LTR Design of Discrete-time Proportional-Integral Observers

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are taken as $\rho_0 = [0.5 \ 0.5]^T$ and the model-switching probability matrix as

$$
\begin{bmatrix}
 p_{11} & p_{12} \\
 p_{21} & p_{22}
\end{bmatrix} = \begin{bmatrix}
 0.85 & 0.15 \\
 0.30 & 0.70
\end{bmatrix}.
$$

(65)

A Monte Carlo simulation with 500 experiments was executed, and the results are presented in Fig. 3. In each experiment, the measurement model in effect at each point of time was randomly chosen according to (65). In Fig. 3(a), the root-mean-square-error (RMSE) in the state estimate versus time is presented. Averaging the RMSE’s over the time interval gives an average error of 10.75 for the IMM filter, 9.32 for the smoother of Method 1, and 9.42 for Method 2. Fig. 3(b) presents the probability of error in the system-structure detection versus time (i.e., the probability of choosing the incorrect measurement model at each point of time). Averaging the probabilities over the time interval gives an average probability of error of 0.19 for the IMM filter, 0.15 for Method 1, and 0.16 for Method 2. The two smoothers provided noticeably better performances than the IMM filter, while the smoother of Method 1 provided slightly better performance than Method 2. The smoother of Method 1 provided the best overall performance because it considered the most hypotheses.

A simulation example comparing the performances of these algorithms in reconstructing the trajectory of a maneuvering target has also been performed. Detailed results are not presented because of space limitations. Both smoothers provided significantly better performance than the IMM filter in estimating the system state. However, unlike the system-structure simulation results above, the Method 1 smoother provided significantly better mode estimates than the Method 2 smoother. The mode estimates from the Method 2 smoother and the IMM filter were comparable.

V. SUMMARY

Suboptimal approaches to the one-step fixed-lag smoothing problem for Markovian switching systems were examined in this paper. Two algorithms for generating one-step fixed-lag smoothed estimates were presented. In the first algorithm, the models over the two most recent sampling periods were considered. For $n$ models, there are $n^2$ possible ways of conditioning on models in two sampling periods, and this algorithm evaluated the $n^2$ hypotheses using $n^2$ parallel one-step smoothers. In the second algorithm, only the models in the most recent sampling period were considered, and it evaluated $n$ hypotheses using $n$ parallel one-step smoothers. A simulation of a system-structure detection problem was used to compare the performances of the two smoothers and an IMM filter. The results show that the smoother of Method 1 provided the best overall performance. Variants of these one-step smoothing algorithms have been used in conjunction with IMM filtering algorithms to develop techniques for the alignment of asynchronous sensors [6]. Finally, approaches similar to the ones presented in this paper have been applied to the fixed-interval smoothing problem for Markovian switching systems [7].

REFERENCES


LTR Design of Discrete-Time Proportional-Integral Observers

B. Shafai, S. Beale, H. H. Niemann, and J. L. Stoustrup

Abstract—This paper applies the proportional-integral (PI) observer in connection with loop-transfer recovery (LTR) design for discrete-time systems. Both the prediction and the filtering versions of the PI observer are considered. We show that a PI observer makes it possible to obtain time recovery, i.e., exact recovery for $t \to \infty$, under mild conditions. Two systematic LTR design methods, one based on an extension of the linear quadratic Gaussian loop-transfer recovery (LQG/LTR) and the other based on linear matrix inequality (LMI), are derived for the discrete-time PI observer case. Explicit expressions for the recovery error when exact recovery is not achievable for all frequencies are also given.

I. INTRODUCTION

Since the appearance of the papers by Doyle and Stein [2], [3] dealing with loop-transfer recovery (LTR), many papers have been written on this topic for both continuous and discrete-time systems. The most notable ones for continuous-time systems are [1], [15], [10], [11], [18], [7], [12], and [13]. Although there are certain similarities between the LTR of continuous and discrete-time systems, there exist also fundamental differences. Without going into the details, it is well known that an arbitrarily specified target loop-transfer function is recoverable if the continuous-time system is minimum phase and left invertible.

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where PI observer as follows:

discrete-time, prediction PI observer is equivalent to the continuous-
possible to derive two versions of the PI observer for the discrete-time
practical systems contain time delays, and they are often nonstrictly
zeros, that computation time is sometimes not negligible, and that
the reference input. The states are estimated by using a proportional-
stabilizable, (C, A) detectable and C, B full rank. It is further
assumed that the system (A, B, C) has no poles or zeros at the origin.

The state feedback gain $A_x$, $B_x$, and $C_x$ are given by (8).

**Lemma 2.1:** Let the recovery matrix $M_{PL}(z)$ be given by

$$M_{PL}(z) = F_x(zI - A_x - K_x C_x)^{-1} B_x$$

where $A_x$, $B_x$, $C_x$, $K_x$, and $F_x$ are given by (5) or (6) for the
filtering observer. Then

$$E_S(z) = S_{TF}(z) M_{PL}(z).$$

**Lemma 2.2:** Let the sensitivity recovery error be given by (7).

Based on the discrete-time LTR formulation, we now give necessary
and sufficient conditions for both exact and time recovery.

**Lemma 2.3:** Exact LTR is obtained if and only if one of the following equivalent
conditions holds:

$$E_S(z) = 0$$

$$M_{PL}(z) = 0.$$
We can then define time recovery for discrete-time PI observer-based systems as follows.

**Definition 2.1:** Let $M_{P1}(z)$ be the recovery matrix. Time recovery is obtained if and only if

$$M_{P1}(1) = 0.$$  

(13)

Analogous to the continuous-time case, the condition for achieving time recovery with a PI observer can now be derived for the discrete-time case. With respect to the prediction PI observer we have the following result.

**Theorem 2.1:** Time recovery is obtained with a prediction PI observer if and only if the largest invariant subspace of $(I - A - KpC)^{-1}B K_1 C$ contained in the controllable subspace of $(I - A - KpC)^{-1}(I - A - KpC)^{-1}(I - A - KpC)^{-1}B)$ corresponding to $z = 1$ is itself contained in the unobservable subspace of $(F, (I - A - KpC)^{-1}B K_1 C)$.

In connection to Theorem 2.1, the following corollary gives a simple matrix condition which can be checked to determine whether or not time recovery is achievable.

**Corollary 2.1:** Let the Jordan normal form of the matrix $(I - A - KpC)^{-1}B K_1 C$ be given by

$$T^{-1}(I - A - KpC)^{-1}B K_1 C T = \begin{bmatrix} J_0 & 0 \\ 0 & J \end{bmatrix}$$

where $J_0$ contains all the Jordan blocks associated with the eigenvalue $z = 1$ according to the partitioning

$$T = \begin{bmatrix} T_1 & T_2 \\ \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} S_1 & S_2 \\ \end{bmatrix}.$$  

(15)

Then time recovery is obtained if and only if

$$F T_1 (S_1 (I - A - KpC)^{-1}B) + J_0 S_1 (I - A - KpC)^{-1}B = 0.$$  

(16)

$$\cdots, J_0^{i-1} S_1 (I - A - KpC)^{-1}B = 0.$$  

(17)

With respect to the filtering PI observer, the only difference is that the target design gain $F$ in Theorem 2.1 and Corollary 2.1 is replaced by $P_1(A + KpC)$.

Again, the condition on $K_1$ for time recovery will generically be satisfied if $K_1 C$ has full row rank. As in the continuous-time case, however, this condition is neither necessary nor sufficient.

### III. LQG/LTR Design of Discrete-Time PI Observers

In the following subsections we extend the LQG and LQGLTR design methods of full-order observers for the case of PI-observers.

#### A. LQG Design

Consider the extended state form of a PI observer-based controller given by (4). To proceed with an LQG design for (1), select weighting matrices $\Gamma$ and $\Sigma$ which satisfy

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} = \begin{bmatrix} I_1 & I_2 \\ I_2 & I_2 \end{bmatrix} \geq 0, \quad \Sigma \geq 0$$

(18)

respectively. Solve the algebraic Riccati equation

$$P = A_x P A_x^T - A_x P C_x^T (\Sigma + C_x P C_x^T)^{-1} C_x P A_x^T + \Gamma$$

(19)

where

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$  

(20)

Then compute $K_1$ by

$$K_1 = -A_x P C_x^T (\Sigma + C_x P C_x^T)^{-1} = \begin{bmatrix} -A_x P_{11} & -BP_{12} C_x^T \\ -P_{21} C_x^T & -P_{22} C_x^T \end{bmatrix} (\Sigma + C_x P C_x^T)^{-1}.$$  

(21)

The integral gain $K_1$ has full rank if and only if $CP_{12}$ has full rank. Rewriting (19) as four (effectively three) simultaneous equations leads to

$$0 = -P_{11} + A_x P_{12} A_x^T + B P_{12} C_x^T + A_x P_{21} B^T + B P_{22} B^T$$

$$-A_x P_{11} C_x^T D^{-1} C_x P_{12} B^T - B P_{12} C_x^T D^{-1} C_x P_{12} B^T - B P_{22} C_x^T D^{-1} C_x P_{22} B^T + \Gamma_{11}$$

(23)

$$0 = -P_{12} + A_x P_{12} A_x^T + B P_{12} C_x^T + A_x P_{21} B^T + B P_{22} B^T$$

$$-B P_{12}^2 C_x^T D^{-1} C_x P_{22} B^T - B P_{22} C_x^T D^{-1} C_x P_{22} B^T + \Gamma_{12}$$

(24)

$$0 = -P_{22} C_x^T D^{-1} C_x P_{22} B^T + \Gamma_{22}.$$  

(25)

From (25) we see that $CP_{12}$ has full rank if and only if $\Gamma_{22} = L_2 L_2^T$ is positive definite. Moreover, $\Gamma_{22}$ is the only submatrix of $\Gamma$ which, via $P_{12}$, influences $K_1$. As in the continuous time case therefore, LQG design of a PI observer generically yields time recovery.

#### B. LQG/LTR Design of Full-Order Observers

Derivation of an LQGLTR design method for discrete-time systems parallels the derivation for continuous-time systems given in [9] with the exception that a design can be obtained with zero weighting on the measurement signals, i.e., $\Sigma = 0$. The solution to the Riccati equation with $\Sigma = 0$ has been given in an explicit form in [14].

**Lemma 3.1:** Assume that the system $(A, B, C)$ satisfies

$$CA^i B = 0, \quad i = 1, \ldots, l - 2.$$  

(26)

$$\det [CA^{-1} B] \neq 0.$$  

(27)

Then the singular stationary Riccati equation ($\Sigma = 0$) for the system $(A, B, C)$ is given by

$$P = AP A^T + \Gamma - APC^T (C P C^T)^{-1} C P A^T$$

(28)

and with $\Gamma = BB^T$ the observer gain $K$ is given by

$$K = -A^T C (A^{-1} B)^{-1}.$$  

(29)

where the system $(A, B, C)$ is minimum phase [it is the minimum phase projection of $(A, B, C)$].

The connection between the minimum phase system $(A, B, C)$ and the nonminimum phase system $(A, B, C)$ is given by

$$G(z) = \hat{G}(z) \hat{B}_s(z) = C(zI - A)^{-1} \hat{B} \hat{B}_s(z)$$

(30)

where $\hat{B}_s(z)$ is stable, has zeros coinciding with the nonminimum phase zeros of $G(z)$, and satisfies $\hat{B}_s(z^{-1})^T \hat{B}_s(z) = 1$. The transfer function $\hat{G}(z)$ is minimum phase and is termed the minimum phase counterpart of $G(z)$. One method for calculating $\hat{G}(z)$ and $\hat{B}_s(z)$ can be found in [19].

Now, using the recovery matrix for the full-order prediction observer

$$M_{P1}(z) = F(z I - A - KpC)^{-1} B$$

(31)

and the gain given in Lemma 3.1 with $\Gamma = LL^T$ and $\Sigma = 0$, we have the following result.
Theorem 3.1: Let the full-order prediction observer gain \( K \) be
\[
K = -A^1 \hat{L}(CA^{-1} \hat{L})^{-1}.
\]
Then the recovery matrix \( M_P(z) \) is given by
\[
M_P(z) = F(zI - A)^{-1}[B - z^{-1}A^1 \hat{L}(CA^{-1} \hat{L})^{-1} \hat{L}(CA^{-1} \hat{L})^{-1} C(zI - A)^{-1} B]
\]
where \((A, \hat{L}, C)\) is the minimum phase projection of \((A, L, C)\).

Proof: Rewriting the recovery matrix as
\[
M_P(z) = F(zI - A - K C)^{-1} B
\]
and substituting \( K = -A^1 \hat{L}(CA^{-1} \hat{L})^{-1} \) in the above equation, we get
\[
M_P(z) = F(zI - A)^{-1}(I - K C(zI - A)^{-1})^{-1} B
\]
where \((A, \hat{L}, C)\) is the minimum phase projection of \((A, L, C)\).

Now, using \( C(zI - A)^{-1} \) the recovery matrix becomes
\[
M_P(z) = F(zI - A)^{-1} \hat{L}(CA^{-1} \hat{L})^{-1}(I + C(zI - A)^{-1} A^1 \hat{L}(CA^{-1} \hat{L})^{-1} C(zI - A)^{-1} B)
\]
and substituting \( K = -A^1 \hat{L}(CA^{-1} \hat{L})^{-1} \) in the above equation, we get
\[
M_P(z) = F(zI - A - K C)^{-1} B
\]
and with the observer gain given in Lemma 3.1, we have directly
\[
M_P(z) = F(zI - A)^{-1}[B - z^{-1}A^1 \hat{L}(CA^{-1} \hat{L})^{-1} \hat{L}(CA^{-1} \hat{L})^{-1} C(zI - A)^{-1} B].
\]

It is now very easy to derive explicit expressions for the recovery matrices for the P-observer. When \( C B \) does not have full rank but satisfies the conditions given in Lemma 3.1, we obtain the expressions for the recovery matrices given in the following lemma.

Lemma 3.2: Let the system \((A, B, C)\) be nonminimum phase, and let the optimal LQG/LTR gain be given by (29). Then the recovery matrices for the full order observers are given by
\[
M_P(z) = F(zI - A)^{-1}[B - z^{-1}A^1 \hat{L}(CA^{-1} \hat{L})^{-1} \hat{L}(CA^{-1} \hat{L})^{-1} C(zI - A)^{-1} B].
\]

Moreover, the different terms in the above equation can be expressed by the following results:
\[
F(zI - A)^{-1} B = F(zI - A)^{-1} B
\]
\[
C(zI - A)^{-1} B = C(zI - A)^{-1} B
\]
\[
A^1 \hat{L} = A^1 \hat{L}
\]
\[
C(zI - A)^{-1} B = C(zI - A)^{-1} B
\]

Proof: Lemma 3.2 follows directly from Theorem 3.1 by using $L = B$.

C. LQG/LTR Design of PI Observers

The LQG/LTR design of a full-order PI observer can be realized by using \( \Gamma = B B^T \) and \( \Sigma = 0 \), as shown in the above section. Similarly, if we let \( \Gamma = B_2 B_2^T \) and \( \Sigma = 0 \) in the PI observer design, we have \( K_I = 0 \) due to the fact that \( \Gamma_{22} = 0 \). Instead, let us use a modified matrix \( \Gamma = L_l L_l^T \), where \( L_l^T = [B^T L_2^T] \). It is assumed that \( L_2 \) is selected such that \( I - L_2 \) has the eigenvalues inside the unit circle. This is a technical assumption which will simplify the equations for the recovery matrices. However, there is no need to introduce nonminimum phase zeros into the system by the selection of the free parameter \( L_2 \). With this choice of \( \Gamma \), we will get a nonzero \( K_I \) in general for a nonzero \( L_2 \).

Let \( \hat{L}_x \) satisfy the minimum phase condition on \((A_x, \hat{L}_x, C_x)\) with \( \hat{L}_x^T = [B^T L_2^T] \).

Then one can compute \( \hat{L}_x \) by applying the method proposed in [19].

Using \( L_x \) and (29), the following PI-observer gain is derived for the nonminimum phase case:
\[
M_P(z) = F(zI - A - K C)^{-1} B
\]
and substituting \( K = -A^1 \hat{L}(CA^{-1} \hat{L})^{-1} \) in the above equation, we get
\[
M_P(z) = F(zI - A)^{-1}(I - K C(zI - A)^{-1})^{-1} B
\]
where \((A, \hat{L}, C)\) is the minimum phase projection of \((A, L, C)\).

Proof: Substitution of \( A_x, B_x, C_x, F, \) and \( \hat{L}_x \) in the general recovery matrix of Theorem 3.1 yields
\[
M_P(z) = F(zI - A)^{-1}[B - z^{-1}A^1 \hat{L}(CA^{-1} \hat{L})^{-1} \hat{L}(CA^{-1} \hat{L})^{-1} C(zI - A)^{-1} B]
\]

It is now reasonable to state the following result.

Theorem 3.2: Let the matrix \( L_2 \) be selected such that \( I - L_2 \) has the eigenvalues inside the unit circle. The recovery matrix \( M_P(z) \) for the prediction PI-observer take the following form when the optimal LQG/LTR gain in (37) is used:
\[
M_P(z) = F(zI - A)^{-1}[B - z^{-1}A^1 \hat{L}(CA^{-1} \hat{L})^{-1} \hat{L}(CA^{-1} \hat{L})^{-1} C(zI - A)^{-1} B]
\]
Note that the assumption on \( L_2 \) has been used in connection with the calculation of the last equation. Thus, using these equations, the recovery matrices \( M_{P_I}(z) \), \( M_{F}(z) \) can be derived directly. Note that we get the recovery matrices for the P-observer from Theorem 3.2 by using \( L_2 = 0 \). As a direct consequence of Theorem 3.2, we achieve time recovery. This is reflected in the following result.

**Theorem 3.3:** Let the recovery matrix for the prediction PI-observer be given as in Theorem 3.2. Then for \( z = 1 \), the recovery matrix satisfies

\[
M_{P_I}(1) = 0.
\]

**Proof:** Since the recovery matrix \( M_{P_I}(z) \) can be expressed by

\[
M_{P_I}(z) = F(zI - A)^{-1}[B - z^{-1}(A^T \hat{B}_1 + A^{-1}BL_2 + \cdots + BL_2)
+ (zI - I)^{-1}BL_2(C, \Phi_2, B_2)]
= F(zI - A)^{-1}[B - z^{-1}(A^T \hat{B}_1 + A^{-1}BL_2 + \cdots + BL_2)
+ (zI - I)^{-1}BL_2((zI - I)C \Phi B_1
+ C \Phi BL_2)^{-1}, C \Phi B]
\]

we simply evaluate it at \( z = 1 \) and the result follows, i.e.,

\[
M_{P_I}(1) = F(I - A)^{-1}[B - BL_2(C \Phi BL_2)^{-1}, C \Phi B]
= F(I - A)^{-1}[B - B]
= 0.
\]

Equivalently, we can calculate the recovery matrix for the filtering PI observer which is given by

\[
M_{P_I, F}(z) = F(zI - A)^{-1}[B - z^{-1}(A^T \hat{B}_1 + A^{-1}BL_2 + \cdots + BL_2)
+ (zI - I)^{-1}BL_2(zI - I)
\cdot (zI - I + L_2)^{-1}].
\] (39)

Here, it is also possible to get time recovery.

In the minimum phase case, we have \( \hat{B}_d(z) = I \) and \( \hat{B}_1 = B \). Thus, the equations for the recovery matrices take the following forms:

\[
M_{P_I}(z) = F(zI - A)^{-1}[B - z^{-1}(A^T \hat{B}_1 + A^{-1}BL_2 + \cdots
+ BL_2 + (zI - I)^{-1}BL_2(zI - I)
\cdot (zI - I + L_2)^{-1}]
\] (40)

\[
M_{P_I, F}(z) = F(zI - A)^{-1}[B - z^{-1}(A^T \hat{B}_1 + A^{-1}BL_2
+ \cdots + BL_2 + (zI - I)^{-1}BL_2)(zI - I)
\cdot (zI - I + L_2)^{-1}].
\] (41)

**D. LMI-Based Solutions for LQG/LTR Design**

The above Riccati equation in Lemma 3.1 has also been considered in [17]. The result derived in [17] is based on an LMI (linear matrix inequality) formulation of the Riccati equation. Let us consider a linear matrix function \( G(Q) \)

\[
G(Q) = \begin{bmatrix}
AQX - Q + BB^T AQ X^T \\
CQA^T \\
CQC^T
\end{bmatrix}.
\] (42)

Let \( Q \) be the largest real symmetric solution of the LMI

\[
G(Q) \geq 0.
\] (43)

It can be shown that whenever \((C, A)\) is detectable, the largest real symmetric solution \( Q \) to (43) exists and is unique. Moreover, the solution \( Q \) to (43) is the solution to the Riccati equation in Lemma 3.1. Knowing the solution \( Q \) to (43), we define the following relation between \( G(Q) \) and \( B_Q, D_Q \):

\[
G(Q) = \begin{bmatrix}
B_Q \\
D_Q
\end{bmatrix} \times \begin{bmatrix}
B_Q^T \\
D_Q
\end{bmatrix}
\] (44)

where the system characterized by the quadruple \((A, B_Q, C, D_Q)\) is the minimum phase projection of \((A, B, C)\), including also the infinite zero structure. The direct matrix \( D_Q \) is nonsingular, so the system \((A, B_Q, C, D_Q)\) has no infinite zeros at all. Furthermore, the observer gain can be expressed in terms of the matrices \( B_Q \) and \( D_Q \) as

\[
K = -B_Q D_Q^{-1}.
\] (45)
Based on the above LMI result for the Riccati equation, we can derive the following factorization of the system \((A, B, C)\):

\[
G(z) = G_{m,\Phi}(z)\hat{B}_{n,\Phi}(z)
\]

\[
= (C(zI - A)^{-1}BQ + DQ)\hat{B}_{n,\Phi}(z)
\]  

(46)

where \(\hat{B}_{n,\Phi}(z)\) is its all pass factor. Note that this factorization takes the infinite zero structure into account.

By using the two equations for the Kalman filter gain, we get directly the connection between the matrices in (30) and (46)

\[
B_Q = A'\hat{B}, \quad D_Q = CA'\hat{B} = \hat{B}_{n,\Phi}(z) = z^{-1}\hat{B}_n(z).
\]  

(47)

The connection between the two all-pass factors follows directly by the observation that the system \((A, B, C)\) has \(1\) zeros at infinity.

If we use the Kalman gain (45) in the recovery matrix for the prediction observer, we obtain directly

\[
M_P(z) = F(zI - A)^{-1}(B - B_Q\hat{B}_{n,\Phi}(z)).
\]  

(48)

Now, let us consider the LMI solution for the PI-observer by considering the extended system \((A_x, L_x, C_x)\) given by

\[
G(Q) = \begin{bmatrix} A_x^T & Q & A_x^T & A_x & C_x^T & C_x^T \end{bmatrix} \succeq \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}
\]  

where

\[
Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}
\]  

(50)

is the largest real symmetric solution of (49). Knowing the solution \(Q\), we get the following relation between \(G(Q)\) and \(L_Q, D_Q\):

\[
G(Q) = \begin{bmatrix} L_Q \\ D_Q \end{bmatrix} \times \begin{bmatrix} L_Q^T \\ D_Q^T \end{bmatrix}.
\]  

(51)

The PI-observer gain is then given by

\[
K_i = -L_QD_Q^{-1} = -\begin{bmatrix} B_Q \\ L_2 \end{bmatrix}D_Q^{-1}.
\]  

(52)

Using this gain in the recovery matrix \(M_{P1}(z)\), we have the following result.

**Theorem 3.4:** Let the matrix \(L_2\) be selected such that \(I - L_2\) has the eigenvalues inside the unit circle. The recovery matrix \(M_{P1}(z)\) for the prediction PI-observer takes the following form when the gain in (52) is used:

\[
M_{P1}(z) = F(zI - A)^{-1}(B - (B_Q + (zI - I)^{-1}BL_2)\hat{B}_{n,\Phi}(z))(zI - I)(zI - I + L_2)^{-1}.
\]  

(53)

where \(\hat{B}_{n,\Phi}(z)\) satisfies

\[
C_x\Phi_xL_x = (D_Q + C_x\Phi_xL_Q)\hat{B}_{n,\Phi}(z).
\]

**Proof:** See the proof of Theorem 3.2.

Note that the equation for the recovery matrix based on the LMI solution has the same structure as the recovery matrix given in Theorem 3.2. Therefore, time recovery is still obtained.

**IV. EXAMPLE**

Consider a nonminimum phase single-input-single output (SISO) system represented by

\[
A = \begin{bmatrix} 1.1036 & 0.0186 \\ -0.4060 & 0.0179 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1544 \\ 0.0498 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0.
\]

The discrete-time system is nonminimum phase with two zeros at \(-1.7989\) and \(-0.1239\). Note also that \(CB\) has full rank. As target design we use \([13]_{\Phi}\)

\[
F = \begin{bmatrix} 7.1222 & 7.5293 & 2.7373 \end{bmatrix}
\]

Applying the optimal LQG/LTR gains for the P and PI observers considered in the above section results in the recovery matrices shown in Fig. 1. The PI-observer has been designed for different values of the gain \(L_2\). As expected, the PI observer achieves time recovery, i.e., the recovery matrix has small gain at low frequencies. It is also important to point out that one can shape the gain of the recovery matrix at low frequencies by the selection of \(L_2\). In Fig. 2, the associated sensitivity transfer functions are shown. Again, the controller based on the PI observer recovers the target sensitivity
function at low frequencies. In Fig. 3, the step responses are shown for the target closed loop and the observer-based closed-loop transfer functions. The advantage of the PI observer, as compared to P observer, is clearly shown in this plot as well. Moreover, the step response corresponding to PI observer-based implementation reaches the steady-state value very fast.

V. CONCLUSION

This paper presented two versions of the discrete-time PI observer: a prediction and a filtering PI-observer. Both LQG and LQGLTR design methods were derived for each observer type with special attention to the time recovery effect of the PI observer. Necessary and sufficient conditions for achieving LTR and time recovery in PI observer-based systems are given.

Moreover, explicit expressions have been derived for the recovery matrices for both the P and the PI-observer in light of optimal LQGLTR design. These explicit forms are derived for both minimum phase as well as for nonminimum phase systems. As a direct consequence of these derivations, it was established that it is always possible to obtain time recovery when PI-observer is applied. Furthermore, the LQGLTR design method does not have to be employed for achieving time recovery.

REFERENCES
