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by

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THE LATTICE OF D-STRUCTURES

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Abstract. The set of d-structures on a topological space form a lattice and in fact a locale. There is a Galois connection between the lattice of subsets of the space and the lattice of d-structures. Variation of the d-structures induces change in the spaces of directed paths. Hence variation of d-structures and variation of the "forbidden area" may be considered together via for instance (co)homology and homotopy sequences.

1. Introduction

In directed topology, a topological space X is equipped with an extra structure, $\vec{P} \subset X^I$, the *d-structure* or the dipaths, a subset of the set X^I of all paths from the unit interval I satisfying certain properties 2.1. A given topological space will support many such directed structures, and clearly, inclusion of d-structures provides a partial order on the set of d-structures. In Thm. 3.8, we show, that the d-structures on a fixed space form a complete distributive lattice, in fact even a locale. A d-structure is closed, if the set of dipaths is closed as a subset of the set of all paths with the compact open topology. Such a closed structure, $\vec{P}$, has a complement $\neg\vec{P}$, which is also a d-structure. Hence, all paths may be written as a (possibly countably infinite) concatenation of dipaths from $\vec{P}$ and its complement, and moreover the intersection of $\vec{P}$ and $\neg\vec{P}$ contains only the constant paths.

In [5], a geometric model for concurrent computing is studied. The model is a directed space, which in many cases may be modelled as a product of directe graphs minus a "forbidden area". In section 4, we take the point of view, that the forbidden area is not removed from the space, but instead, no directed paths (except the constant ones) enter this area. This gives a correspondence between subsets of the space and directed structures: Given a subset F, the associated d-structure $\mu(F)$ is the maximal structure avoiding F. Given a d-structure, $\vec{P}$ the associated forbidden area $\nu(\vec{P})$ is the set of points which are only intersected by constant paths in $\vec{P}$. The pair (μ, ν) is a Galois connection, see Thm. 4.4. In Section 6, we study the n-cube with a product-d-structure. A dipath is required to increase in a subset of the coordinates. Such structures correspond to relaxing the order (letting time run backwards) in a subset of the processors running in parallel. The inclusions of path spaces corresponding to the sublattice structure

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may be studied via homology-sequences of pair or triples. The relative homology groups carry information about the effect of iteratively relaxing the orders.

2. Definitions - closed d-structures

The definitions here are not new. Most of them may be found in [8].

Definition 2.1. A *d*-space is a topological space X with a set of paths $\vec{P} \subset X^I$ such that

- $\vec{P}$ contains all constant paths.
- $\gamma, \mu \in \vec{P}$ implies $\gamma \star \mu \in \vec{P}$, where $\star$ is concatenation.
- If $\varphi : I \to I$ is monotone, $t \leq s \Rightarrow \varphi(t) \leq \phi(s)$, and $\gamma \in \vec{P}$, then $\gamma \circ \varphi \in \vec{P}$. I.e., $\vec{P}$ is closed under monotone reparametrization and subpath.

φ is a *di-reparametrization* - not necessarily surjective.
The d-space is *saturated* if whenever $\varphi : I \to I$ a monotone surjection and $\gamma \circ \varphi \in \vec{P}$, then $\gamma \in \vec{P}$.

A d-map or dimap $f : X \to Y$ is a continuous map, such that if $\alpha \in \vec{P}$ then $f \circ \alpha \in \vec{P}(Y)$.

The set of distinguished paths, $\vec{P}$ are called the *dipaths*. They are d-maps from the ordered interval $\vec{I}$ to X.

For $\gamma : I \to X$, we denote $\gamma(0)$ the source and $\gamma(1)$ the target of γ.
For subsets $A, B \in X$, let $\vec{P}(X, A, B)$ denote dipaths with source $\gamma(0) \in A \subseteq X$ and target $\gamma(1) \in B \subseteq X$.
$\vec{T}(X, \vec{P}, A, B)$ denotes the *trace space*, i.e., the dipaths up to di-reparametrization.

The category of d-spaces is denoted **d-Top**

Example 2.2. When $\vec{P} = X^I$, then $(X, \vec{P})$ is a d-space with *trivial d-structure.* If $\vec{P}$ is the constant maps, X has the *discrete d-structure.* Note that the d-maps *from* a space with discrete d-structure are the continuous maps. The d-maps *to* a space with trivial d-structure are the continuous maps.

Example 2.3. A subspace $Y \subset X$ of a d-space has an induced d-structure in the obvious way.
The product $X \times Y$ of two d-spaces has a product d-structure: $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ is a dipath if both components are.

Example 2.4. Let I be the unit interval. Then $\vec{I}$ denotes I with d-structure $\{\gamma : I \to I | t_1 \leq t_2 \Rightarrow \gamma(t_1) \leq \gamma(t_2)\}$. The other direction, $\overleftarrow{I}$ is I with d-structure $\{\gamma : I \to I | t_1 \leq t_2 \Rightarrow \gamma(t_1) \geq \gamma(t_2)\}$. The trivial d-structure is denoted $\overleftrightarrow{I}$ and the discrete structure is I^d.
These structures immediately give rise to different d-structures on the unit n-cube such as $\vec{I}^k \times \overleftarrow{I}^{n-k}$ and $\vec{I}^k \times \overleftrightarrow{I}^{n-k}$.
This example is studied in detail in Section 6

3. Lattice structure

The set of all d-structures on a topological space form a lattice under the subset order. That lattice is complete, distributive and in fact a Heyting algebra. We give a concrete description of the pseudo-complement.This is in general not a complement as we show in Ex. 3.11. A d-structure $\vec{P}$, which is closed as a subset of the path space, has a complement under countable concatenation of dipaths. Hence, in that case, any path $\gamma \in X^I$ may be written as a countable concatenation of dipaths in $\vec{P}$ and $\neg\vec{P}$, where $\neg\vec{P} \cap \vec{P}$ is the set of constant paths.

Definition 3.1. Let X be a topological space and let $\mathbb{P}(X)$ be the set of d-structures on X. The lattice structure on $\mathbb{P}(X)$ is as follows: Let $\vec{P}, \vec{Q} \in \mathbb{P}(X)$ then

- $\vec{P} \leq \vec{Q}$ if the identity map $id : X \to X$ is a d-map from $(X, \vec{P})$ to $(X, \vec{Q})$.
- Meet is defined by $\vec{P} \wedge \vec{Q} = \vec{P} \cap \vec{Q}$
- Join is $\vec{P} \vee \vec{Q} = (\vec{P} \cup \vec{Q})^*$, where $*$ is closure under subpath, finite concatenation and di-reparametrization.

Remark 3.2. Since $\vec{P}$ and $\vec{Q}$ are both closed under subpath and di-reparametrization the closure $*$ is only needed to ensure that whenever $\gamma \star \mu \in (\vec{P} \cup \vec{Q})^*$ then so are paths like $\eta(t) = \gamma(ut)$ for $0 \leq t \leq 1/u$ and $\eta(t) = \mu(u - ut)$ for $1/u \leq t \leq 1$. Hence, we need only close off under piecewise linear di-reparametrization. In fact, all dipaths in $\vec{P} \vee \vec{Q}$ have the form $\mu_1 \diamond \mu_2 \diamond \cdots \diamond \mu_k \circ \alpha$, where

- $\xi = \mu_1 \diamond \mu_2 \diamond \cdots \diamond \mu_k : [0, k] \to X$ is defined by $\xi(t) = \mu_j(t + (j - 1))$ for $t \in [j - 1, j]$.
- There is a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$, s.t. $\alpha : [0, 1] \to [0, k]$ is given by $\alpha(t) = \frac{t - t_j}{t_j - t_{j-1}} + j$ for $t \in [t_{j-1}, t_j]$
- $\mu_j \in \vec{P} \cup \vec{Q}$.

Definition 3.3. Let X be a set. A dTop-structure on X is a topology τ and a set af dipaths $\vec{P} \in X^I$, where X^I are the paths $\gamma : I \to X$ continuous wrt. τ. We define a lattice structure on the dTop-structures:

- $(\tau, \vec{P}) \leq (\sigma, \vec{Q})$ if the identity map $id : X \to X$ induces a d-map from $(X, \tau, \vec{P})$ to $(X, \sigma, \vec{Q})$.
- The meet operation is $(\tau, \vec{P}) \wedge (\sigma, \vec{Q}) = ((\tau \cup \sigma)^c, \vec{P} \cap \vec{Q})$, where $(\tau \cup \sigma)^c$ is the topology generated by $\tau \cup \sigma$.
- The join is $(\tau, \vec{P}) \vee (\sigma, \vec{Q}) = (\tau \cap \sigma, (\vec{P} \cup \vec{Q})^*)$.

Denote this lattice $\mathbb{TP}(X)$.

Remark 3.4. The following are easy facts about the above lattices:

- The relation $\leq$ is inclusion for the lattice of d-structures. For dTop-structures it is inclusion and containment: $(\tau, \vec{P}) \leq (\sigma, \vec{Q})$ if $\vec{P} \subseteq \vec{Q}$ and $\tau \supseteq \sigma$.

- The top element $\top$ of $\mathbb{P}(X)$ is X^I and the bottom $\perp$ is the constant paths.
- The top element of $\mathbb{TP}(X)$ is the trivial topology, $\tau = \{X, \emptyset\}$ and all maps $I \to X$ in **Set**. The bottom element is the discrete topology and the constant paths.

Definition 3.5. A lattice L is

- Complete, if every non-empty subset $\{L_j | j \in J\} \subset L$ has a supremum and an infimum.
- Bounded, if there is a top and a bottom.
- Distributive if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in L$.
- A Heyting Algebra, if it is bounded and for all $a \in L$, the function $d_a(x) = a \wedge x$ has a right (or upper) adjoint. The right adjoint g_a is then called an *implication* and $g_a(y)$ is denoted $a \to y$.
- A Boolean algebra, if it is bounded, distributive and every element x has a *complement*, $\neg x$, s.t. $x \vee \neg x = \top$ and $x \wedge \neg x = \perp$.
- A complete Heyting algebra is called a locale.

Remark 3.6. Proofs of the following properties may be found for instance in [7].

- L is a Heyting algebra if and only if every element x has a *pseudo-complement* $\neg x$, s.t. $\neg x \wedge x = \perp$ namely $\neg x = (x \to \perp)$.
- A Boolean algebra is a Heyting algebra.
- A Heyting algebra is Boolean if and only if $\neg\neg x = x$ for all $x \in L$

Lemma 3.7. *Let X be a topological space, then $\mathbb{P}(X)$ is complete.*

Proof. Let $\{\vec{P}_j | j \in J\} \subset \mathbb{P}(X)$. Define $\bigvee_{j \in J} \vec{P}_j = (\bigcup_{j \in J} \vec{P}_j)^*$ where again $()^*$ is closure under finite concatenation and non-decreasing reparametrization. $\bigwedge_{j \in J} \vec{P}_j = \bigcap_{j \in J} \vec{P}_j$. $\square$

Theorem 3.8. *Let X be a topological space, then $\mathbb{P}(X)$ is a complete Heyting algebra, i.e., a locale.*

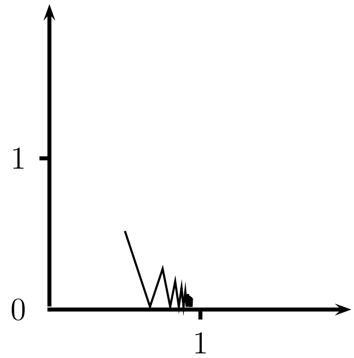
Proof. By [7] p.10, it suffices to check the infinite distributive law

$$\vec{P} \wedge \bigvee_{j \in J} \vec{Q}_j = \bigvee_{j \in J} (\vec{P} \wedge \vec{Q}_j).$$

Suppose $\gamma \in \vec{P} \wedge \bigvee_{j \in J} \vec{Q}_j$ Then $\gamma \in \vec{P}$ and $\gamma = \mu_1 \diamond \mu_2 \cdots \diamond \mu_k \circ \varphi$, where $\mu_i \in \vec{Q}_{j_i}$ and φ is a di-reparametrization, which may be assumed to be piecewise linear as in Rem. 3.2. Since $\vec{P}$ is closed under subpaths, $\mu_i \in \vec{P}$, so $\gamma \in \bigvee_{j \in J} (\vec{P} \wedge \vec{Q}_j)$.

Now suppose $\gamma \in \bigvee_{j \in J} (\vec{P} \wedge \vec{Q}_j)$. Then $\gamma = \mu_1 \diamond \mu_2 \cdots \diamond \mu_k \circ \varphi$ where $\mu_i \in \vec{P} \wedge \vec{Q}_{j_i}$. By concatenation, $\gamma \in \vec{P}$ and clearly $\bigvee_{j \in J} (\vec{P} \wedge \vec{Q}_j) \subset \bigvee_{j \in J} \vec{Q}_j$, so we are done. $\square$

Corollary 3.9. *For $\vec{P}, \vec{Q} \in \mathbb{P}$ there is an implication, $\vec{P} \to \vec{Q}$. Moreover, $\vec{P}$ has a pseudo-complement $\neg \vec{P}$ such that $\vec{P} \wedge \neg \vec{P} = \vec{0}$.*



Proof. See [7] p. 24 Lemma 3.16. The point is, that $\vec{P} \to \vec{Q}$ or $g_{\vec{P}}(\vec{Q})$ is

$$\bigvee\{\vec{R}|\vec{P} \wedge \vec{R} \leq \vec{Q}\}.$$

□

Lemma 3.10. *There is a concrete description of the pseudo-complements:*

$$\neg\vec{P} = \{\gamma \in X^I | \forall t_1 \leq t_2 \in I : \gamma_{[t_1,t_2]} \in \vec{P} \Rightarrow \gamma([t_1, t_2]) = \gamma(t_1)\}.$$

Proof. 1) $\neg\vec{P}$ is a d-structure: It contains the constant maps, is closed under reparametrization, concatenation and subpath.
2) $\neg\vec{P} \wedge \vec{P} = \perp$: If $\gamma \in \neg\vec{P} \cap \vec{P}$, then $\gamma \in \neg\vec{P}$ and $\gamma_{|[0,1]} \in \vec{P}$, so clearly, γ is constant.
3) If $\vec{Q} \wedge \vec{P} = \perp$, then $\vec{Q} \leq \neg\vec{P}$: Suppose $\mu \notin \neg\vec{P}$. Then there is a $t_1 < t_2$ s.t. $\mu_{[t_1,t_2]} \in \vec{P}$ and not constant. If $\mu \in \vec{Q}$, then $\mu_{[t_1,t_2]} \in \vec{Q}$, since $\vec{Q}$ is closed under subpath. Hence $\mu_{[t_1,t_2]} \in \vec{P} \cap \vec{Q}$ and not constant, a contradiction. □

Example 3.11. The pseudo-complement is in general not a complement, hence the locale of d-structures is not a Boolean algebra:
We give an example where $\neg\vec{P} \vee \vec{P} \neq X^I$. Let $X = \vec{I} \times \vec{I}$, i.e., $\gamma \in \vec{P}$ if $t_1 \leq t_2 \to \gamma_i(t_1) \leq \gamma_i(t_2)$ for $i = 1, 2$. $\mu \in \neg\vec{P}$ if for $t_1 < t_2 \in I$, either $\mu_{[t_1,t_2]}$ is constant or there are a, b, $t_1 \leq a \leq b \leq t_2$ s.t. $\mu_1(a) > \mu_1(b)$ or $\mu_2(a) > \mu_2(b)$.
An example of a curve not in $\neg\vec{P} \vee \vec{P}$ is the piecewise linear graph, see Fig. 3 connecting the points $(1 - \frac{1}{2n}, \frac{1}{2n})$ and $(1 - \frac{1}{2n+1}, 0)$ for $t \in [1 - \frac{1}{2n}, 1 - \frac{1}{2n+1}]$, and connecting $(1 - \frac{1}{2n+1}, 0)$ and $(1 - \frac{1}{2(n+1)}, \frac{1}{2(n+1)})$ for $t \in [1 - \frac{1}{2n+1}, 1 - \frac{1}{2(n+1)}]$ and s.t. $\gamma(1) = (1, 0)$. This saw tooth curve is not a *finite* concatenation of paths in $\vec{P} \cup \neg\vec{P}$. It is, however, a countable concatenation of such pieces.

Definition 3.12. A d-structure $\vec{P}$ is closed, if it is a closed subset of X^I in the compact-open topology.

Theorem 3.13. *Let $\vec{P}$ be a closed d-structure. Then $(\vec{P}\cup\neg\vec{P})^\omega = X^I$ where $(\vec{Q}\cup\vec{R})^\omega$ is the closure under countable concatenation and monotone reparametrization.*

Proof. Let $\gamma \in X^I$. For $t_0 \in I$, $\gamma|_{[t_0,t_0]}$ is constant and hence in $\vec{P}$. For all $t_0 \in I$ let

$$a_{t_0} = \inf\{t \in I | \gamma_{[t,t_0]} \in \vec{P}\}$$
$$b_{t_0} = \sup\{t \in I | \gamma_{[t_0,t]} \in \vec{P}\}$$

The sets are non-empty and since $\vec{P}$ is closed, a_{t_0} is a minimum and b_{t_0} is a maximum, so $\gamma_{[a_{t_0},b_{t_0}]} \in \vec{P}$.

If $s \in [a_{t_0}, b_{t_0}]$, then $[a_s, b_s] = [a_{t_0}, b_{t_0}]$, but the *set* $\{[a_t, b_t] | a_t < b_t\}$ is at most countable by Lemma 3.16. Enumerate and order these s.t. $a_k < b_k < a_{k+1}$ for $k \in \mathbb{N}$. This gives the decomposition: $\gamma_{[a_k,b_k]} \in \vec{P}$ and $\gamma_{[b_k,a_{k+1}]} \in \neg\vec{P}$. $\square$

Example 3.14. The d-structure on $X = \vec{I}^n$ is closed: Let γ_n be a sequence in $\vec{P}$ and suppose $\gamma_n \to \gamma$ in X^I. Suppose $\gamma \notin \vec{P}$. Then there are $t_1 < t_2$ s.t. for some coordinate, i, $\gamma^i(t_1) > \gamma^i(t_2)$, where γ^i is the i'th component function. Suppose wlog, that $i = 1$ and denote $\gamma^1(t_j) = x_j$. Let $\varepsilon < \frac{x_1-x_2}{2}$ and let $U_j =]x_j - \varepsilon, x_j + \varepsilon[\times I^{n-1}$. By continuity, there are $\delta_j > 0$ s.t. $\gamma([t_j - \delta_j, t_j + \delta_j]) \subset U_j$. Let $K_j = [t_j - \delta_j, t_j + \delta_j]$ and let $\Phi(K_j, U_j)$ be the open set of functions mapping the compact set K_j to U_j. Then $\gamma \in \Phi(K_1, U_1) \cap \Phi(K_2, U_2)$ and hence there is $N \in \mathbb{N}$ s.t. for $n \geq N$ $\gamma_n \in \Phi(K_1, U_1) \cap \Phi(K_2, U_2)$, i.e., γ_n is not in $\vec{P}$ for $n \geq N$. The pseudo-complement $\neg\vec{P}$ is not closed: Let $X = \vec{I}^2$ and let $\gamma_n(t) = (t, \frac{1}{n}(1 - 2t) + 1/2)$. Then $\gamma_n \in \neg\vec{P}$ and $\lim_{n\to\infty} \gamma_n$ is $\gamma(t) = (t, 1/2) \in \vec{P}$ and not constant. Notice, that there is something to prove. The pseudo-complement is not the set theoretic complement in X^I.

Remark 3.15. Hence, a closed d-structure has a complement, if we require d-structures to be closed under infinite concatenation. However, the complement of a closed d-structure is in general not closed; so the lattice of closed d-structures with infinite concatenation is not a Boolean algebra.

Lemma 3.16. *Let S be a set of disjoint closed non-trivial subintervals of I. Then S is countable.*

Proof. Each element of S contains a rational number. Hence, the identity $I \to I$ induces a surjection from a subset of the rationals to S. $\square$

4. Galois connection

Important examples of d-structures arise in concurrency theory, see [5], namely $\vec{I}^n \setminus F$, where F is a subset - the "forbidden" area. In general, the model is a product of directed graphs minus a forbidden area. Instead of considering $\vec{I}^n \setminus F$ as a subset of $\vec{I}^n$ with the induced d-structure, one may consider $\vec{I}^n$ and "halt" the dipaths in F, i.e., only allow constant paths at points in F. Similarly, given a d-structure, one may ask, which points are forbidden, in the sense that only constant dipaths meet them. This gives a Galois connection between the lattice of subsets and the lattice of dipaths. See Thm. 4.4.

Definition 4.1. Let X be a topological space. Then the set of subsets constitute a complete lattice 2^X under inclusion; with $A \vee B = A \cup B$ and $A \wedge B = A \cap B$. There is a top, namely X and a bottom, $\emptyset$.

Definition 4.2. Let $(X, \vec{P})$ be a d-space. The sublattice in $\mathbb{P}$ of d-structures below $\vec{P}$ is denoted $\downarrow \vec{P}$.

Let $F \subset X$. The d-structure $\{\gamma \in \vec{P} | \gamma([0,1]) \cap F \neq \emptyset \Rightarrow \gamma \text{ is constant } \}$, the dipaths avoiding F, is denoted $\mu(F)$. $\mu : 2^X \to \mathbf{P}$.

For $\vec{Q} \in \downarrow \vec{P}$, let $\nu(\vec{Q}) = \{x \in X | \vec{Q}(x, -) = \vec{Q}(-, x) = \star\}$.

Remark 4.3. $\nu(\vec{Q})$ is the set of points fixed under $\vec{Q}$ in the sense that no non-constant dipaths run through them. The definition of ν may be extended to any subset of $\vec{P}$ in an obvious way.
$\mu(F)$ is the maximal d-structure in $\downarrow \vec{P}$ which fixes F.

Theorem 4.4. *(μ, ν) is a Galois connection:*

$$\mu : (2^X, \mathit{inclusion})^{op} \rightleftarrows (\downarrow \vec{P}, \mathit{inclusion}) : \nu$$

Proof. We have to see

(1) $F_1 \subseteq F_2 \Rightarrow \mu(F_1) \supseteq \mu(F_2)$
(2) $\vec{Q}_1 \subseteq \vec{Q}_2 \Rightarrow \nu(\vec{Q}_1) \supseteq \nu(\vec{Q}_2)$
(3) $\vec{Q} \subseteq \mu(F) \Leftrightarrow \nu(\vec{Q}) \supseteq F$

1) and 2) establishes, that μ and ν are increasing maps. 3) Says, that (μ, ν) is a Galois connection. This is all straight forward and left to the reader. □

Remark 4.5. If we consider a lattice as a category in the standard way - replace all $\leq$ by a morphisms - then the theorem may be reformulated as : μ is a right adjoint to ν.

In the following, we will consider μ, ν as in 4.4. Hence, the lattice structure on 2^X is the opposite structure, $A \wedge B = A \cup B$ and $A \vee B = A \cap B$, and the d-space is $(X, \vec{P})$.

Proposition 4.6. *μ preserves arbitrary $\wedge$ and ν preserves arbitrary $\vee$:*

$$\mu(\bigcup_{j \in J} F_j) = \bigwedge_{j \in J} \mu(F_j)$$

and

$$\nu(\bigvee_{j \in J} \vec{Q}_j) = \bigcap_{j \in J} \nu(\vec{Q}_j)$$

for all families of $F_j \in 2^X$ and of $\vec{Q}_j \in \downarrow \vec{P}$.

Proof. By [7] Thm. 3.2, this follows when μ, ν is a Galois connection. We give the argument for finite families to illustrate that result in this particular case. $\mu(F_1 \cup F_2) = \mu(F_1) \wedge \mu(F_2)$: Let $\gamma \in \vec{P}$ be non-constant. Then $\gamma \in \mu(F_1) \cup \mu(F_2)$ if and only if $\gamma(I) \cap (F_1 \cup F_2) = \emptyset$. Now $\gamma(I) \cap (F_1 \cup F_2) = (\gamma(I) \cap F_1) \cup (\gamma(I) \cap F_2)$ which is empty if and only if $\gamma(I) \cap F_1 = \emptyset$ and $\gamma(I) \cap F_2 = \emptyset$. The latter is equivalent to $\gamma \in \mu(F_1) \cap \mu(F_2) = \mu(F_1) \wedge \mu(F_2)$. The constant paths are in all d-structures.

$\nu(\vec{Q}_1 \vee \vec{Q}_2) = \nu(\vec{Q}_1) \cap \nu(\vec{Q}_2)$: Let $x \in \nu(\vec{Q}_1 \vee \vec{Q}_2)$, i.e., $x \notin \gamma(I)$ for every non-constant dipath $\gamma = \mu_1 \star \mu_2 \cdots \star \mu_k \circ \alpha$ with $\mu_i \in \vec{Q}_1 \cup \vec{Q}_2$. Or, equivalently, $x \notin \eta_i(I)$ for all non-constant $\eta_i \in \vec{Q}_1 \cup \vec{Q}_2$. That is, $\nu(\vec{Q}_1 \vee \vec{Q}_2) = \nu(\vec{Q}_1 \cup \vec{Q}_2)$, where we extend ν to all subsets of $\vec{P}$.

The (set theoretic) complement $(\nu(\vec{Q}_1) \cap \nu(\vec{Q}_2))^c = \nu(\vec{Q}_1)^c \cup \nu(\vec{Q}_2)^c$ is $\{x \in X | \exists \gamma \in \vec{Q}_1 : x \in \gamma(I) \neq \{x\} \vee \exists \mu \in \vec{Q}_2 : x \in \mu(I) \neq \{x\}\} = \{x \in X | \exists \gamma \in \vec{Q}_1 \cup \vec{Q}_2 : x \in \gamma(I) \neq \{x\}\} = (\nu(\vec{Q}_1 \cup \vec{Q}_2))^c = (\nu(\vec{Q}_1 \vee \vec{Q}_2))^c$. □

Proposition 4.7. *$\mu(F_1 \cap F_2) \supset \mu(F_1) \vee \mu(F_2)$ for all $F_i \subset X$. Suppose $F_i \in X$ and $X \setminus F_i$ are compact for $i = 1, 2$. Then $\mu(F_1 \cap F_2) = \mu(F_1) \vee \mu(F_2)$.*

Proof. We tacitly assume that the paths we study are in $\vec{P}$ and non-constant. $\mu(F_1 \cap F_2) \supset \mu(F_1) \vee \mu(F_2)$: Let $\gamma \in \mu(F_1) \vee \mu(F_2)$, $\gamma = \eta_1 \star \eta_2 \cdots \star \eta_r \circ \alpha$, where $\eta_i \in \mu(F_1) \cup \mu(F_2)$. Then $\eta_i(I) \cap F_1 = \emptyset$ or $\eta_i(I) \cap F_2 = \emptyset$, so $\eta_i(I) \cap (F_1 \cap F_2) = \emptyset$ wherefore $\gamma(I) \cap (F_1 \cap F_2) = \emptyset$.

$\mu(F_1 \cap F_2) \subset \mu(F_1) \vee \mu(F_2)$ when $X \setminus F_i$ is compact: Let $\gamma \in \mu(F_1 \cap F_2)$. Then $\gamma^{-1}(X \setminus F_1) = \bigsqcup_{i=1}^m [a_i, b_i]$ and $\gamma^{-1}(X \setminus F_2) = \bigsqcup_{j=1}^k [c_j, d_j]$ by compactness. Since $\gamma^{-1}(F_1 \cap F_2) = \emptyset$, $\gamma(I) \subset X \setminus (F_1 \cap F_2) = X \setminus F_1 \cup X \setminus F_2$. I.e., $I = \bigcup_{i=1}^m [a_i, b_i] \cup \bigcup_{j=1}^k [c_j, d_j]$. Hence γ is a finite concatenation of dipaths avoiding F_1 and dipaths avoiding F_2. □

Remark 4.8. Instead of requiring $X \setminus F_i$ compact, a "finite return" condition would suffice: F_i is open and for all $\mu \in \vec{P}$, $\mu^{-1}(F_i)$ is a finite union of open intervals.

Proposition 4.9. *For all $\vec{Q}_1, \vec{Q}_2 \in \downarrow \vec{P}$, $\nu(\vec{Q}_1 \wedge \vec{Q}_2) \supset \nu(\vec{Q}_1) \cup \nu(\vec{Q}_2)$.*

Proof. As in 4.4, we study the (set theoretic) complements.

$$(\nu(\vec{Q}_1 \wedge \vec{Q}_2))^c = \{x \in X | \exists \gamma \in \vec{Q}_1 \cap \vec{Q}_2 : \gamma \text{ not constant and } x \in \gamma(I)\}$$

and

$$(\nu(\vec{Q}_i))^c = \{x \in X | \exists \eta_i \in \vec{Q}_i : \eta_i \text{ not constant and } x \in \eta_i(I)\}.$$

So

$$(\nu(\vec{Q}_1))^c \cap (\nu(\vec{Q}_2))^c = \{x \in X | \exists \eta_1 \in \vec{Q}_1, \eta_2 \in \vec{Q}_2 : \eta_i \text{ not constant and } x \in \eta_i(I)\}$$

Hence $(\nu(\vec{Q}_1 \wedge \vec{Q}_2))^c \subset (\nu(\vec{Q}_1))^c \cap (\nu(\vec{Q}_2))^c$. □

Remark 4.10. It is not obvious which condition to put on $\vec{Q}_i$ to get an equality in 4.9. It has to do with the germs of dipaths at points. One option is to require for all $x \in X$, either $\vec{Q}_i$ fixes x or there is a neighborhood U_x s.t. whenever $z \in U_x$, $\vec{Q}_i(x, z) = \vec{P}(x, z)$ and $\vec{Q}_i(z, x) = \vec{P}(z, x)$. A "locally full" requirement.

Notation: For X, Y topological spaces and subsets $A \subset X$, $B \subset Y$, let $\Phi(A, B) = \{f : X \to Y | f(A) \subset B\}$. With A is compact and B is open, the subsets $\Phi(A, B)$ are a basis for the compact open topology.

Proposition 4.11. *Let $(X, \vec{P})$ be a d-space where X is Hausdorff and $\vec{P}$ is closed. Let $F \subset X$ be an open subset. Then $\mu(F)$ is closed.*

Proof. Suppose $\gamma(I) \cap F \neq \emptyset$. Then, since F is open, there are $c < d$ s.t. $\gamma(]c, d[) \subset F$ and for $c < a < b < d$, we get $\gamma([a, b]) \subset F$. Thus $\gamma \in \Phi([a, b], F)$ and if $\gamma_n \to \gamma$, then $\gamma_n \in \Phi([a, b], F)$ for n large enough, in particular, $\gamma_n(I) \cap F \neq \emptyset$. We have to see, that if γ is not constant, then neither is γ_n for n large. This is standard for a Hausdorff target space: Suppose $x_1 = \gamma(t_1) \neq \gamma(t_2) = x_2$, that U_i is open, $i = 1, 2$, $x_i \in U_i$ and $U_1 \cap U_2 \neq \emptyset$. Then $\gamma \in \Phi(t_1, U_1) \cap \Phi(t_2, U_2)$ and hence, so does γ_n for n large enough, i.e., $\gamma_n(t_1) \neq \gamma_n(t_2)$. □

5. The paths between pairs of points

For a pair of points $s, t \in X$ the dipaths $\vec{P}(X, s, t)$ and the sets of traces $\vec{T}(\vec{P}, s, t)$ depend on the d-structure $\vec{P}$ and $\vec{P} \leq \vec{Q}$ implies $\vec{P}(X, s, t) \subset \vec{Q}(X, s, t)$ and similarly for traces. This induces a map of lattices $\rho_{s,t} : \mathbb{P} \to P(X, s, t)$. Given a set of paths $A \subset P(X, s, t)$, there is a minimal d-structure $\sigma(A) \in \mathbb{P}$ containing A. The pair $(\rho_{s,t}, \sigma)$ is a Galois connection.

Definition 5.1. Let $s, t \in X$, X a topological space. Let $\vec{P} \leq \vec{Q}$. Then $\vec{P}(X, s, t) \subset \vec{Q}(X, s, t)$ and $\vec{T}(X, \vec{P}, s, t) \subset \vec{T}(X, \vec{Q}, s, t)$, so the hierarcy of d-structures induces a partial order on $\mathbb{P}(X, s, t) = \{\vec{P}(X, s, t) | \vec{P} \in \mathbb{P}\}$ and on $\mathbb{T}(X, s, t) = \{\vec{T}(X, \vec{P}, s, t) | \vec{P} \in \mathbb{P}\}$ via inclusion.

Example 5.2. The lattice structure on the set $\{\vec{P}(X, s, t) | \vec{P} \in \mathbb{P}\}$ is *not* inherited from the lattice of d-structures. This is easily seen from an example: Suppose $\vec{P}(X, s, t) = \vec{Q}(X, s, t) = \emptyset$; then the least upper bound in the lattice of sets is $\vec{P}(X, s, t) \cup \vec{Q}(X, s, t) = \emptyset$. Suppose there is a point r s.t. $\vec{P}(X, s, r) \neq \emptyset$ and $\vec{Q}(X, r, t) \neq \emptyset$. Then $\vec{P} \vee \vec{Q}(X, s, t) \neq \emptyset$, since $\vec{P} \vee \vec{Q}$ contains the union and is closed under concatenation.

However, it is always true that $\vec{P} \wedge \vec{Q}(X, s, t) = \vec{P}(X, s, t) \cap \vec{Q}(X, s, t)$.

Proposition 5.3. *Let $\vec{P} \leq \vec{Q}$ be d-structures on X and suppose $s, t \in X$. Then there is an exact sequence in homology*

$$\to H_{k+1}(\vec{P}(X, s, t)) \to H_{k+1}(\vec{Q}(X, s, t)) \to H_{k+1}(\vec{Q}(X, s, t), \vec{P}(X, s, t)) \to$$

$$H_k(\vec{P}(X,s,t)) \to H_k(\vec{Q}(X,s,t)) \to \cdots$$

Proof. This is just the usual sequence for the pair $\vec{P}(X,s,t) \subseteq \vec{Q}(X,s,t)$. □

Remark 5.4. Notice, that variation of the d-structure may be induced by introduction of a forbidden region. Hence, the homology sequences may be used iteratively in adding or removing parts of the forbidden area. If $F_1 \subset F_2 \subset \cdots \subset F$ is a filtration of the forbidden area F, then $\mu(F_1) \geq \mu(F_2) \geq \cdots \geq \mu(F)$, and there are associated sequences in homology and cohomology of path spaces.

6. A hierarchy of structures on the n-cube

The d-structures on the n-cube introduced in Ex. 2.4 give rise to a sublattice of d-structures. For a pair of points s, t in the cube, the corresponding spaces of directed paths $\vec{P}(I^n,s,t)$ may be compared via homology and homotopy sequences.
In [1] and [2], reversible computing is considered in a concurrent setting. Allowing a process to reverse corresponds to relaxing the order in the corresponding coordinate, i.e., replacing $\vec{I}$ with $\overleftrightarrow{I}$.

Definition 6.1. $\vec{P}^{i_1,\dots,i_k}$ denotes the product d-structure on I^n, where the j'th interval is $\vec{I}$ if $j \notin \{i_1,\dots,i_k\}$ and $\overleftrightarrow{I}$ otherwise.

We restrict to the case where the i'th coordinate function of a dipath either has to increase or has no requirements. Hence, the discrete order and the decreasing order is not considered here.

Proposition 6.2. *The d-structures $\vec{P}^{i_1,\dots,i_k}$ constitute a complete bounded sublattice of the full lattice of d-structures on I^n. The induced structure is*

- $\vec{P}^{i_1,\dots,i_k} \leq \vec{P}^{j_1,\dots,j_l}$ *if* $\{i_1,\dots,i_k\} \subset \{j_1,\dots,j_l\}$
- $\vec{P}^{i_1,\dots,i_k} \vee \vec{P}^{j_1,\dots,j_l} = \vec{P}^{\{i_1,\dots,i_k\}\cup\{j_1,\dots,j_l\}}$
- $\vec{P}^{i_1,\dots,i_k} \wedge \vec{P}^{j_1,\dots,j_l} = \vec{P}^{\{i_1,\dots,i_k\}\cap\{j_1,\dots,j_l\}}$
- *The bottom is* $\perp = \vec{I}^n = \vec{P}^{\emptyset}$ *and the top is* $\top = \overleftrightarrow{I} = \vec{P}^{1,2,\dots,n}$

Proof. An easy check. □

The following example is a combination of the hierarchy in 6.2 with the removal of forbidden areas. The overall hierarchy of d-structures sets this in the same framework.

Example 6.3. In [5] we introduced this example, see Fig. 6. Let $F \subset I^3$ be the union of $F_1 =]1,3[\times[0,10]\times]1,3[$, $F_2 =]4,6[\times]4,6[\times]2,8[$ and $F_3 =]7,9[\times[0,10]\times]7,9[$, where $I = [0,10]$.
Variation of the d-structure: In $\vec{I}^3 \setminus (F_1 \cup F_3)$, $\vec{P}^{\emptyset}(\mathbf{0},\mathbf{1})$ has 4 components. $\vec{P}^{\emptyset}(\mathbf{0},\mathbf{1}) \subset \vec{P}^{k}(\mathbf{0},\mathbf{1})$ and the latter still has 4 components for $k \in \{1,2,3\}$. However $\vec{P}^{1,3}(\mathbf{0},\mathbf{1})$ has infinitely many components, since we introduce loops around

F_1 and F_3.

Variation of the forbidden area: By 5.1 $\vec{P}^{\emptyset}(I^3 \setminus (F_1 \cup F_2 \cup F_3), \mathbf{0}, \mathbf{1}) \subset \vec{P}^{\emptyset}(I^3 \setminus (F_1 \cup F_3), \mathbf{0}, \mathbf{1})$. The set A of paths going through $[0,1] \times [0,10] \times \{3\}$ and $[9,10] \times [0,10] \times \{7\}$, form a connected component of $\vec{P}^{\emptyset}(I^3 \setminus (F_1 \cup F_3), \mathbf{0}, \mathbf{1})$ (the following argument spells out an argument from [9]):

$$\gamma \in A \Rightarrow \gamma(t) \in T = I^3 \setminus \{(x,y,z) | (x_1 > 1 \wedge x_3 < 3) \vee (x_1 < 9 \wedge x_3 > 7)\}$$

and vice versa:

$$\gamma \in \vec{P}(T, \mathbf{0}, \mathbf{1}) \Rightarrow \gamma \in A$$

since $\mathbf{0}, \mathbf{1}$ are in different components of $T \setminus [0,1] \times [0,10] \times \{3\}$ and of $T \setminus [9,10] \times [0,10] \times \{7\}$

$p, q \in T \Rightarrow p \vee q \in T$, where $p \vee q = (max(p_1, q_1), max(p_2, q_2), max(p_3, q_3))$. Moreover, the line from p to $p \vee q$ is in T

If $\gamma_1, \gamma_2 \in A$, then $\gamma_1 \vee \gamma_2(t) = \gamma_1(t) \vee \gamma_2(t) \in A$ by the above. The family $\mu_s(t) = s\gamma_1(t) + (1-s)\gamma_1 \vee \gamma_2(t)$ is a path in A from γ_1 to $\gamma_1 \vee \gamma_2$ and similarly, γ_2 is connected to $\gamma_1 \vee \gamma_2$.

Increasing the forbidden area: When we remove F_2, the set of paths in $T \setminus F_2$ has two components represented by $\gamma_1 : (0,0,0) \to (1,0,3) \to (10,0,3) \to (10,10,10)$ and $\gamma_2 : (0,0,0) \to (1,10,3) \to (10,10,3) \to (10,10,10)$ (going in front or behind F_2).

Varying the d-structure: In $\vec{P}^{\{3\}}(I^3 \setminus (F_1 \cup F_3), \mathbf{0}, \mathbf{1})$, the dipaths γ_1 and γ_2 are in the same component of the path space. Connected by the family of dipaths

$\gamma_s : (0,0,0) \to (1,0,3) \to (3,0,3) \to (4,0,3-6s) \to (10,0,3-6s) \to (10,10,10)$ for $0 \leq s \leq 1/3$

$\gamma_s : (0,0,0) \to (1,30s-10,3) \to (3,30s-10,3) \to (4,30s-10,1) \to (10,30s-10,1) \to (10,10,10)$ for $1/3 \leq s \leq 2/3$

$\gamma_s : (0,0,0) \to (1,10,3) \to (3,10,3) \to (4,10,6s-3) \to (10,10,6s-3) \to (10,10,10)$ for $2/3 \leq s \leq 1$

all paths are piecewise linear - the 6 points are connected by lines with the parameter t in the intervals $[\frac{k}{5}, \frac{k+1}{5}]$ for $k = 0 \dots 4$.

7. Conclusion

The hierarchy of d-structures on a fixed space X allows the comparison of $X \setminus F$ for different forbidden areas and of a variation of the directed structure itself as in Section 6; and because of the overall framework these variations may be combined. Relative homology appears naturally as an invariant one should study. In particular, a calculation in the simpler cases seems to be both feasible and relevant. Other directed invariants such as the universal dicovering spaces [3], [4] and [6] are related via the hierarchy. This is another direction, which

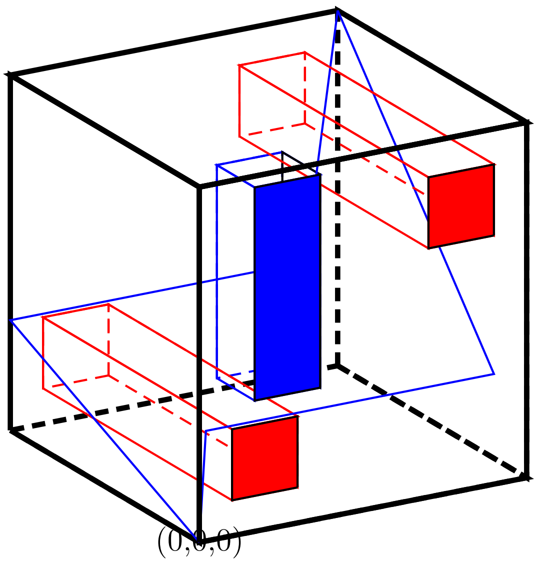


Figure 1. Three holes in a three dimensional cube. And two directed paths.

should be explored. This may then be compared with for instance the approach to reversible computing taken in [1] and [2].

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