Two applications of the footprint (or Δ-set) bound

Estimation of generalized Hamming weights

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Based on joint works with T. Høholdt and H. E. Andersen

1 The Footprint (or Δ-set) bound

Definition 1 Let $\prec$ be a monomial ordering on $\mathcal{M}(X_1,\ldots,X_m)$ and k a field. Given an ideal $I \subseteq k[X_1,\ldots,X_m]$ the set

$$\Delta_\prec(I) = \{M \in \mathcal{M}(X_1,\ldots,X_m) \mid \text{ there does not exist any } F \in I \text{ with } \mathrm{lm}(F) = M\}$$

is called the footprint of I

Theorem 2 If $\Delta_\prec(I)$ is finite then $\#\mathbf{V}_{\bar{k}}(I) \leq \#\Delta_\prec(I)$ holds. Equality holds if I is radical. In particular $\#\mathbf{V}_{\mathbb{F}_q}(I) = \#\Delta_\prec(I \cup \langle X_1^q - X_1, \ldots, X_m^q - X_m\rangle)$.

2 Generalized Hamming weights

Definition 3 The tth generalized Hamming weight of a code C is

$$d_t(C) = \min\{\#\mathrm{Sup}(U) \mid U \text{ is a linear subcode of } C \text{ of dimension } t\}$$

Let $\{P_1,\ldots,P_n\} = \mathbf{V}_{\mathbb{F}_q}(\langle G_1,\ldots G_g\rangle)$ and $\mathrm{ev}(F) = (F(P_1),\ldots,F(P_n))$.

$$A = \begin{bmatrix} \mathrm{ev}(F_1) \\ \vdots \\ \mathrm{ev}(F_a) \end{bmatrix}$$

$$[F_i] = \{F_i + \textstyle\sum_{j=1}^{i-1} \alpha_j F_j \mid \alpha_j \in \mathbb{F}_q\}$$

$$D_{\{[F_{i_1}],\ldots,[F_{i_s}]\}} = \max\{\#\{P_j \in V \mid F'_{i_1}(P_j) = \cdots = F'_{i_s}(P_j) = 0\} \mid F'_{i_t} \in [F_{i_t}], t = 1,\ldots s\}$$

$$D_s = \max\{D_{\{[F_{i_1}],\ldots,[F_{i_s}]\}} \mid 1 \leq i_1 < \cdots < i_s \leq r\}.$$

Theorem 4 Let C be a code with parity check matrix A (not necessarily of full rank) then for $d^* \leq a+t$, $t \leq k$, $d \leq n$ we have

$$d_t \geq d^* \Leftrightarrow D_{a-d^*+t+1} \leq d^* - 2$$
$$d_t \leq d^* \Leftrightarrow D_{a-d^*+t} \geq d^*$$

Theorem 5 Let C be a code with generator matrix A (assumed to be of full rank) then for $t = 1,\ldots k$ we have $d_t = n - D_t$.

Observation 6

$$D_{\{[F_{i_1}],\ldots,[F_{i_s}]\}} = \max\{\#\Delta_\prec(\langle F'_{i_1},\ldots,F'_{i_s},G_1,\ldots,G_g, X_1^q - X_1,\ldots,X_m^q - X_m\rangle) \mid F'_{i_t} \in [F_{i_t}], t = 1,\ldots s\}$$

$$\leq \#\Delta_\prec(\langle \mathrm{lm}(F_{i_1}),\ldots,\mathrm{lm}(F_{i_s}),G_1,\ldots,G_g,$$

3 Weighted degree orderings

Definition 7 Given weights $w(X_1),\ldots,w(X_m) \in \mathbb{R}_+$ define $\prec_w$ on $\mathcal{M}(X_1,\ldots,X_m)$ by $X_1^{i_1}\cdots X_m^{i_m} \prec_w X_1^{j_1}\cdots X_m^{j_m}$ if one of following conditions holds

(1) $w(X_1^{i_1}\cdots X_m^{i_m}) < w(X_1^{j_1}\cdots X_m^{j_m})$
(2) $w(X_1^{i_1}\cdots X_m^{i_m}) = w(X_1^{j_1}\cdots X_m^{j_m})$ and $X_1^{i_1}\cdots X_m^{i_m} \prec_{lex} X_1^{j_1}\cdots X_m^{j_m}$

Proposition 8 Define a weighted degree monomial ordering by the weights $w(X) = b$ and $w(Y) = a$ and consider

$$F(X,Y) = X^a + \alpha Y^b + F'(X,Y)$$
$$G(X,Y) = X^i Y^j + G'(X,Y)$$

where α is non-zero and $a,b > 0$, $w(F') < ab$, and $w(G') < bi + aj$. The equation set $F(X,Y) = G(X,Y) = 0$ has at most $bi + aj$ solutions

4 Parity check matrix description

Improved Hermitian codes
Let V be the 64 points on the Hermitian curve $X^5 + Y^4 + Y$ over $\mathbb{F}_{16}$. Let parity check matrix be

$$\begin{bmatrix} \mathrm{ev}(1) \\ \mathrm{ev}(X) \\ \mathrm{ev}(Y) \\ \mathrm{ev}(X^2) \\ \mathrm{ev}(XY) \\ \mathrm{ev}(Y) \\ \mathrm{ev}(X) \\ \mathrm{ev}(Y+X^4) \end{bmatrix}$$

$D_{\{[XY],[Y+X^4]\}} \leq 7$ follows from Proposition 8
$D_{\{[Y^2],[Y^3+X^4]\}} \leq 8$ follows by choosing $w(X) = 1$ and $w(Y) = 1.1$
$D_{\{[X],[X],[Y+X^4]\}} \leq 6$ follows by choosing $w(X) = 1$ and $w(Y) = 1.4$

Going through all combinations gives $D_1 \leq 16$, $D_2 \leq 8$, $D_3 \leq 6$ and $D_4 \leq 4$.

This implies $d_1 \geq 6$, $d_2 \geq 8$, $d_i \geq i+7$ for $i = 3,\ldots,9$ and $d_i = i+8$ for $i = 10,\ldots,56$. Not only minimum distance is improved.

5 Generator matrix description

Hermitian codes over $\mathbb{F}_{16}$ Defining polynomial $X^5 + Y^4 + Y$ has 64 zeros which gives an evaluation map $\mathrm{ev} : \mathbb{F}_{16}[X,Y] \to \mathbb{F}_{16}^{64}$.

Choose $w(X) = 5$, $w(Y) = 4$ and lexicographic ordering with $X \prec_{lex} Y$. By standard results

$$\mathrm{ev}(\Delta_{\prec_w}(\langle X^5 + Y^4 + Y, X^{16} + X, Y^{16} + Y\rangle))$$

constitutes a basis for $\mathbb{F}_{16}^{64}$. Below is listed

$$\#(\Delta_{\prec_w}(\langle X^iY^j, X^5 + Y^4\rangle) \cap \Delta_{\prec_w}(\langle X^5 + Y^4 + Y, X^{16} + X, Y^{16} + Y\rangle))$$

for all

$$X^iY^j \in \Delta_{\prec_w}(\langle X^5 + Y^4 + Y, X^{16} + X, Y^{16} + Y\rangle)$$

.

15	19	23	27	31	35	39	43	47	51	55	59	60	61	62	63
10	14	18	22	26	30	34	38	42	46	50	54	56	58	60	62
5	9	13	17	21	25	29	33	37	41	45	49	52	55	58	61
0	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60

Traditional codes corresponds to linear span of all $\mathrm{ev}(X^iY^j)$ with $w(X^iY^j) \leq s$.

Improved codes corresponds to linear span of all $\mathrm{ev}(X^iY^j)$ with Δ-size at most some chosen number.

	k	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9
Improved	55	6	8	9	11	12	14	15	16	**18**
Traditional	55	4	8	9	12	13	14	**16**	**17**	**18**

	k	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8
Improved	51	9	12	14	15	17	18	19	**21**
Traditional	51	8	12	13	16	17	18	**20**	**21**
Traditional	50	9	13	14	17	18	19	**21**	**22**

Certainly, minimum distances are improved, but higher weights need NOT be.

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