without waves), described by the $(x, z)$-system. Although the wave seems to be moving, the mathematical problem is simplified by assuming the wave to be steady in the $(x_r, z_r)$-system and that the still water body is propagating with $c_r$ in the negative $x$-direction.

![Figure 30: Observation of the fluid domain from the $(x_r, z_r)$-system, which follows the wave, while the still water body fixed $(x, z)$-system propagates at a speed of $c_r$ in the negative $x$-direction.](image)

From this perspective, not only (93) has to be fulfilled in the fluid domain, but also the boundary conditions need to be fulfilled. However, first they need to be transformed into the new computational fluid domain, namely the $(x_r, z_r)$-system.

Assuming a flat and impermeable seabed, the kinematic bottom boundary condition is introduced as

$$
\psi(x_r, -d) = Q
$$

where the discharge $Q = \int_{-h}^{\eta} u \, dz$ and the kinematic boundary condition at the free surface assumes particles to remain at the surface by
\[ \psi(x_r, \eta) = 0 \] (95)

and additionally makes sure that the difference between the adjacent stream lines is equal to the discharge. The above defined kinematic boundary conditions are possible only in the \((x_r, z_r)\)-system because stream lines follow the seabed and the free surface, and the flow is everywhere tangent to the stream lines.

At the free surface the pressure is assumed constant and since nothing evolves during time, the steady Bernoulli equation may express the dynamic free surface boundary condition as

\[ g\eta + \frac{1}{2} (u^2 + w^2) = R \] (96)

where \(R\) is known as the Bernoulli constant. By assuming the wave to be periodic and that the wave crest coincide with the origin of the \((x_r, z_r)\)-system, the stream function is assumed to be expressed as

\[ \psi(x_r, z) = c_r(z + d) + \sum_{j=1}^{N} B_j \frac{\sinh(jk(z + d))}{\cosh(jkd)}\cos(jkx_r) + Q \] (97)

which automatically fulfills (93), (94) and the periodicity condition. However, the free surface boundary conditions (95) and (96) are not automatically fulfilled. They are fulfilled by solving a system of equations, which will enforce the last boundary conditions at a number of \(i = N+1\) equidistant points in the interval \(x_r = [0, \frac{L}{2}]\). The assumed stream function in (97) is substituted into the free surface boundary conditions, which gives

\[ \psi(x_r, \eta_i) = c_r(\eta_i + d) + \sum_{j=1}^{N} B_j \frac{\sinh(jk(\eta_i + d))}{\cosh(jkd)}\cos(jkx_r) + Q = 0 \] (98)

and

\[
\begin{align*}
g\eta_i & + \frac{1}{2} \left[ -c_r - k \sum_{j=1}^{N} j B_j \frac{\cosh(jk(\eta_i + d))}{\cosh(jkd)}\cos(jkx_r) \right]^2 \\
& + \frac{1}{2} \left[ -k \sum_{j=1}^{N} j B_j \frac{\sinh(jk(\eta_i + d))}{\cosh(jkd)}\sin(jkx_r) \right]^2 \\
& = R
\end{align*}
\] (99)

and a total of \(2N+5\) unknowns \((\eta_i, B_j, c_r, k, Q \text{ and } R)\) are present, but only a total of \(2N+2\) equations. Therefore, three additional equations have to be formulated for the system of equations to yield an unique solution. From the condition of incompressibility the mean free surface elevation has to be zero

\[ \bar{\eta} = \frac{1}{L} \int_0^L \eta dx_r = 0 \] (100)

and by defining the wave height as

\[ H = \eta_{\text{max}} - \eta_{\text{min}} \] (101)

and wave length as

\[ L = c_r \cdot T \] (102)

the last equations are found and thereby the system of equations are no longer under-determined. The established system of equations – namely (98), (99), (101) and (102) – can be solved by appropriate initial guesses and the Newton-Raphson method [19]. After the unknowns have been determined, the stream function is determined by (97), from which the particle kinematics and free surface elevation may be derived. Further details are given in [19], where constraints on current and discharge are also described.
4 Hydrodynamic and gravitational forces

Offshore structures exposed to fluid-induced loads, originating from waves and current. Consequently, the structures may displace and induce additional fluid-structure induced loads. In general hydrodynamic loads consist of both inertial- and viscous forces. In the following sections, these two types of loading are introduced, in relation to floating structures.

4.1 Inertial forces

By means of Bernoulli’s equation for unsteady potential flow, the pressure in a potential flow is determined by

\[ p = -\rho \frac{\partial \phi}{\partial t} - \rho \frac{1}{2} \left| \nabla \phi \right|^2 - \rho g z \]  

(103)

where \( \phi \) is the total velocity potential. In linear theory, the convective pressure \( \rho \frac{1}{2} \left| \nabla \phi \right|^2 \approx 0 \) and the total velocity potential can be divided into three components

\[ \phi = \phi_I + \phi_D + \phi_R \]  

(104)

where \( \phi_I \) is the undisturbed wave potential, \( \phi_D \) is the diffracted wave potential and \( \phi_R \) is the radiated wave potential. The corresponding pressure field becomes

\[ p = -\rho \left( \frac{\partial \phi_I}{\partial t} + \frac{\partial \phi_D}{\partial t} + \frac{\partial \phi_R}{\partial t} + g z \right) = p_I + p_D + p_R + p_H \]  

(105)

where \( p_I \) is the unsteady pressure due to undisturbed waves, \( p_D \) is the unsteady pressure due to wave diffraction, \( p_R \) is the unsteady pressure due to wave radiation and \( p_H \) is the hydrostatic pressure.

The resulting forces, which originate from the pressure \( p \) on the submerged surface, are found by

\[ \mathbf{F} = -\int_{A_s} \rho n dA \quad \wedge \quad \mathbf{M} = -\int_{A_s} \rho (\mathbf{r} \times n) dA \]  

(106)

where \( n \) is an unit vector perpendicular to surface and \( \mathbf{r} \) is the vector from the point of interest to the unit vector at the body surface (see figure 31).

Froude-Krylov force

For relative transparent structures, where \( \frac{\text{characteristic length}}{\text{wave length}} \ll 1 \), the flow may not be disturbed significantly. Thus, \( \phi \approx \phi_I \) is an acceptable assumption, generating a resulting force

\[ \mathbf{F}_{FK} = -\int_{A_s} p_I n dA \]  

(107)

which is known as the Froude-Krylov force. The Froude-Krylov force is generated by the unsteady pressure field, due to the undisturbed wave potential. Expressing (107) in terms of the divergence theorem, the Froude-Krylov force is given by

\[ \int_{A_s} \mathbf{F}_{FK} = -\int_{\mathbf{r}} \mathbf{F} dA \]
\[ \mathbf{F}_{FK} = -\int_{A_t} p_t n dA = -\int_{V_t} \nabla p_t dV \quad (108) \]

where \( \nabla p_t \) may be assumed constant within the submerged volume, and hence, approximated at the body center as \( \nabla p_{I_c} \). Thus, an approximate for the Froude-Krylov force is given by

\[ \mathbf{F}_{FK} \approx -\nabla p_{I_c} \int_{V_s} dV = -\nabla p_{I_c} V = \rho \ddot{u}_c V \quad (109) \]

where \( \ddot{u}_c \) is the fluid acceleration, measured at the body center. This can be verified by considering figure 32 showing a cylinder exposed to a flow in the x-direction.

From figure 32 it is often convenient to express the Froude-Krylov force, in (109), as an in-line force

\[ f_{FK} \approx \int_0^{2\pi} p_t \cos \theta d\theta = \rho \pi R^2 \ddot{u}_c = \rho A \ddot{u}_c \quad (110) \]

This relation will later show to be quite useful, in the application of loads to slender members, by means of the Morison equation.

**Hydrodynamic mass force**

Applying a force \( \mathbf{F} \) to a mass \( m \) will introduce an acceleration \( a \), which according to Newton’s 2nd has to fulfill

\[ F = ma \quad (111) \]

In general, accelerating the mass \( m \), the surrounding medium reacts on this, introducing accelerations of the medium too, so that

\[ F = (m + m_a) a \quad (112) \]

where \( m_a \) is the added mass, accounting for the mass of a surrounding medium, which also has to accelerate in relation to the point mass \( m \). In air, added mass is negligible, however, in water the added mass may introduce significant forces, called hydrodynamic mass forces.
Consider the accelerating circular cross section in figure 33 with radius $R$ and acceleration $\ddot{x}$. The hydrodynamic mass force, parallel to the acceleration in x-direction, is determined by integration of the pressure

$$f_a = \int p ds = \int pR \cos \theta d\theta$$

(113)

where $p = -\rho \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right)$. The velocity potential for a flow around a fixed cylinder [20] is in polar coordinates expressed as

$$\phi = \ddot{u}(r + \frac{R^2}{r}) \cos \theta$$

(114)

which for the inverse relation of a resting fluid at infinity ($r \to \infty$), and cylinder with forward speed $\dot{x} = \ddot{u}$, the velocity potential is expressed by

$$\phi = \frac{R^2}{r} \cos \theta$$

(115)

Inserting the velocity potential into (113), at the cylinder surface ($r = R$), the hydrodynamic mass force is expressed as

$$f_a = \int_0^{2\pi} p \cos \theta R d\theta = \int_0^{2\pi} \left( \ddot{x} R \cos \theta + \frac{1}{2} \dot{x}^2 \right) \cos \theta R d\theta$$

(116)

while multiplication, and moving constants in front, leads to

$$f_a = -\rho \ddot{x} R^2 \int_0^{2\pi} \cos^2 \theta d\theta - \rho \frac{1}{2} \dot{x}^2 \int_0^{2\pi} \cos \theta d\theta = -\rho \pi R^2 \ddot{x}$$

(117)

from this, the added mass is found by realizing the force $f_a$ is caused by acceleration of the surrounding fluid, which by means of (111) is given as

$$f_a = -\rho \pi R^2 \ddot{x} = -m_a \ddot{x}$$

(118)

From this, it is noticed that the hydrodynamic mass force acts opposite to the acceleration. The added mass of the circular cross section is now isolated from (118) and is pr. unit length

$$m_a = \rho \pi R^2 = \rho C_a A$$

(119)

where $C_a = 1$ is the theoretical added mass coefficient of a circular cross sectional. An in-line hydrodynamic mass force pr. unit length is finally expressed as

$$f_a = -\rho C_a A \ddot{x}$$

(120)

where $\ddot{x}$ is the structural acceleration. In situations where the structural element is flexible or flexible constrained, a relative hydrodynamic mass force is expressed as

$$f_a = \rho C_a A (\ddot{u} - \ddot{x})$$

(121)

where $\ddot{u}$ is the fluid particle acceleration.

4.2 Viscous forces

Bernoulli’s equation in (103), may not predict realistic pressures for general cases. Consider the cylinder in figure 33 now instead of applying an acceleration to the cylinder, we assume the cylinder is fixed and the flow is uniform. The velocity potential of a fixed cylinder in a uniform flow is stated in (113) as

$$\phi = \ddot{u}(r + \frac{R^2}{r}) \cos \theta$$

(122)

where $\ddot{u}$ is the velocity of the steady uniform flow. A resulting force in the flow direction are now found by means of pressure integration, along the perimeter of the cylinder
\[ F_D = -\int_0^{2\pi} pR\cos\theta d\theta \]  (123)

where the pressure \( p = \frac{1}{2} \rho \hat{u}^2 (1 - 4\sin^2 \theta) \) is found from the steady Bernoulli equation \( \frac{\partial \phi}{\partial t} = 0 \). Carrying out the integration, a resulting force is found

\[ F_D = -\int_0^{2\pi} \frac{1}{2} \rho \hat{u}^2 (1 - 4\sin^2 \theta) R\cos\theta d\theta = 0 \]  (124)

From (124) it is noticed, that from potential theory, no resulting force is introduced when exposing a fixed cylinder to a steady fluid flow. This solution, with lack of physical meaning, is referred to as d’Alembert’s paradox.

**Viscous drag forces**

In reality, viscous drag forces are introduced, as the fluid interacts with the structural surface. In figure 34, the potential flow pressure distribution is compared to an experimental. Because the potential flow theory assumes irrotational and inviscid flow, no flow separation will occur. This do not disqualify potential flow theory, but restrict it to problems where viscous effects are negligible.

From figure 34 it is obvious that a resulting force is introduced by the steady flow. Therefore, a drag coefficient \( C_D \) is introduced, if viscous effects are present. In this manner a viscous drag force pr. unit length is determined by

\[ f_D = \frac{1}{2} C_D \rho D \hat{u}^2 = \frac{1}{2} C_D \rho D |\hat{u}| \]  (125)

where \( D \) is the diameter of the cylinder and \( \hat{u}^2 = \hat{u} |\hat{u}| \), in order to preserve the flow direction. The drag coefficients can be determined from experiments or CFD analyses. From an experimental point of view, the Reynolds number \( Re = \frac{uD}{\nu} \) where \( \nu \) is the kinematic viscosity, is used to distinguish between different flow regimes, as shown in figure 35.

From figure 34 it is obvious that a resulting force is introduced by the steady flow. Therefore, a drag coefficient \( C_D \) is introduced, if viscous effects are present. In this manner a viscous drag force pr. unit length is determined by

\[ f_D = \frac{1}{2} C_D \rho D \hat{u}^2 = \frac{1}{2} C_D \rho D |\hat{u}| \]  (125)

where \( D \) is the diameter of the cylinder and \( \hat{u}^2 = \hat{u} |\hat{u}| \), in order to preserve the flow direction. The drag coefficients can be determined from experiments or CFD analyses. From an experimental point of view, the Reynolds number \( Re = \frac{uD}{\nu} \) where \( \nu \) is the kinematic viscosity, is used to distinguish between different flow regimes, as shown in figure 35.

**Figure 34:** Pressure distributions due to a (dotted lines) potential flow and a (full lines) viscous flow.

**Figure 35:** Different flow regimes and corresponding Reynolds numbers. [21]
The in-line viscous drag force \( f_D \) therefore depend on the flow conditions. As for the hydrodynamic mass force, the viscous drag force may be expressed in a relative form by

\[
f_D = \frac{1}{2} C_D \rho D (\dot{u} - \dot{x}) |\dot{u} - \dot{x}|
\]

where \( \dot{x} \) is the structural velocity. This will include viscous damping, which may contribute significantly to the total hydrodynamic loading on floating structures.

4.3 Buoyancy and gravity

Floating structures are indeed exposed to buoyancy forces, but are however also exposed to gravity. The resulting forces we define as gravitational forces, as they are introduced due to the gravitational acceleration.

For some structures the buoyancy forces may act as restoring forces, but for others the buoyancy forces may introduce destabilizing moments, which necessitate other kinds of restoring forces. Therefore, a floating structure in equilibrium is not exposed to restoring forces, but is indeed exposed to forces due to either buoyancy, gravity or other forces, such as mooring line forces. The restoring forces from buoyancy and gravity are included by applying buoyancy forces based on the instantaneous wetted area, as well as gravity for the whole structure.

A buoyancy force is constituted by the weight of the fluid displaced by the body, cf. Archimedes’ principle. An example is given for a fully submerged circular cross section, in figure 36.

Mathematically, Archimedes’ principle is verified by integrating the hydrostatic pressure, on the perimeter of the cross section. This leads to a buoyancy force pr. unit length

\[
f_b = \int_0^{2\pi} p_H ds = \int_0^{2\pi} \rho g R_o^2 \sin^2 \theta d\theta = \rho g R_o^2 \frac{\pi}{4} = \rho g A_s
\]

where \( \rho \) is density of water, \( g \) gravitational acceleration and \( A_s \) is the submerged cross sectional area, based on the outer radius \( R_o \). From this a distribution of buoyancy force may be defined, based on the instantaneous submerged cross sectional area.

In general the buoyancy force may vary due to internal flooding, which means that the distributed buoyancy force is not proportional to the outer radius \( R_o \). Thus, a more general formulation of the buoyancy force is
\[ f_b = C_b \rho g A_s \]  \hspace{1cm} (128)

where \( C_b \) is a buoyancy coefficient, which may be used to define the correct proportion of buoyancy, based on the actual setting. If the cylinder is flooded, then the following must be true

\[ f_b = C_b \rho g A_s = \rho g A \]  \hspace{1cm} (129)

which yield a buoyancy coefficient \( C_b = \frac{A}{A_s} \), where \( A \) is the actual cross sectional area. It must be noted that for deep water applications, the internal stresses due to hydrostatic pressure must be accounted for, which are not taken into account by the current formulation.

Finally, the gravitational forces are determined in a similar manner as in \((127)\). Now the distributed gravitational forces are expressed as

\[ f_g = \rho m g A \]  \hspace{1cm} (130)

where \( \rho m \) is the material density of the structural member. The resultant gravitational forces may be found directly by multiplication of the mass matrix and an acceleration vector equal to the gravitational acceleration. The resulting gravitational forces are expressed by

\[ f_G = f_b + f_g \]  \hspace{1cm} (131)

and is calculated in a similar manner, as described in the following section.

### 4.4 Morison equation

In previous sections, inertial and viscous forces was introduced. In the following, a total in-line force is established, based on the inertial and viscous force contributions. The main idea, established by Morison \[6\], is to sum the inertial and viscous force contributions into a total in-line force

\[ f_w = f_I + f_V \]  \hspace{1cm} (132)

where \( f_I = f_{FK} + f_a \) is the inertial forces from equations \((110)\) and \((121)\), while \( f_V = f_D \) is the viscous forces from \((125)\). For a vertical cylinder, the relative Morison equation is stated as

\[ f_w = \rho A \ddot{u} + \rho C_a A (\ddot{u} - \ddot{x}) + \frac{1}{2} C_D \rho D (\dot{u} - \dot{x}) | \dot{u} - \dot{x} | \]  \hspace{1cm} (133)

accounting for Froude-Krylov forces, hydrodynamic mass forces, viscous drag forces and viscous damping. In general, the in-line force \( f_w \) will vary with time, as particle kinematics are functions of time and structural members may displace, due to the incoming waves. Hence, for the sake of generality, the particle kinematics are divided into components, perpendicular and tangential to the structure.

The global fluid kinematics are transformed by the base triad (section \(2.1\)), to the local frame of reference by

\[ [\ddot{u}_l, \ddot{v}_l, \ddot{w}_l]^T = E^T [\ddot{u} + \dot{u}_c, \dot{v}, \dot{w}]^T \]  \hspace{1cm} (134)

Consider the structural member in figure \(37\) which is exposed to an incoming wave. The wave loading is considered up to the free surface elevation \( \eta \). Further, as arbitrary element orientation is possible, the wave loading is decomposed into local components as \( f_w = [f_{wu}, f_{wn}, f_{wn}]^T \).

The distributed wave loading perpendicular to the member \( f_{wn} \) is determined by

\[ f_{wn} = \rho A \ddot{u}_n + \rho C_a A (\ddot{u}_n - \ddot{x}_n) + \frac{1}{2} C_D \rho D (\ddot{u}_n - \ddot{x}_n) | \ddot{u}_n - \ddot{x}_n | \]  \hspace{1cm} (135)

where \( A \) is the cross sectional area, \( \ddot{u}_n \) is the normal fluid acceleration, \( C_a \) is the added mass coefficient, \( \ddot{x}_n \) is the normal structural acceleration, \( C_D \) is the perpendicular drag coefficient, \( \ddot{u}_n \) is the normal fluid velocity and \( \ddot{x}_n \) is the normal structural velocity.
Additionally, the distributed wave loading tangential to the beam axis $f_{wt}$ is determined by

$$f_{wt} = \frac{1}{2} \rho_w C_{Dt} D (\dot{u}_t - \dot{x}_t) |\dot{u}_t - \dot{x}_t|$$

(136)

where $C_{Dt}$ is the tangential drag coefficient, $\dot{u}_t$ is the tangential fluid velocity and $\dot{x}_t$ is the tangential structural velocity. As noticed, no inertial forces are included in the tangentially distributed hydrodynamic loading, but may be included in a similar manner.
References


