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Ding, Tao; Lu, Runzhao; Yang, Yongheng; Blaabjerg, Frede

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A Condition of Equivalence Between Bus Injection and Branch Flow Models in Radial Networks

Tao Ding, *Member, IEEE*, Runzhao Lu, *Student Member, IEEE*, Yongheng Yang, *Senior Member, IEEE*, and Frede Blaabjerg, *Fellow, IEEE*

***Abstract*—This paper presents an exact bijection between the branch flow model (BFM) and bus injection model (BIM) in radial systems. Moreover, the equivalence and the corresponding condition are investigated and rigorously proved. The exploration reveals that the bijection exists if and only if the network is connected and there is no zero-impedance branch.**

***Index Terms*—Branch flow model, bus injection model, bijective function, convexification**

I. Introduction

THE OPTIMAL power flow problem is fundamental in power system operations, which is practical in economic dispatch, unit commitment, transmission system expansion planning and reactive power operation, etc. Extensive research efforts have been made since the first formulation of the economic dispatch problem by Carpentier in 1962 [1]. The AC Optimal Power Flow (ACOPF) problem can accurately describe the power flow, but it is nonconvex due to the nonconvex AC power flow equation, challenging the global optimality.

Recently, many papers have discussed the convex relaxation of the AC power flow equation to attain the global optimality of ACOPF [2-25]. The majority of convex relaxation models are based on the second-order cone programming (SOCP) and semidefinite programming (SDP). These convex formulations have gained much attractiveness, as they can: 1) guarantee the global optimality, 2) certify the infeasibility, and 3) offer a lower bound for the optimization [2]. Particularly, for radial networks, SOCP and SDP relaxation techniques are proved equivalent, while the SOCP is more efficient in terms of convergence speed and it is thus suitable for radial networks [3-6].

In general, there are two types of SOCP formulations to model the power flow equation. One is the bus injection model (BIM) and the other is the branch flow model (BFM). In the BIM, the optimization variables are nodal variables including voltages, currents and power injections [7-12]. In the BFM, the corresponding optimization variables are currents and powers of the branches [3-6, 13-15].

It has been presented that the BIM and BFM are equivalent in [4, 16], especially for the equivalence between the relaxed BIM and BFM. However, certain questions remain unsolved:

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T. Ding and R. Lu are with the State Key Laboratory of Electrical Insulation and Power Equipment, Department of Electrical Engineering, Xi'an Jiaotong University, Xi'an, Shaanxi, 710049, China (tding15@mail.xjtu.edu.cn);

Y. Yang and F. Blaabjerg are with the Department of Energy Technology, Aalborg University, Aalborg DK-9220, Denmark.

1) Are the BIM and BFM always equivalent?
2) What is the exact relationship between the BIM and BFM?
3) Are the two models still equivalent after relaxation?

In this paper, it has been found that the BIM and BFM are equivalent under the normal and relaxation if and only if the network is connected and there is no zero-impedance branch. Notably, the zero-impedance problem has widely been studied in power systems [26-29] and the islanding problem may exist in faulty or emergent conditions. Moreover, an exact bijective function is given and proved. It is interesting to find that only one injection $\Gamma_{BIM}\rightarrow\Gamma_{BFM}$ exists for the case where the network is disconnected; and only one injection $\Gamma_{BFM}\rightarrow\Gamma_{BIM}$ exists for the case where there is zero impedance. For the above, the surjection does not exist.

The rest of this paper is organized as follows. Section II presents the explicit derivation of the BIM and BFM. Then, a bijection and the corresponding condition from the BIM to BFM is given in Section III. Numerical results are provided in Section IV and conclusions are drawn in Section V.

II. Bus Injection and Branch Flow Models

A. General AC power flow model

For a power grid, the general power flow is described as

$$\begin{cases} U_0 = U_0^{ref}, \quad \theta_0 = \theta_0^{ref} \\ p_i^c = U_i \sum_{j=1}^{n} U_j \left(G_{ij}\cos\theta_{ij} + B_{ij}\sin\theta_{ij} \right), \quad i \in \mathbb{B}, ij \in \mathbb{L} \\ q_i^c = U_i \sum_{j=1}^{n} U_j \left(G_{ij}\sin\theta_{ij} - B_{ij}\cos\theta_{ij} \right) \end{cases} \quad (1)$$

where U_i and θ_i are the voltage magnitude and angle of bus *i*; G_{ij} and B_{ij} are real and imaginary parts of the admittance matrix; p_i^c and q_i^c are controllable active and reactive power; $\mathbb{B}$ and $\mathbb{L}$ are the sets of branches and buses, respectively; *n* is the number of buses; $\theta_{ij}=\theta_i-\theta_j$ is the angle difference of the branch with the *from* bus *i* and *to* bus *j*; U_0^{ref} and θ_0^{ref} are voltage magnitude and angle of the reference bus.

Let r_{ij} and x_{ij} be the resistance and reactance of branch *ij*. According to the definition of admittance matrix, G_{ij}, B_{ij}, r_{ij} and x_{ij} can be correlated as

$$G_{ij} = -\frac{r_{ij}}{r_{ij}^2 + x_{ij}^2}, \qquad B_{ij} = \frac{x_{ij}}{r_{ij}^2 + x_{ij}^2}, \quad G_{ij} = G_{ji}, \quad B_{ij} = B_{ji} \quad (2)$$

which helps to derive the following:

$$\begin{cases} x_{ij}B_{ij} - r_{ij}G_{ij} = \dfrac{x_{ij}^2}{x_{ij}^2 + r_{ij}^2} - \dfrac{-r_{ij}^2}{x_{ij}^2 + r_{ij}^2} = 1 \\ r_{ij}B_{ij} + x_{ij}G_{ij} = \dfrac{r_{ij}x_{ij}}{x_{ij}^2 + r_{ij}^2} + \dfrac{-x_{ij}r_{ij}}{x_{ij}^2 + r_{ij}^2} = 0 \end{cases} \tag{3}$$

For convenience, the self-admittance can be expressed as $G_{s,i} - \sum_{j=1, j\neq i}^{n} G_{ij} = G_{ii}$ and $B_{s,i} - \sum_{j=1, j\neq i}^{n} B_{ij} = B_{ii}$, where $G_{s,i}$ and $B_{s,i}$ are shunt admittance. Then, (1) can be reformulated as

$$\begin{cases} U_0 = U_0^{ref}, \quad \theta_0 = \theta_0^{ref} \\ -G_{s,i}U_i^2 + p_i^c = U_i \sum_{j=1, j\neq i}^{n} U_j (G_{ij}\cos\theta_{ij} + B_{ij}\sin\theta_{ij}) \\ B_{s,i}U_i^2 + q_i^c = U_i \sum_{j=1, j\neq i}^{n} U_j (G_{ij}\sin\theta_{ij} - B_{ij}\cos\theta_{ij}) \end{cases}, \ i \in \mathbb{B}, ij \in \mathbb{L} \tag{4}$$

For a radial network, the voltage angel of the BIM and BFM can be eliminated since there is no circle and the angles can be uniquely determined by the voltage magnitudes and the branch flows. The transformations can be derived as follows.

B. Bus Injection Model

A transformation is given as

$$\begin{cases} u_i = U_i^2 \\ T_{ij} = U_i U_j \sin\left(\theta_{ij}\right) \\ R_{ij} = U_i U_j \cos\left(\theta_{ij}\right) \end{cases}, \quad ij \in \mathbb{L}, \quad i, j \in \mathbb{B} \tag{5}$$

which utilizes the squared voltage magnitudes, voltage inner and cross products of the connected buses for each line. Taking the square for T_{ij} and R_{ij} will eliminate the voltage angle. This gives the BIM formulation as

$$-G_{s,i}u_i + p_i^c = -u_i \sum_{ij\in\{\mathbb{L}\}} G_{ij} + \sum_{ij\in\{\mathbb{L}\}} \left(G_{ij}R_{ij} + B_{ij}T_{ij}\right), \ i \in \mathbb{B} \tag{6a}$$

$$B_{s,i}u_i + q_i^c = u_i \sum_{ij\in\{\mathbb{L}\}} B_{ij} + \sum_{ij\in\{\mathbb{L}\}} \left(G_{ij}T_{ij} - B_{ij}R_{ij}\right), \ i \in \mathbb{B} \tag{6b}$$

$$R_{ij}^2 + T_{ij}^2 = u_i u_j, \ ij \in \mathbb{L} \tag{6c}$$

$$R_{ij} = R_{ji}, T_{ij} = -T_{ji}, \ ij \in \mathbb{L} \tag{6d}$$

$$u_0 = \left(U_0^{ref}\right)^2 \tag{6e}$$

which is defined as the feasible region of the BIM being Γ_{BIM}. In addition, the variables in the BIM should include (R_{ij}, T_{ij}) for each line ij, and (u_i, p_i^c, q_i^c) for each bus i.

C. Branch Flow Model

The BFM formulation employs the squared voltage and current magnitudes to eliminate the voltage angles, following

$$\begin{cases} v_i = U_i^2 \\ l_{ij} = I_{ij}^2 \end{cases}, \ ij \in \mathbb{L}, i, j \in \mathbb{B} \tag{7}$$

Subsequently, the BFM formulation can be derived from (1) as

$$-G_{s,i}v_i + P_i^c = \sum_{k:i\to k} P_{ik} - \sum_{j:j\to i} \left(P_{ji} - r_{ji}l_{ji}\right), i \in \mathbb{B} \tag{8a}$$

$$B_{s,i}v_i + Q_i^c = \sum_{k:i\to k} Q_{ik} - \sum_{j:j\to i} \left(Q_{ji} - x_{ji}l_{ji}\right), i \in \mathbb{B} \tag{8b}$$

$$v_j = v_i - 2\left(r_{ij}P_{ij} + x_{ij}Q_{ij}\right) + \left(r_{ij}^2 + x_{ij}^2\right)l_{ij}, ij \in \mathbb{L} \tag{8c}$$

$$l_{ij}v_i = P_{ij}^2 + Q_{ij}^2, ij \in \mathbb{L} \tag{8d}$$

$$v_0 = \left(U_0^{ref}\right)^2 \tag{8e}$$

where P_i^c and Q_i^c are the controllable active and reactive powers in the BFM. Here, the feasible region of the BFM in (8) is defined as Γ_{BFM}. Moreover, the variables in the BFM should include (P_{ij}, Q_{ij}, l_{ij}) for the line ij, and (v_i, P_i^c, Q_i^c) for each bus i. It should be noted that l_{ij} for each line is just as an intermediate variable that can be uniquely determined by P_{ij}, Q_{ij} and v_i. Thus, l_{ij} is not considered in the computing variables.

Consequently, it can be found that the number of the computing variables for the BIM and BFM is equal. Moreover, by the transformations in (5) and (7), only the voltage magnitudes are preserved, while the voltage angles are eliminated. For radial systems, since the number of angle differences equals to the number of unknown voltage angles, the voltage angles can be uniquely determined and therefore this angle elimination can be exactly recovered once the branch flows and voltage magnitudes are determined [3-6].

III. Equivalence of BIM and BFM by a Linear Bijection

A. Proof on the Equivalence of the BIM and BFM

As shown in Fig. 1, proving the equivalence of the BIM and BFM requires finding a bijective function between them, such that for any feasible point in one set is uniquely paired with one feasible point of the other set [13, 16].

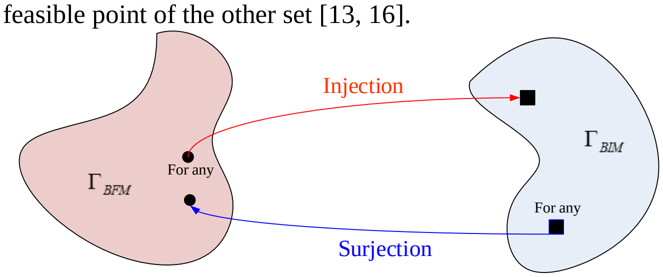

Fig. 1. Bijective function between Γ_{BIM} and Γ_{BFM}.

<u>Lemma 1 [30]:</u> Assume $\mathcal{F}$: $\mathcal{V}\rightarrow\mathcal{W}$ is an injection and $\mathcal{F}$ is a bijection if and only if $\mathcal{F}$ is invertible.

According to Lemma 1, we can define an injection $\mathcal{F}$ between the BIM and BFM as (9). In the following, we will prove that $\mathcal{F}$ is a bijection between Γ_{BFM} and Γ_{BIM}, i.e., $\mathcal{F}$ is an invertible injection from Γ_{BFM} to Γ_{BIM}.

$$\mathcal{F}: \begin{cases} v_i = u_i & i \in \mathbb{B} \\ P_{ij} = -u_i G_{ij} + G_{ij}R_{ij} + B_{ij}T_{ij} & ij \in \mathbb{L} \\ Q_{ij} = u_i B_{ij} + G_{ij}T_{ij} - B_{ij}R_{ij} & ij \in \mathbb{L} \\ P_i^c = p_i^c, \quad P_i^c = q_i^c & i \in \mathbb{B} \end{cases} \tag{9}$$

(i) Prove $\mathcal{F}$ is a $\Gamma_{BIM}\rightarrow\Gamma_{BFM}$ linear injection

At first, it can be found that $\mathcal{F}$ is a linear injection. Now, we want to show that for any given feasible solution $\left(\overline{u_i}, \overline{u_j}, \overline{R_{ij}}, \overline{T_{ij}}, \overline{p_i^c}, \overline{q_i^c}\right)\big|_{\forall ij\in\mathbb{L}} \in \Gamma_{BIM}$, the image solution $\left(\overline{v_i}, \overline{v_j}, \overline{P_{ij}}, \overline{Q_{ij}}, \overline{P_i^c}, \overline{Q_i^c}\right)\big|_{\forall ij\in\mathbb{L}}$ obtained by (9) belongs to Γ_{BIM}.

$$\left(\overline{v_i},\overline{v_j},\overline{P_{ij}},\overline{Q_{ij}},\overline{P_i^c},\overline{Q_i^c}\right):\begin{cases}\overline{v_i}=\overline{u_i}\\ \overline{P_{ij}}=-\overline{u_i}G_{ij}+G_{ij}\overline{R_{ij}}+B_{ij}\overline{T_{ij}}\\ \overline{Q_{ij}}=\overline{u_i}B_{ij}+G_{ij}\overline{T_{ij}}-B_{ij}\overline{R_{ij}}\\ \overline{P_i^c}=\overline{p_i^c},\quad \overline{Q_i^c}=\overline{q_i^c}\end{cases} \tag{10}$$

Proof: As $\left(\overline{u_i},\overline{u_j},\overline{R_{ij}},\overline{T_{ij}},\overline{p_i^c},\overline{q_i^c}\right)\Big|_{\forall ij\in\mathbb{L}}\in\Gamma_{BIM}$, it satisfies

$$-G_{s,i}\overline{u_i}+\overline{p_i^c}=-\overline{u_i}\sum_{ij\in\{\mathbb{L}\}}G_{ij}+\sum_{ij\in\{\mathbb{L}\}}\left(G_{ij}\overline{R_{ij}}+B_{ij}\overline{T_{ij}}\right),\ i\in\mathbb{B} \tag{11a}$$

$$B_{s,i}\overline{u_i}+\overline{q_i^c}=\overline{u_i}\sum_{ij\in\{\mathbb{L}\}}B_{ij}+\sum_{ij\in\{\mathbb{L}\}}\left(G_{ij}\overline{T_{ij}}-B_{ij}\overline{R_{ij}}\right),\ i\in\mathbb{B} \tag{11b}$$

$$\overline{R}_{ij}^2+\overline{T}_{ij}^2=\overline{u_i}\,\overline{u_j}\ ,ij\in\mathbb{L} \tag{11c}$$

$$\overline{R_{ij}}=\overline{R_{ji}}\ ,\ \overline{T_{ij}}=-\overline{T_{ji}}\ ,ij\in\mathbb{L} \tag{11d}$$

$$\overline{u_0}=\left(U_0^{ref}\right)^2 \tag{11e}$$

Additionally, the transmission losses can be formulated as

$$\begin{aligned}P_{loss}&=r_{ij}\overline{l_{ij}}=\overline{P_{ij}}+\overline{P_{ji}}\\&\overset{(10)}{=}\left(-\overline{u_i}G_{ij}+G_{ij}\overline{R_{ij}}+B_{ij}\overline{T_{ij}}\right)+\left(-\overline{u_i}G_{ji}+G_{ji}\overline{R_{ji}}+B_{ij}\overline{T_{ji}}\right)\\&=\left(-\overline{u_i}G_{ij}+G_{ij}\overline{R_{ij}}+B_{ij}\overline{T_{ij}}\right)+\left(-\overline{u_j}G_{ij}+G_{ij}\overline{R_{ij}}-B_{ij}\overline{T_{ij}}\right)\\&=G_{ij}\left(2\overline{R_{ij}}-\overline{u_i}-\overline{u_j}\right)\end{aligned} \tag{12a}$$

$$\begin{aligned}Q_{loss}&=x_{ij}\overline{l_{ij}}=\overline{Q_{ij}}+\overline{Q_{ji}}\\&\overset{(10)}{=}\left(\overline{u_i}B_{ij}+G_{ij}\overline{T_{ij}}-B_{ij}\overline{R_{ij}}\right)+\left(\overline{u_j}B_{ji}+G_{ji}\overline{T_{ji}}-B_{ji}\overline{R_{ji}}\right)\\&=\left(\overline{u_i}B_{ij}+G_{ij}\overline{T_{ij}}-B_{ij}\overline{R_{ij}}\right)+\left(\overline{u_j}B_{ij}-G_{ij}\overline{T_{ij}}-B_{ij}\overline{R_{ij}}\right)\\&=B_{ij}\left(\overline{u_i}+\overline{u_j}-2\overline{R_{ij}}\right)\end{aligned} \tag{12b}$$

Then, it derives,

$$\left(r_{ij}^2+x_{ij}^2\right)\overline{l_{ij}}=r_{ij}P_{loss}+x_{ij}Q_{loss}=\overline{u_i}+\overline{u_j}-2\overline{R_{ij}} \tag{13}$$

① To show that $\left(\overline{v_i},\overline{v_j},\overline{P_{ij}},\overline{Q_{ij}},\overline{P_i^c},\overline{Q_i^c}\right)$ satisfies (8a)

For any $i\in\mathbb{B}$, substitute P_{ij} by the linear injection (10) into (8a) and its right-hand expression can be obtained by

$$\begin{aligned}&\sum_{k:i\to k}P_{ik}-\sum_{j:j\to i}\left(P_{ji}-r_{ji}l_{ji}\right)\overset{(10)}{=}\\&\sum_{k:i\to k}\left(-\overline{u_i}G_{ik}+G_{ik}\overline{R_{ik}}+B_{ik}\overline{T_{ik}}\right)-\sum_{j:j\to i}\left(-\overline{u_j}G_{ji}+G_{ji}\overline{R_{ji}}+B_{ji}\overline{T_{ji}}-r_{ji}\overline{l_{ji}}\right)\\&\overset{(12a)}{=}\sum_{k:i\to k}\left(-\overline{u_i}G_{ik}+G_{ik}\overline{R_{ik}}+B_{ik}\overline{T_{ik}}\right)-\\&\qquad\sum_{j:j\to i}\left(-\overline{u_j}G_{ij}+G_{ij}\overline{R_{ij}}-B_{ij}\overline{T_{ij}}-G_{ij}\left(2\overline{R_{ij}}-\overline{u_i}-\overline{u_j}\right)\right)\\&=\sum_{k:i\to k}\left(-\overline{u_i}G_{ik}+G_{ik}\overline{R_{ik}}+B_{ik}\overline{T_{ik}}\right)+\sum_{j:j\to i}\left(-G_{ij}\overline{u_i}-G_{ij}\overline{R_{ij}}+B_{ij}\overline{T_{ij}}\right)\\&=-\overline{u_i}\sum_{ij\in\{\mathbb{L}\}}G_{ij}+\sum_{ij\in\{\mathbb{L}\}}\left(G_{ij}\overline{R_{ij}}+B_{ij}\overline{T_{ij}}\right)\overset{(11a)}{=}-G_{s,i}\overline{u_i}+\overline{p_i^c}\overset{(10)}{=}-G_{s,i}\overline{v_i}+\overline{P_i^c}\end{aligned}$$

② To show that $\left(\overline{v_i},\overline{v_j},\overline{P_{ij}},\overline{Q_{ij}},\overline{P_i^c},\overline{Q_i^c}\right)$ satisfies (8b)

For any $i\in\mathbb{B}$, substitute Q_{ij} by the linear injection (10) into (8b) and its right-hand expression can be obtained by

$$\begin{aligned}&\sum_{k:i\to k}Q_{ik}-\sum_{j:j\to i}\left(Q_{ji}-x_{ji}l_{ji}\right)\overset{(10)}{=}\\&\sum_{k:i\to k}\left(\overline{u_i}B_{ik}+G_{ik}\overline{T_{ik}}-B_{ik}\overline{R_{ik}}\right)-\sum_{j:j\to i}\left(\overline{u_j}B_{ji}+G_{ji}\overline{T_{ji}}-B_{ji}\overline{R_{ji}}-x_{ji}\overline{l_{ji}}\right)\\&\overset{(12b)}{=}\sum_{k:i\to k}\left(\overline{u_i}B_{ik}+G_{ik}\overline{T_{ik}}-B_{ik}\overline{R_{ik}}\right)-\\&\qquad\sum_{j:j\to i}\left(\overline{u_j}B_{ij}-G_{ji}\overline{T_{ij}}-B_{ij}\overline{R_{ij}}-B_{ij}\left(\overline{u_i}+\overline{u_j}-2\overline{R_{ij}}\right)\right)\\&=\sum_{k:i\to k}\left(\overline{u_i}B_{ik}+G_{ik}\overline{T_{ik}}-B_{ik}\overline{R_{ik}}\right)+\sum_{j:j\to i}\left(B_{ij}\overline{u_i}+G_{ij}\overline{T_{ij}}-B_{ij}\overline{R_{ij}}\right)\\&=\overline{u_i}\sum_{ij\in\{\mathbb{L}\}}B_{ij}+\sum_{ij\in\{\mathbb{L}\}}\left(G_{ij}\overline{T_{ij}}-B_{ij}\overline{R_{ij}}\right)\overset{(11b)}{=}B_{s,i}\overline{u_i}+\overline{q_i^c}\overset{(10)}{=}B_{s,i}\overline{v_i}+\overline{Q_i^c}\end{aligned}$$

③ To show that $\left(\overline{v_i},\overline{v_j},\overline{P_{ij}},\overline{Q_{ij}},\overline{P_i^c},\overline{Q_i^c}\right)$ satisfies (8c)

For any $ij\in\mathbb{L}$, substitute v_i, v_j, P_{ij} and Q_{ij} by the linear injection (10) into (8c), and it gives

$$\begin{aligned}&v_j-\left(v_i-2\left(r_{ij}P_{ij}+x_{ij}Q_{ij}\right)+\left(r_{ij}^2+x_{ij}^2\right)l_{ij}\right)\\&\overset{(10)}{=}\overline{u_j}-\left(\overline{u_i}-2\left(r_{ij}\overline{P_{ij}}+x_{ij}\overline{Q_{ij}}\right)+\left(r_{ij}^2+x_{ij}^2\right)\overline{l_{ij}}\right)\\&=\overline{u_j}-\overline{u_i}+2\left(r_{ij}\overline{P_{ij}}+x_{ij}\overline{Q_{ij}}\right)-\left(r_{ij}^2+x_{ij}^2\right)\overline{l_{ij}}\\&\overset{(13)}{=}\overline{u_j}-\overline{u_i}+2r_{ij}\left(-\overline{u_i}G_{ij}+G_{ij}\overline{R_{ij}}+B_{ij}\overline{T_{ij}}\right)+\\&\qquad 2x_{ij}\left(\overline{u_i}B_{ij}+G_{ij}\overline{T_{ij}}-B_{ij}\overline{R_{ij}}\right)-\left(\overline{u_i}+\overline{u_j}-2\overline{R_{ij}}\right)\\&=-2\overline{u_i}+2\left(x_{ij}B_{ij}-r_{ij}G_{ij}\right)\overline{u_i}+\\&\qquad 2\left(x_{ij}B_{ij}-r_{ij}G_{ij}-1\right)\overline{R_{ij}}+2\left(r_{ij}B_{ij}+x_{ij}G_{ij}\right)\overline{T_{ij}}\end{aligned} \tag{14}$$

According to (3), (14) can be simplified as

$$\overline{u_j}-\left(\overline{u_i}-2\left(r_{ij}\overline{P_{ij}}+x_{ij}\overline{Q_{ij}}\right)+\left(r_{ij}^2+x_{ij}^2\right)\overline{l_{ij}}\right)=0 \tag{15}$$

④ To show that $\left(\overline{v_i},\overline{v_j},\overline{P_{ij}},\overline{Q_{ij}},\overline{P_i^c},\overline{Q_i^c}\right)$ satisfies (8d)

For any $ij\in\mathbb{L}$, substitute v_i, v_j, P_{ij} and Q_{ij} by the linear injection (10) into (8d), and it derives

$$\begin{aligned}&\overline{P}_{ij}^2+\overline{Q}_{ij}^2-\overline{l_{ij}}\,\overline{v_i}\overset{(10)}{=}\left(-\overline{u_i}G_{ij}+G_{ij}\overline{R_{ij}}+B_{ij}\overline{T_{ij}}\right)^2\\&\qquad-\left(\overline{u_i}B_{ij}+G_{ij}\overline{T_{ij}}-B_{ij}\overline{R_{ij}}\right)^2-\frac{\overline{u_i}+\overline{u_j}-2\overline{R_{ij}}}{r_{ij}^2+x_{ij}^2}\overline{v_i}\\&=\left(G_{ij}^2+B_{ij}^2\right)\left(\overline{u_i^2}+\overline{R_{ij}^2}+\overline{T_{ij}^2}-2\overline{u_i}\,\overline{R_{ij}}\right)-\frac{\overline{u_i}+\overline{u_j}-2\overline{R_{ij}}}{r_{ij}^2+x_{ij}^2}\overline{u_i}\\&=\frac{\overline{u_i^2}+\overline{R_{ij}^2}+\overline{T_{ij}^2}-2\overline{u_i}\,\overline{R_{ij}}-\overline{u_i^2}-\overline{u_j}\,\overline{u_i}+2\overline{R_{ij}}\,\overline{u_i}}{r_{ij}^2+x_{ij}^2}=\frac{\overline{R_{ij}^2}+\overline{T_{ij}^2}-\overline{u_j}\,\overline{u_i}}{r_{ij}^2+x_{ij}^2}\end{aligned}$$

According to (11d), it holds that

$$\overline{P}_{ij}^2+\overline{Q}_{ij}^2-\overline{l_{ij}}\,\overline{v_i}=0\Leftrightarrow\overline{P}_{ij}^2+\overline{Q}_{ij}^2=\overline{l_{ij}}\,\overline{v_i} \tag{16}$$

⑤ To show that $\left(\overline{v_i},\overline{v_j},\overline{P_{ij}},\overline{Q_{ij}},\overline{P_i^c},\overline{Q_i^c}\right)$ satisfies (6-e)

According to (8e), (11e) and the injection (10), we have

$$\begin{cases}\overline{u_i}=\left(U_i^{sch}\right)^2\\ \overline{v_i}=\overline{u_i}\end{cases}\Rightarrow\overline{v_i}=\left(U_i^{sch}\right)^2 \tag{17}$$

With the above proof (①-⑤), it is shown that for any given feasible solution of Γ_{BIM}, $\left(\overline{u_i},\overline{u_j},\overline{R_{ij}},\overline{T_{ij}},\overline{p_i^c},\overline{q_i^c}\right)\Big|_{\forall ij\in\mathbb{L}}\in\Gamma_{BIM}$, the corresponding image solution $\left(\overline{v_i},\overline{v_j},\overline{P_{ij}},\overline{Q_{ij}},\overline{P_i^c},\overline{Q_i^c}\right)\Big|_{\forall ij\in\mathbb{L}}$ by the linear injection $\mathscr{F}$ belongs to Γ_{BFM}. Therefore, $\mathscr{F}$ is a linear injection from Γ_{BIM} to Γ_{BFM}.

(ii) Prove $\mathscr{F}$ is an invertible injection

The linear injection in (9) $\mathscr{F}$: $\Gamma_{BIM}\rightarrow\Gamma_{BFM}$, can be written as

$$\begin{pmatrix} P_i^c \\ Q_i^c \\ v_i \\ P_{ij} \\ Q_{ij} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -G_{ij} & G_{ij} & B_{ij} \\ 0 & 0 & B_{ij} & -B_{ij} & G_{ij} \end{pmatrix} \begin{pmatrix} p_i^c \\ q_i^c \\ u_i \\ R_{ij} \\ T_{ij} \end{pmatrix} \tag{18}$$

The determinant of the coefficient matrix can be calculated by

$$\det\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -G_{ij} & G_{ij} & B_{ij} \\ 0 & 0 & B_{ij} & -B_{ij} & G_{ij} \end{pmatrix} = \det\left(G_{ij}G_{ij}+B_{ij}B_{ij}\right) \tag{19}$$

<u>Theorem 1:</u> If the network is connected and there is no zero-impedance, the injection (9) is invertible and the linear bijection $\Gamma_{BIM}\rightarrow\Gamma_{BFM}$ exists.

If **Theorem 1** holds, the impedance of each branch is infinite and non-zero, so the coefficient matrix of (19) is invertible, whose inverse matrix can be formulated as

$$\begin{pmatrix} p_i^c \\ q_i^c \\ u_i \\ R_{ij} \\ T_{ij} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -r_{ij} & -x_{ij} \\ 0 & 0 & 0 & x_{ij} & -r_{ij} \end{pmatrix} \begin{pmatrix} P_i^c \\ Q_i^c \\ v_i \\ P_{ij} \\ Q_{ij} \end{pmatrix} \tag{20}$$

<u>Case1:</u> If the branch ij is opened and the network is not connected, the values of r_{ij} and x_{ij} are infinite (e.g., G_{ij} and B_{ij} are zero). At this time, P_{ij} and Q_{ij} should be strictly zero, no matter what $\left(u_i,u_j,R_{ij},T_{ij},p_i^c,q_i^c\right)$ is. As a result, we could only find a non-invertible injection $\Gamma_{BIM}\rightarrow\Gamma_{BFM}$ by

$$\begin{pmatrix} P_i^c \\ Q_i^c \\ v_i \\ P_{ij} \\ Q_{ij} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_i^c \\ q_i^c \\ u_i \\ R_{ij} \\ T_{ij} \end{pmatrix} \tag{21a}$$

<u>Case 2:</u> If the impedance of the branch ij is zero, the values of r_{ij} and x_{ij} are zero (e.q., G_{ij} and B_{ij} are infinite). At this time, u_i and u_j are strictly equal, no matter what $\left(v_i,v_j,P_{ij},Q_{ij},P_i^c,Q_i^c\right)$ is. This gives R_{ij}=v_i and T_{ij}=0. As a result, we could only find a non-invertible injection $\Gamma_{BFM}\rightarrow\Gamma_{BIM}$ by

$$\begin{pmatrix} p_i^c \\ q_i^c \\ u_i \\ R_{ij} \\ T_{ij} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} P_i^c \\ Q_i^c \\ v_i \\ P_{ij} \\ Q_{ij} \end{pmatrix} \tag{21b}$$

For the two cases, only the injection exists but the bijection does not exist. Take **Case 1** for illustration. If the network is not connected, the injection $\Gamma_{BIM}\rightarrow\Gamma_{BFM}$ exists in (21a) and it is unique, but there are many surjections $\Gamma_{BFM}\rightarrow\Gamma_{BIM}$. This suggests that the two formulations are not equivalent anymore.

B. Proof on Equivalence of Relaxed BIM and BFM by SOCP

The feasible region of the BIM and BFM is non-convex due to the quadratic equalities. To convexify the feasible regions, conic relaxation techniques were utilized for the BFM [3-6, 13-15] and BIM [7-12] to relax the quadratic equalities into inequalities, such that

$$\text{BIM:}\quad R_{ij}^2+T_{ij}^2=u_iu_j \xrightarrow{relax} R_{ij}^2+T_{ij}^2\le u_iu_j,\ ij\in\mathbb{L} \tag{22a}$$

$$\text{BFM:}\quad l_{ij}v_i=P_{ij}^2+Q_{ij}^2 \xrightarrow{relax} l_{ij}v_i\ge P_{ij}^2+Q_{ij}^2,\ ij\in\mathbb{L} \tag{22b}$$

Replacing (6c) by (22a) and (8d) by (22b), the relaxed feasible region of the BIM and BFM is denoted as $\overline{\Gamma}_{BIM}$ and $\overline{\Gamma}_{BFM}$, respectively. If **Theorem 1** holds, we will prove that the function $\mathscr{F}$ in (7) is also a bijection from $\overline{\Gamma}_{BIM}$ to $\overline{\Gamma}_{BFM}$.

Proof: Suppose $\forall\left(\overline{u_i},\overline{u_j},\overline{R_{ij}},\overline{T_{ij}},\overline{p_i^c},\overline{q_i^c}\right)\Big|_{\forall ij\in\mathbb{L}}\in\Gamma_{BIM}$ and its image solution is $\left(\overline{v_i},\overline{v_j},\overline{P_{ij}},\overline{Q_{ij}},\overline{P_i^c},\overline{Q_i^c}\right)\Big|_{\forall ij\in\mathbb{L}}$ obtained by (9). According to ①②③⑤, it suggests that (6a), (6b), (6c) and (6e) are still satisfied. As for the relaxed inequalities (22), if (22a) holds, we have $\overline{R_{ij}^2}+\overline{T_{ij}^2}\le\overline{u_i}\,\overline{u_j}$. According to ④, it gives

$$\overline{P}_{ij}^2+\overline{Q}_{ij}^2-\overline{l_{ij}}\,\overline{v_i}=\frac{\overline{R_{ij}^2}+\overline{T_{ij}^2}-\overline{u_j}\,\overline{u_i}}{r_{ij}^2+x_{ij}^2} \tag{23}$$

The constraint $\overline{R_{ij}^2}+\overline{T_{ij}^2}\le\overline{u_i}\,\overline{u_j}$ leads to $\overline{P}_{ij}^2+\overline{Q}_{ij}^2-\overline{l_{ij}}\,\overline{v_i}\le 0$ and the image solution satisfies (22b). Therefore, the image solution belongs to $\overline{\Gamma}_{BFM}$. $\mathscr{F}$ is a $\overline{\Gamma}_{BIM}\rightarrow\overline{\Gamma}_{BFM}$ linear injection. If **Theorem 1** holds, $\mathscr{F}$ is a $\overline{\Gamma}_{BIM}\rightarrow\overline{\Gamma}_{BFM}$ linear bijection and the two relaxed models are equivalent.

IV. NUMERICAL SIMULATIONS

An example two-bus system with three sets of branch impedance is depicted in Fig. 2, where Bus 1 is the reference bus with the voltage being $1\angle 0°$ and Bus 2 is connected to a microgrid that can be operated in an island. The controllable power (p_1^c, q_1^c) is equal to the branch flow (P_{12}, Q_{12}) since the shunt impedance is ignored. Three cases are considered (C1, C2, and C3). C1 is a normal case; C2 contains the zero impedance; and C3 is disconnected since the impedance is infinite. The power flow of the three cases is shown in Table I.

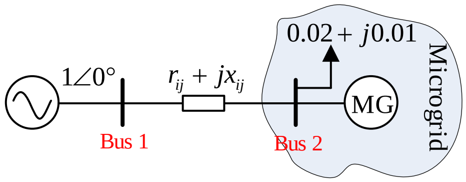

Fig. 2. A simple two-bus example.

For C1, $\left(p_1^c, q_1^c, u_1, R_{12}, T_{12}\right)=\left(0.0205, 0.0111, 1, 0.9573, 0.0300\right)$ and $\left(P_1^c, Q_1^c, v_1, P_{12}, Q_{12}\right)=\left(0.0205, 0.0111, 0.0205, 0.0111\right)$. The linear bijection between Γ_{BIM} and Γ_{BFM} in (18) and (20) is satisfied.

For C2, the impedance of the branch is zero and the shunt is neglected, so the voltage of Bus 2 is equal to Bus 1, i.e., $R_{12} = 1$, $T_{12} = 0$ and the branch flow (P_{12}, Q_{12}) should be equal to the power injection of Bus 2, yielding ($P_{12}=p_1^c$, $Q_{12}=q_1^c$). At this time, the injection $\Gamma_{BFM}\rightarrow\Gamma_{BIM}$ is (21b), but the surjection is not unique, which can be expressed as (24). Choosing different α and β yields different image solutions.

$$\begin{cases} P_{12}=p_2^c+\alpha u_2-\alpha R_{12}+\beta T_{12} \\ Q_{12}=q_2^c+\alpha u_2-\alpha R_{12}+\beta T_{12} \end{cases}, \quad \forall \alpha, \beta \tag{24}$$

For C3, the branch is opened and there is no power flow on the branch, such that ($P_{12} = 0$, $Q_{12} = 0$). Meanwhile, Bus 2 is in the microgrid and the voltage magnitude is determined by the island (i.e., any possible value). Thus, the injection $\Gamma_{BIM}\rightarrow\Gamma_{BFM}$ is (21a), but the surjection is not unique, which can be expressed as (25). Choosing different voltage values of Bus 2 yields different (R_{ij}, T_{ij}) and therefore different image solutions.

$$\begin{cases} R_{12}=\alpha v_2+\beta P_{12}+\gamma Q_{12} \\ T_{12}=\sqrt{1-\alpha^2}\, v_2+\beta P_{12}+\gamma Q_{12} \end{cases}, \quad \forall \alpha, \beta, \gamma \tag{25}$$

Table I. Power flow solutions under different parameters

	Parameters		Power Flow Solutions (p.u.)					
	r_{12}	x_{12}	U_2	θ_2	P_{12}	Q_{12}	R_{12}	T_{12}
C1	1	2	0.9577	-1.7950	0.0205	0.0111	0.9573	0.0300
C2	0	0	1	0	0.02	0.01	1	0
C3	inf	inf	--	--	0	0	--	--

V. Conclusions

This paper aims to investigate the condition on the equivalence between the BIM and BFM in radial systems. It has been demonstrated that if the network is connected and there is no zero-impedance, the bijection exists and the two models are strictly equivalent under both normal and relaxation. Moreover, it has also been indicated that if the condition does not hold, only the injection exists and the two formulations are not equivalent anymore.

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