Downsampling of DFT Precoded Signals for the AWGN Channel

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Outline

- **Motivation for downsampling/undersampling**
- Detection of downsampled signals
- Summary and discussion

Outline

- Compressed sensing (Candès, Romberg & Tao 2006, Donoho 2006)
  - Not always necessary to take N linear measurements to reconstruct a length N signal if the signal obeys certain structure.
  - Most focus on signals with sparse structure but other structure is also considered now (Davenport, Boufounos, Wakin & Baraniuk 2010, Donoho & Tanner 2010*b*).
- Can we use compressed sensing ideas to
  - reduce receiver ADC sampling rate?
  - form new error resistant constellation schemes?
- What is the cost?
  - Performance loss?
  - More complex detection algorithms?

Model

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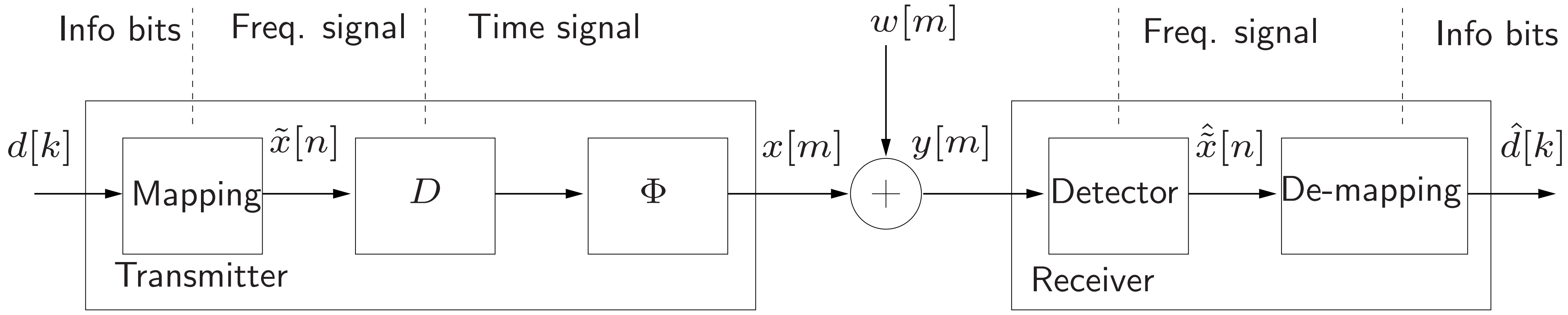

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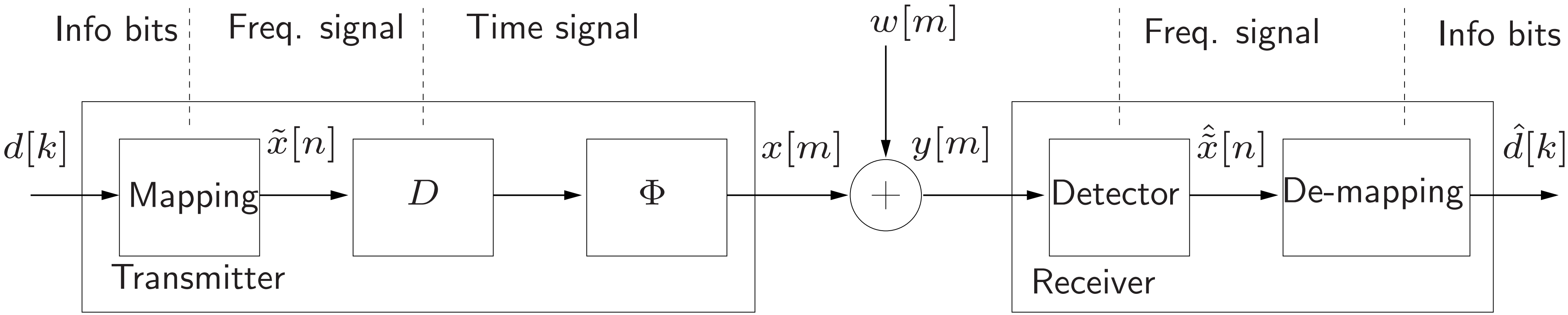

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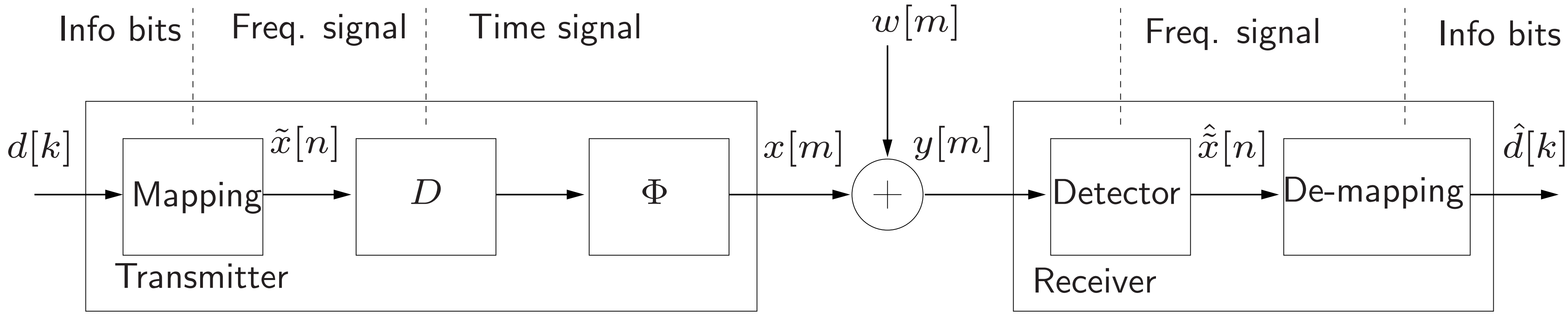

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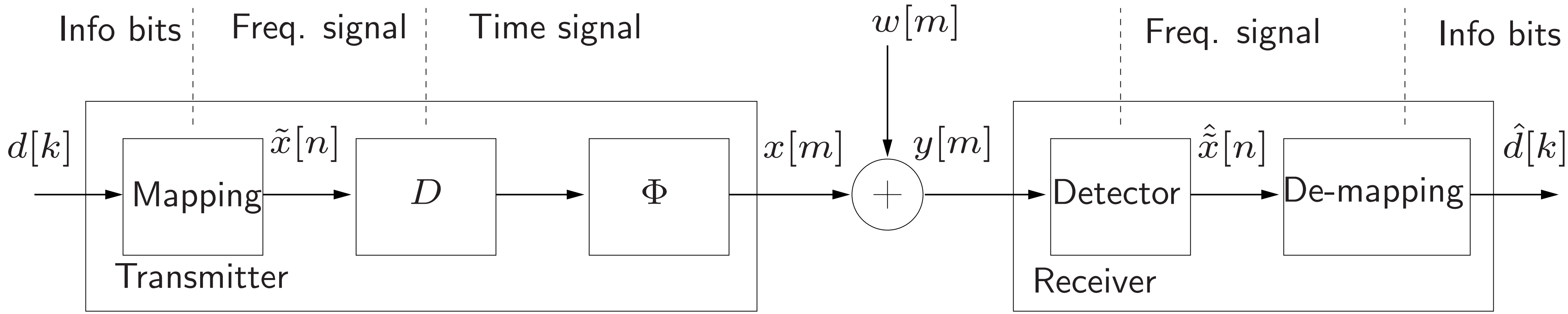

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- Information of each symbol is spread over the whole time domain.

Downsampling Matrix

- The time symbols are then *randomly punctured* by block processing of N to M time symbols with Φ.

- One realization is (each row of Φ contains exactly one element with a one):

$$\Phi = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & \cdots & & & 1 & 0 \end{bmatrix} . \quad (1)$$

- This gives the *downsampling factor*

$$\kappa = \frac{M}{N} . \quad (2)$$

Signal-to-noise Ratio

- The signal-to-noise-ratio (SNR) is given by

$$\frac{E_\mathrm{s}}{N_0} = \frac{E[\,|x[m]|^2\,]}{E[\,|w[m]|^2\,]} = \frac{1}{\sigma^2}\,. \tag{3}$$

- The SNR per information bit is given by

$$\frac{E_\mathrm{b}}{N_0} = \frac{E_\mathrm{s}}{N_0}\frac{1}{R} = \frac{\kappa}{\sigma^2 \log_2(L)}, \quad R = \frac{\log_2(L)}{\kappa}\,. \tag{4}$$

- The rate R both depends on the constellation size L as well as the downsampling factor, such that the energy per bit is reduced by the factor κ.

Detection Problem

- The maximum likelihood detection problem for $\Phi = I$ is

$$\hat{\tilde{x}}_{\mathrm{ML}} = \underset{\tilde{x} \in \mathcal{S}^N}{\operatorname{argmin}} \|D\tilde{x} - y\|_2 = \underset{\tilde{x} \in \mathcal{S}^N}{\operatorname{argmin}} \|\tilde{x} - \tilde{y}\|_2 = h(\tilde{y}) \tag{5}$$

- $\tilde{y} = D^H y$, $\tilde{x} = [\tilde{x}[n'], \cdots \tilde{x}[n' + N - 1]]^T$, and the other symbols are formed in a similar way due to the block processing.
- The maximum likelihood detector for the downsampled signal is

$$\hat{\tilde{x}}_{\mathrm{ML}} = \underset{\tilde{x} \in \mathcal{S}^N}{\operatorname{argmin}} \|\Phi D\tilde{x} - y\|_2 \tag{6}$$

- In general this is NP-hard (Verdú 1989).
- How to form relaxations and analyze these?

Structured Description

- Consider so-called s-*simple* descriptions (Donoho & Tanner 2010*a*).
- Vectors $z \in [0,1]^J$ with $J - s$ elements equal to exactly 0 or 1
- Let $A \in \mathbb{R}^{H \times J}$, $H \leq J$

$$b = Az\,. \tag{7}$$

- We can then reconstruct z with the convex feasibility problem

$$\hat{z} = \underset{Az=b,0\leq z_i\leq 1,i=1,\cdots,J}{\operatorname{argmin}} 0\,. \tag{8}$$

- Then $\hat{z} = z$ with high probability for certain settings of the triplet (s, H, J) and A (Donoho & Tanner 2010*a*).
- Note, not a sparse description.

Relation to Downsampling DFT Precoded Signals

- Consider 0-simple vectors (all elements are exactly 0 or 1)
- BPSK: $\Re(\tilde{x}) = 2z - \mathbb{1}$, $\Im(\tilde{x}) = 0, \quad \mathbb{1} = [1, 1, \ldots, 1]^T$
- 0-simple vectors have phase-transition at $\frac{H}{J} = \frac{1}{2}$ (succeeds with high probability at higher rate and fails at lower rate).
- For the noise-less case $w[m] = 0$ $(\sigma^2 = 0)$ we then have
  - BPSK phase-transition at $\kappa = \frac{M}{N} = \frac{1}{4}$
  - QPSK phase-transition at $\kappa = \frac{M}{N} = \frac{1}{2}$

Detection via Convex Relaxation

- Consider the following ML estimation of the model $b = Az + e$ with e *i.i.d.* Gaussian noise

$$\hat{z}_{\mathrm{ML}} = \underset{z_i \in \{0,1\}, i=1,\cdots,J}{\operatorname{argmin}} \|Az - b\|_2 \,. \tag{9}$$

- Convex relaxation

$$\hat{z}_{\mathrm{ML-CR}} = \underset{0 \le z_i \le 1, i=1,\cdots,J}{\operatorname{argmin}} \|Az - b\|_2 \,. \tag{10}$$

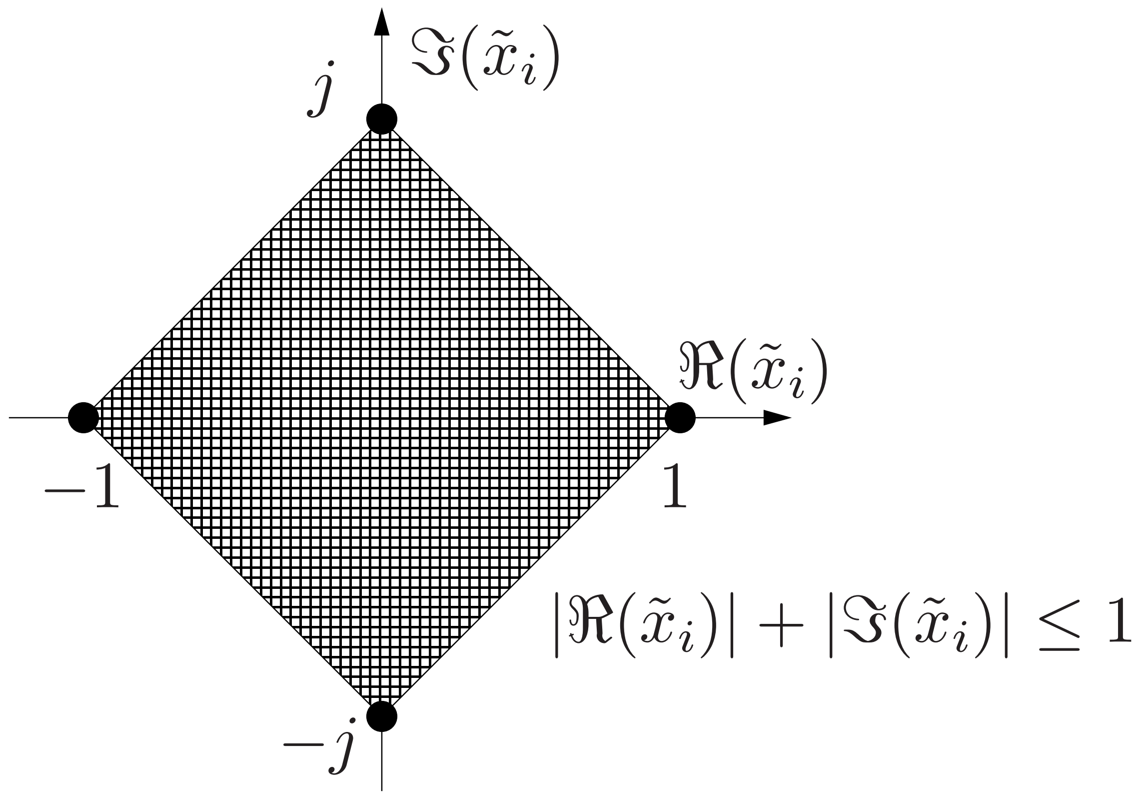
- BPSK detection is then

$$\hat{\tilde{x}}_{\mathrm{ML-CR}} = \underset{-1 \le \Re(\tilde{x}_i) \le 1,\, \Im(\tilde{x}_i)=0,\, i=1,\cdots,N}{\operatorname{argmin}} \|\Phi D \tilde{x} - y\|_2 \,. \tag{11}$$

Interpretation of the Convex Relaxation

- QPSK detection is then

$$\hat{\tilde{x}}_{\mathrm{ML-CR}} = \underset{|\Re(\tilde{x}_i)|+|\Im(\tilde{x}_i)|\leq 1, i=1,\cdots,N}{\operatorname{argmin}} \|\Phi D\tilde{x} - y\|_2 \,. \qquad (12)$$



Detection via Semidefinite Relaxation

- QPSK detection can be written as the quadratically constrained quadratic program

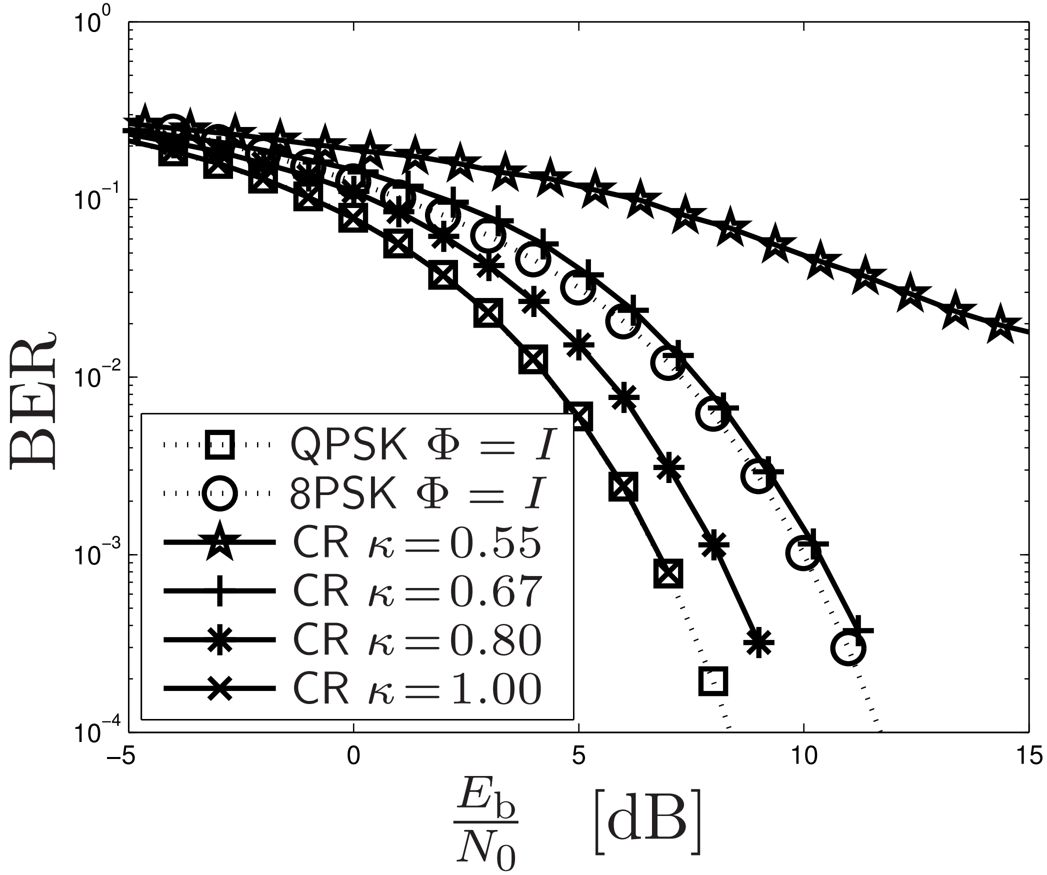
$$\begin{array}{ll} \text{minimize} & \|\Phi D\tilde{x} - y\|_2^2 \\ \text{subject to} & \Re(\tilde{x}_i)^2 + \Im(\tilde{x}_i)^2 = 1,\ \forall\, i = 1, \cdots, N \\ & \Re(\tilde{x}_i)\Im(\tilde{x}_i) = 0,\ \forall\, i = 1, \cdots, N\,. \end{array} \tag{13}$$

- We then form a standard semidefinite relaxation, see *e.g.*, (Luo, Ma, So, Ye & Zhang 2010), $C \in \mathbb{R}^{2N+1\times 2N+1}$, $X \in \mathbb{R}^{2N+1\times 2N+1}$

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}\,(CX) \\ \text{subject to} & X_{i,i} + X_{i+N,i+N} = 1,\ \forall\, i = 1, \cdots, N \\ & X_{i,i+N} = 0,\ \forall\, i = 1, \cdots, N \\ & X_{2N+1,2N+1} = 1 \\ & X \succeq 0 \end{array} \tag{14}$$

Simulations – BER – Convex Relaxation

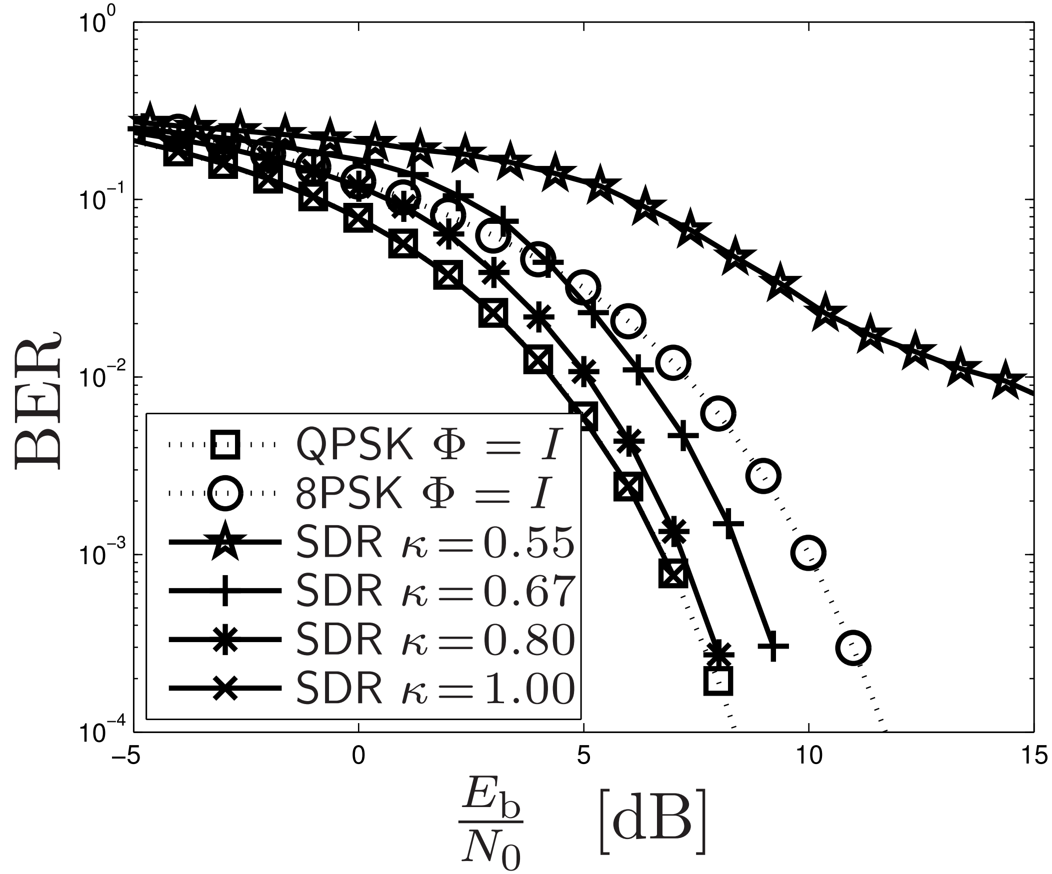
- Uncoded BER *i)* QPSK, 8PSK using $\Phi = I$ and optimal detection, *ii)* QPSK based CR detectors and downsampling factor κ.



- $\kappa = 0.67$ gives the same number of bits per symbol as 8PSK.

Simulations – BER – Semidefinite Relaxation

- Uncoded BER *i)* QPSK, 8PSK using $\Phi = I$ and optimal detection, *ii)* QPSK based SDR detectors and downsampling factor κ.



- $\mathrm{BER} \approx 10^{-3}$: improves 8PSK with $1.8\,\mathrm{dB}$ (suboptimal detection).

Undersampling

- Place Φ (or ΦD) at the receiver instead

$$y = \Phi(Dx + w) = \Phi Dx + \tilde{w}, \quad \tilde{w} = \Phi w\,. \tag{15}$$

- Then $E[\tilde{w}\tilde{w}^{\mathrm{H}}] = E[\Phi w w^{\mathrm{H}} \Phi^{\mathrm{H}}] = \sigma^2 I$, *i.e.*, the noise is still *i.i.d.* Gaussian.

- Goal: reduce sampling rate of the ADC.

- Using semidefinite relaxation:
  - Undersample $\kappa = 0.67$ with an SNR penalty of $3.5\,\mathrm{dB}$.
  - Seems as a high penalty. Only exploit in case of excessive SNR.

Summary and Discussion

- Showed how to use structured descriptions to describe downsampled/undersampled detection problems.
- It is possible to form a more error-resilient signal than standard 8PSK using a downsampled DFT precoded QPSK signal with semidefinite based detection.
- The resulting detection problem is more computationally demanding but still polynomial-time.
- Future work:
  - How can the setup be used with other constellation schemes?
  - How efficiently can the detection problem be solved?

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