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Encounter probability of significant wave height

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Abstract

The determination of the design wave height (often given as the significant wave height) is usually based on statistical analysis of long-term extreme wave height measurement or hindcast. The result of such extreme wave height analysis is often given as the design wave height corresponding to a chosen return period. Sometimes confidence band of the design wave height is also given in order to include sample variability and measurement/hindcast error.

In the reliability based design of coastal structures, encounter probability, defined as the exceedence probability of the design wave height within the structure lifetime, is preferred.

Return period can be converted to encounter probability by several theoretical formulae. The paper gives the derivation of these formulae. It is found that in the normal design situations all formulae give almost the same results.

However, confidence band related to return period cannot be directly interpreted in terms of encounter probability, for which reason it is difficult to choose a certain design confidence band. The paper discusses determination of the design wave height corresponding to a certain encounter probability, taking into consideration the statistical vagrancy of nature, the sample variability and the measurement/hindcast error. The First Order Reliability Method (FORM) is used for this purpose. By comparing with the conventional return period approach, e.g. including 80% confidence band, it is found that the conventional approach is misleading because such an upper bound of the band corresponds to a much lower encounter probability than specified in the design level. Hence structures designed accordingly are too much on the conservative side.

The influence of the inclusion of sample variability and measurement/hindcast error on the design wave height for a given encounter probability is demonstrated by a practical example.

1 Introduction

The determination of the design wave height (often given as the significant wave height) is usually based on statistical analysis of long-term extreme wave height measurement or hindcast. The sources of uncertainty contributing to the uncertainty of the design wave height are (Burcharth 1986):

1) Statistical vagrancy of nature, i.e. the extreme wave height X is a random variable
2) Sample variability due to limited sample size
3) Errors related to measurement, visual observation or hindcast
4) Choice of distribution as a representative of the unknown true long-term distribution
5) Variability of algorithms (choice of threshold, fitting method etc.)
6) Climatological changes

The sources 1, 2 and 3 and their influence on the design wave height are discussed in this paper.

An example is used to demonstrate how the design wave height is conventionally determined. The data consist of 17 significant wave heights corresponding to the peaks of the 17 most severe storms in a period of 20 years for a deep water location in the Mediterranean Sea. Fig.1 shows the data set and a Gumbel distribution fitted to the data.

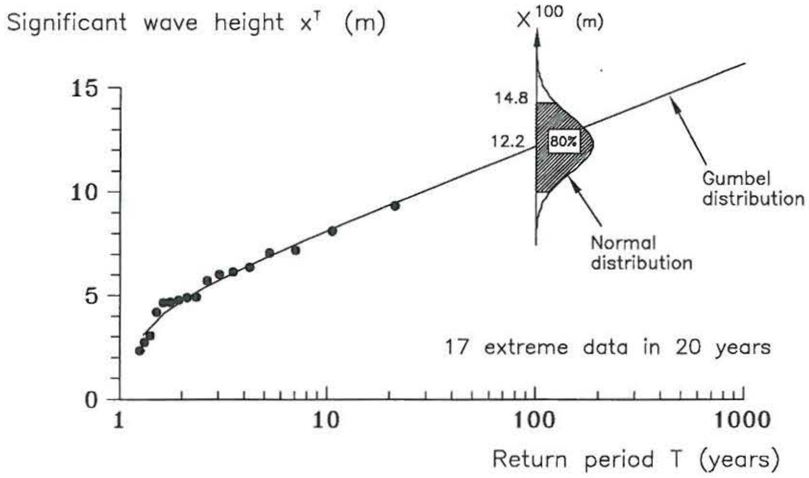

Fig.1. Example of extreme distribution of the significant wave height.

If the design level for the design wave height is a return period of 100 years, i.e. $T = 100$, according to Fig.1 the design wave height is $x^{100} = 12.2\ m$. This means that on average the 12.2 m design wave height will be exceeded once in every 100 years.

In the reliability based design of coastal structures it is more meaningful to use encounter probability, i.e. the probability that the design wave height will be exceeded within the structure lifetime. For example, if the structure lifetime L is 25 years, the encounter probability of the design wave height x^{100} is

$$p \;=\; 1 \;-\; exp\left(-\frac{L}{T}\right) \;=\; 22\% \tag{1}$$

Eq (1) is derived as eq (12) in section 2.

This means that the 12.2 m design wave height will be exceeded with 22% probability within a structure lifetime of 25 years.

If the sample variability is included, the design wave height x^{100} becomes a random variable. The distribution of the design wave height x^{100}, which is usually assumed to follow the normal distribution, can be obtained by numerical simulation, cf. Fig.1. The upper bound of the 80% confidence band is often suggested as the design level. In that case the design wave height is 14.8 m. Note the significant increase of the design wave height after the sample variability is included.

Measurement/hindcast error has been considered in the same way and its significant influence on the design wave height has also been observed (Le Mehaute et al. 1984).

However, the exceedence probability of the 14.8 m design wave height within the structure lifetime is unknown.

In this paper the First Order Reliability Method (FORM) is used to determine the design wave height corresponding to a certain exceedence probability within the structure lifetime (encounter probability). This includes the statistical vagrancy of nature, sample variability and the uncertainty due to measurement/hindcast error.

The results reveal that the application of the 80% confidence band as illustrated in Fig.1 is misleading because the upper bound of the confidence band corresponds to a much lower encounter probability than specified in the design. In other words the structures designed according to the confidence band are too much on the conservative side.

A practical example shows that in normal design situations the inclusion of sample variability and measurement/hindcast error has a significant influence on the design wave height given the same encounter probability.

In the case where only the statistical vagrancy of nature is considered, there are several encounter probability formulae. The paper gives the basic assumptions and the derivation of each formula and provides a comparison of the formulae as well as a recommendation.

2 Encounter probability related to the statistical vagrancy of nature

Even if we had an infinite quantity of historic true wave data and knew the related distribution precisely, there would still be uncertainty as to the largest wave which will occur during any period of time - simply due to the statistical vagrancy of nature.

In the case where only the statistical vagrancy of nature is considered, the encounter probability of design wave height can be calculated by one of several encounter probability formulae.

In order to discuss the various encounter probability formulae it is convenient to start with the definition of return period.

Return period

The following notation is used

X	Significant wave height, which is a random variable due to the statistical vagrancy of nature.
x	Realization of X.
$F(x)$	Cumulative distribution function of X, $F(x) = \text{Prob}(X \leq x)$.
t	Number of years of observation of X.
N	Number of observations in a period of t.
λ	Sample intensity, $\lambda = N/t$.

Fig.2 illustrates the cumulative distribution function of X. The non-exceedence probability of x is $F(x)$, or the exceedence probability of x is $(1 - F(x))$. In other words with $(1 - F(x))$ probability an observed significant wave height will be larger than x.

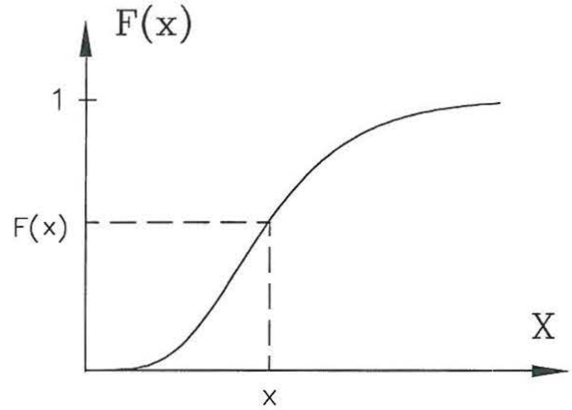

Fig.2. Cumulative distribution function of X.

If the total number of observations is N, the expected number of observations where $(X > x)$ is

$$k = N(1 - F(x)) = t\lambda(1 - F(x)) \tag{2}$$

The return period T of x is defined as

$$T = t\Big|_{k=1} = \frac{1}{\lambda(1 - F(x))} \tag{3}$$

i.e. on average x will be exceeded once in every T years.

Encounter probability formula 1

Based on the fact that on average x will be exceeded once in every T years, it is assumed that the exceedence probability of x in 1 year is $1/T$. Therefore

non-exceedence probability of x in 1 year $\quad \text{Prob}(X \leq x) = 1 - \frac{1}{T}$

non-exceedence probability of x in 2 years $\quad \text{Prob}(X \leq x) = \left(1 - \frac{1}{T}\right)^2$

non-exceedence probability of x in L years $\quad \text{Prob}(X \leq x) = \left(1 - \frac{1}{T}\right)^L$

and the encounter probability, i.e. the exceedence probability of x within a structure lifetime of L years is

$$p = 1 - \left(1 - \frac{1}{T}\right)^L \tag{4}$$

Note that eq (4) cannot be used in the case $T < 1$. In the case of large T, say $T > 20$ years, eq (4) can be approximated by

$$p = 1 - exp\left(-\frac{L}{T}\right) \tag{5}$$

Encounter probability formula 2

Assume that the number of the extreme events is N within the structure lifetime L. X^1 denotes the maximum value in these N independent trials. Then the distribution function of X^1 is

$$F_{X^1}(x) \;=\; P\left(X^1 < x\right) \;=\; (F_X(x))^N \tag{6}$$

Note that F_{X^1} can be interpreted as the non-occurrence of the event $(X > x)$ in any of N independent trials.

Assuming that the number of the extreme events $N = \lambda L$, where λ is the sample intensity, and inserting the definition of return period T (eq (3)) into eq (6) is obtained

$$F_{X^1}(x) \;=\; (F_X(x))^{\lambda L} \;=\; \left(1 \;-\; \frac{1}{\lambda T}\right)^{\lambda L} \tag{7}$$

The encounter probability of x is

$$p \;=\; 1 \;-\; F_{X^1}(x) \;=\; 1 \;-\; \left(1 \;-\; \frac{1}{\lambda T}\right)^{\lambda L} \tag{8}$$

Note that eq (8) cannot be used in the case $\lambda T < 1$. In the case of large λT, say $\lambda T > 20$, eq (8) can also be approximated by eq (5)

Encounter probability formula 3

This formula treats the number of the extreme events within the structure lifetime as a random variable. N is usually assumed to follow the Poisson distribution

$$P(N = n) \;=\; \frac{(\lambda L)^n}{n!}\; exp(-\lambda L) \quad \text{n} = 0,\, 1,\, 2,\, \cdots \tag{9}$$

The probability of the event $(X^1 < x)$ within the structure lifetime is

$$\begin{aligned}
F_{X^1}(x) \;&=\; P(X^1 < x) \;=\; \sum_{n=0}^{\infty} [\; P(N = n)\; F_{X^1}(x, n)\;] \\
&=\; \sum_{n=0}^{\infty} \left[\; \frac{(\lambda L)^n}{n!}\; exp(-\lambda L)\; (\; F_X(x)\;)^n \;\right] \\
&=\; \sum_{n=0}^{\infty} \left[\; \frac{(\lambda\; L\; F_X(x)\;)^n}{n!}\; exp(-\lambda L) \;\right] \\
&=\; exp(-\lambda L)\; \sum_{n=0}^{\infty} \left[\; \frac{(\lambda\; L\; F_X(x)\;)^n}{n!} \;\right] \\
&=\; exp(-\lambda\; L)\; exp(\lambda\; L\; F_X(x)\;) \\
&=\; exp\,[\lambda\; L\; (\; F_X(x) \;-\; 1)]
\end{aligned} \tag{10}$$

Inserting eq (3) into eq (10) is obtained

$$F_{X^1}(x) \;=\; exp\left(-\frac{L}{T}\right) \tag{11}$$

The encounter probability of x is

$$p \;=\; 1 \;-\; F_{X^1}(x) \;=\; 1 \;-\; exp\left(-\frac{L}{T}\right) \tag{12}$$

Note that this encounter probability formula is identical to the approximation of the formulae 1 and 2, eq (5).

Concluding remarks on encounter probability formulae

From the above derivation of the 3 encounter probability formulae it can been seen that, beside the general assumption of independency among extreme events, each formula has its own special assumption.

In normal situations, the encounter probability formula 3 is the approximation of the formulae 1 and 2. Therefore all the formulae give almost the same results.

Outside the usual case the formula 2 deviates from the formulae 1 and 3, see Fig.3 where a very low sample intensity (λ) value is applied.

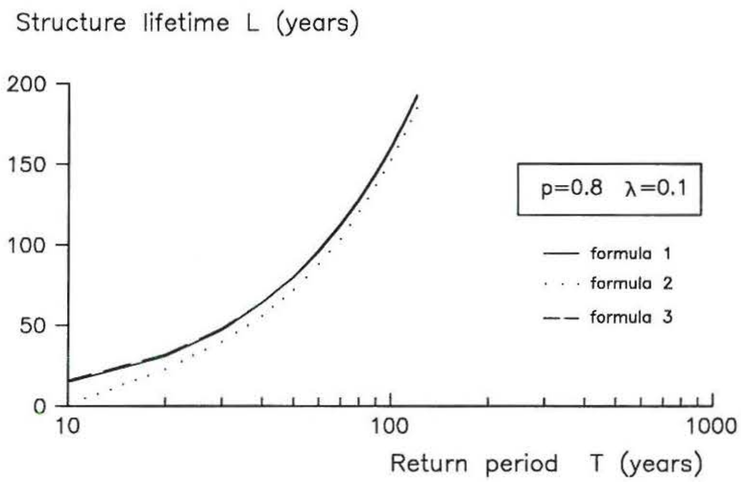

Fig.3. Comparison of encounter probability formulae ($p = 0.8,\ \lambda = 0.1$)

It is recommended that the encounter probability formula 3 (eq (12)) be applied, because it is most simple and the extra assumption involved seems most reasonable. Moreover, it can be used to determine the encounter probability of the design wave height corresponding to $T < 1$ or $\lambda T < 1$.

3 Encounter probability related to the statistical vagrancy of nature, sample variability and measurement/hindcast error

When other uncertainties are involved, the encounter probability cannot be analytically expressed, but can be estimated by the First Order Reliability Method (FORM). To exemplify the discussion, it is assumed that the extreme wave height follows the Gumbel distribution

$$F = F_X(x) = P(X < x) = exp\left(-exp\left(-(\frac{x-B}{A})\right)\right) \tag{13}$$

where X is the extreme wave height which is a random variable, x a realization of X, and A and B the Gumbel distribution parameters.

Principle

Due to the sample variability and measurement/hindcast error, the Gumbel distribution parameters A and B become random variables, and the maximum wave height within the structure lifetime, X^1, becomes a conditional random variable $X^1|_{A,B}$. The probability of $X^1 \geq x_0$ within the structure lifetime is

$$P(X^1|_{A,B} \geq x_0) \;=\; P(x_0 - X^1|_{A,B} \leq 0) \tag{14}$$

Now consider the failure function

$$g(x^1, a, b) \;=\; x_0 - X^1|_{A,B} \begin{cases} < 0 & \text{failure} \\ = 0 & \text{limit state} \\ > 0 & \text{no failure} \end{cases} \tag{15}$$

It can be seen that the failure probability of the failure function is actually the exceedence probability of the design wave height x_0 within the structure lifetime.

By the use of the Rosenblatt transformation, the Hasofer and Lind reliability index β for the failure function can be estimated by the First Order Reliability Theory (FORM), and the failure probability, i.e. the probability of $X^1 \geq x_0$ within the structure lifetime, is calculated by

$$P(X^1|_{A,B} \geq x_0) \;\approx\; \Phi(-\beta) \tag{16}$$

where Φ is the standard normal distribution. The procedure for the calculation of β is detailed in the Appendix.

Numerical simulation of the distribution of A and B

The only unknown in the calculation of β is the distribution of the Gumbel distribution parameters A and B.

Due to the sample variability, i.e. the influence of limited number of data, the Gumbel distribution parameters A and B, estimated from a sample, are subject to uncertainty.

Wave data set contains measurement/hindcast error. Measurement error is from malfunction and non-linearity of instruments, such as accelerometers and pressure cells, while hindcast error occurs when the often limited and uncertain sea-level atmospheric pressure fields are converted to wind data and further to wave data. The accuracy of such conversion depends on the quality of the pressure data and on the technique which is used to synthesize the data into the continous wave field. Burcharth (1986) gives an overview on the variational coefficient C (standard deviation over mean value) of measurement/hindcast error. Table 1 is excerpted from Burcharth (1986).

Table 1. variational coefficient of extreme data C

Methods of determination	Accelerometer buoy Pressure cell Vertical radar	Horizontal radar	Hindcast by SPM	Hindcast by numerical model	Visual
Variational Coef.	0.05-0.1	0.15	0.12-0.2	0.1-0.2	0.2

In order to account the sample variability and measurement/hindcast error, A and B are assumed to follow the normal distribution. The mean values μ_A and μ_B and the standard deviations σ_A and σ_B are obtained by numerical simulations, taking into account the sample variability and the measurement/hindcast error, as explained in the following.

A sample with size N, obtained by measurement or hindcast is fitted to the Gumbel distribution

$$F_X(x) = P(X < x) = exp\left(-exp\left(-(\frac{x-B}{A})\right)\right) \tag{17}$$

The obtained Gumbel distribution parameters A_{true} and B_{true} are assumed to be the true values. Numerical simulation is applied to get the mean values and the standard deviations of the estimators A and B, taking into account the sample variability corresponding to the sample size N. The numerical procedure is as follows:

1) Generate a random number between 0 and 1. Let the non-exceedence probability F_1 equal to that number. The single extreme data x is obtained by

$$x = F_X^{-1}(F_1) = A_{\text{true}}\left[-\ln\left(-\ln F_1\right)\right] + B_{\text{true}} \tag{18}$$

2) Repeat step 1) N times. Thus we obtain a sample belonging to the distribution of eq (17) for sample size N.
3) Fit the sample to the Gumbel distribution and get the new estimated distribution parameters A and B.
4) Repeat steps 2) and 3), say, 10,000 times. Thus we get 10,000 values of A and B.
5) Calculate the mean values (μ_A and μ_B), the standard deviations (σ_A and σ_B) and the correlation coefficient ρ between A and B.

In order to include the measurement/hindcast error an extra step can be added after step 1). This step is to modify each extreme data x generated by step 1), based on the assumption that the hindcast error follows the normal distribution, cf. Fig.4

1*) Generate a random number between 0 and 1. Let the non-exceedence probability F_2 equal to that number. The modified extreme data x_{modified} is obtained by

$$x_{\text{modified}} = x + C\, x\, \Phi^{-1}(F_2) \tag{19}$$

where Φ is the standard normal distribution and C is the coefficient of variation of the measurement/hindcast error. C ranges usually from 0.05 to 0.2, cf. Table 1.

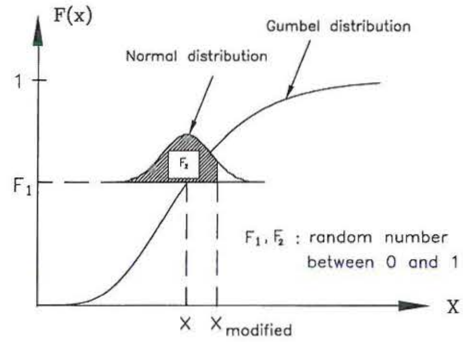

Fig.4. Simulated wave height taking into account measurement/hindcast error.

4 Examples

The deep water wave data presented in Fig.1 is used as an example to demonstrate the determination of the design wave height and the influence of sample variability and measurement/hindcast error. The sample intensity is $\lambda = 17/20$.

By fitting the extreme data to the Gumbel distribution we obtain the Gumbel distribution parameters $A = 1.73$ and $B = 4.53$. The fitting is shown in Fig.1. By inserting the definition of return period (eq (3)) into the Gumbel distribution (eq (13)), we obtain the design wave height corresponding to a certain return period T

$$x^T = A\left(-\ln\left(-\ln\left(1 - \frac{1}{\lambda T}\right)\right)\right) + B \tag{20}$$

If only the statistical vagrancy of nature is considered, i.e. A and B are exact values, by eq (20) the design wave height corresponding to a return period of 100 years is $x^{100} = 12.2$ m, which, by eq (12), corresponds to 22% exceedence probability within a structure lifetime of 25 years,

Conventional design wave height based on confidence band

If sample variability is included, the design wave height x^{100} becomes a random variable. The distribution of the design wave height x^{100}, which is usually assumed to follow the normal distribution, can be obtained by numerical simulation, cf. Fig.5.

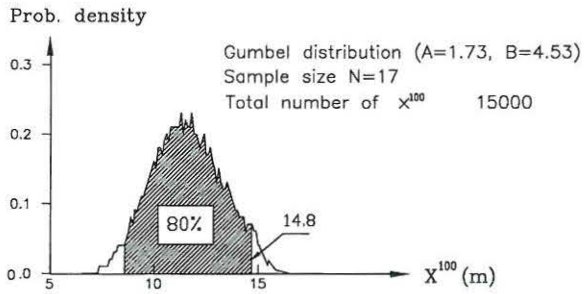

Fig.5. Simulated distribution of x^{100} (sample variability).

In order to account sample variability, an 80% confidence band is often applied. It is found that the design wave height is 14.8 m if the upper bound of the 80% confidence band is taken as the design level.

Encounter probability including sample variability

Taking into account the sample variability, the Gumbel distribution parameters A and B become random variables. Their mean and standard deviation are obtained by the numerical simulation and given in Table 2. Fig.6 shows an example of the distribution.

Table 2. Mean value and standard deviation of A and B.

Sample size N	μ_A	σ_A	μ_B	σ_B	Correlation coefficient ρ
10	1.72	0.54	4.61	0.50	0.173
17	1.72	0.42	4.56	0.45	0.163
25	1.73	0.35	4.56	0.37	0.160
50	1.73	0.25	4.55	0.26	0.129
100	1.73	0.18	4.54	0.19	0.126
1000	1.73	0.06	4.54	0.06	0.121

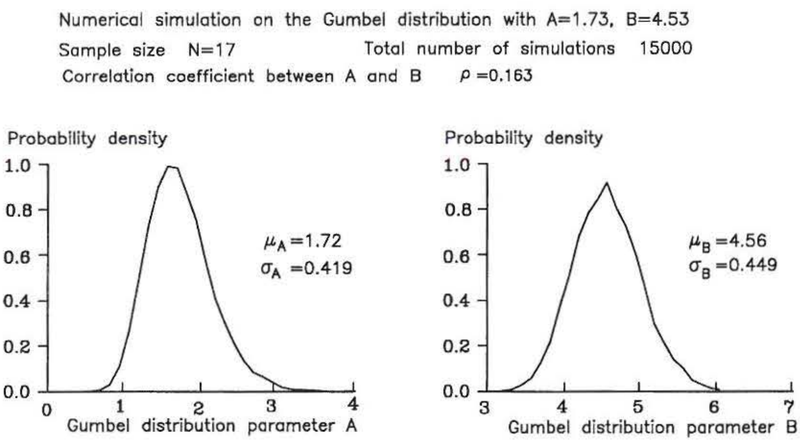

Fig.6. Distribution of A and B due to sample variability (sample size N=17).

The probability density and the non-exceedence probability of the maximum significant wave height within 25 years can be estimated by the First Order Reliability Method, cf. Fig.7, which includes also the case without sample variability obtained by use of eq (12).

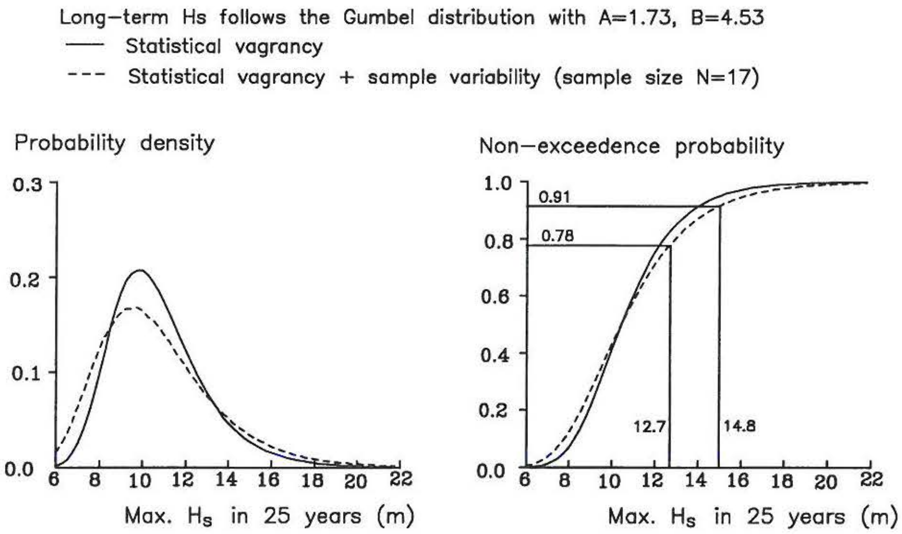

Fig.7. Distribution of maximum significant wave height (sample size $N = 17$).

If the design level is the significant wave height corresponding to 22% exceedence probability within 25 years (i.e. $T = 100$ years), Fig.7 shows that the design wave height with inclusion of sample variability is 12.7 m, which is a little larger than that without sample variability (12.2 m).

It can also be seen that the design wave height of 14.8 m (which corresponds to the upper bound of 80% confidence band of H_s with $p = 22\%$, cf. Fig.5) corresponds to 9% exceedence probability within 25 years. This exceedence probability is much lower than the 22% exceedence probability as specified in the design level. In other words, the design wave height, chosen based on the 80% confidence band, corresponds to a much smaller encounter probability than that specified in the design level. This example shows that the design level should be expressed in terms of encounter probability rather than in terms of confidence band of the return values, because the upper bound of the confidence band cannot be directly interpreted in terms of encounter probability.

In the case of a bigger sample size, there is almost no difference between the design wave height with and without sample variability, cf. Fig.8. For comparison the same λ value is applied.

Table 5. Influence of sample variability and hindcast error
on design significant wave height.

Encounter probability p within $L = 25$ years	0.8	0.5	0.2	0.1	0.05
Return period T (years)	16	36	112	237	487
	Significant wave height (m)				
Statistical vagrancy of nature	8.9	10.4	12.4	13.7	15.0
statistical vagrancy of nature + sample variability (N=17)	8.6	10.4	12.9	14.6	16.2
statistical vagrancy of nature + sample variability (N=17) + hindcast error (C=0.05)	8.6	10.4	12.9	14.6	16.2
statistical vagrancy of nature + sample variability (N=17) + hindcast error (C=0.10)	8.7	10.6	13.2	14.9	16.6
statistical vagrancy of nature + sample variability (N=17) + hindcast error (C=0.20)	9.0	11.1	13.9	15.8	17.6
statistical vagrancy of nature + sample variability (N=17) + hindcast error (C=0.50)	10.6	13.7	17.9	20.7	23.4

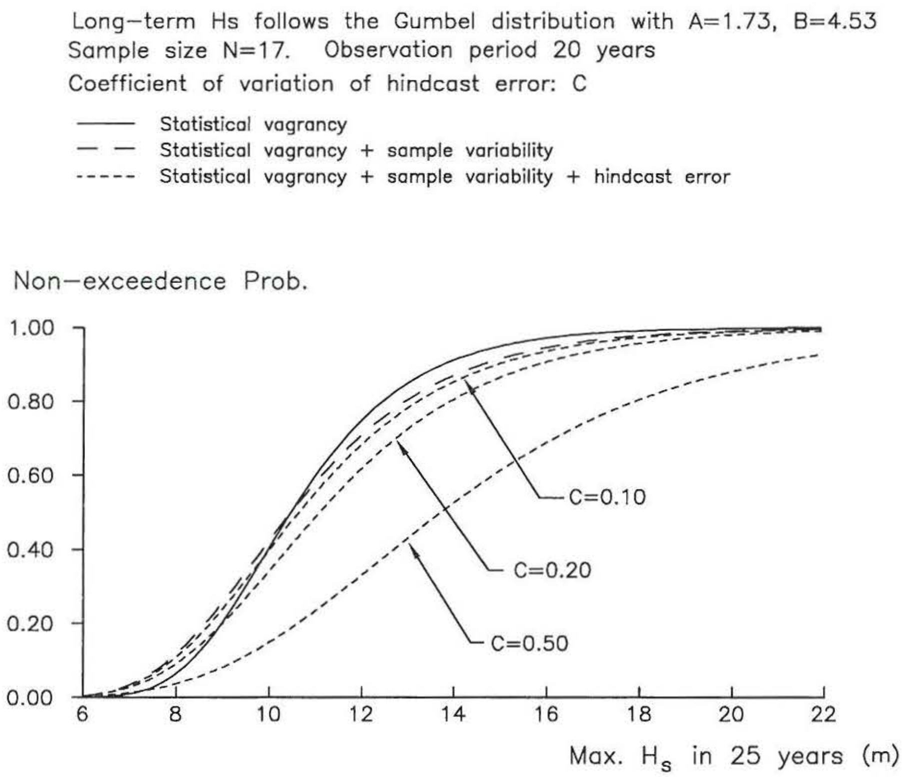

Fig.9. Distribution of maximum significant wave height.

It can be seen from Table 5 and Figure 9 that the influence of the hindcast error on the design wave height depends very much on the C values and the encounter probability.

In practice, a return period of 100 years is often chosen. This corresponds to app. 20% encounter probability for a structure lifetime of 25 years. The variational coefficient of measurement/hindcast error is usually not larger than 0.20, cf. Table 1, it is shown in Table 5 that the inclusion of hindcast error gives 2% and 8% increase in the design significant wave height, corresponding to $C = 0.1$ and $C = 0.2$, respectively.

5 Conclusions

The paper concentrates on the determination of encounter probability, i.e. exceedence probability of the design wave height within the structure lifetime.

If only the statistical vagrancy of the nature is included, encounter probability can be analytically expressed by several formulae. The paper gives the detailed derivation of each formula and their assumptions. It is recommended to use the following encounter probability formula

$$p = 1 - exp\left(-\frac{L}{T}\right)$$

When both sample variability and measurement/hindcast error are considered, the following can be concluded:

1) The First Order Reliability Method (FORM) can be applied to determine the encounter probability.
2) A practical example shows that the design level should be expressed in terms of encounter probability rather than in terms of confidence band of the return values, because the upper bound of the confidence band cannot be directly interpreted in terms of encounter probability.
3) A practical example shows that, in the normal design condition, the inclusion of sample variability has limited influence (less than 5%) on the design wave height, and the influence of the measurement/hindcast error on the design wave height can be up to approximately 8%. Keep in mind the importance of the design wave height in the design of coastal structures, a large sample size with good quality of data is desired in order to reduce the uncertainty related to the design significant wave height.

It should be stressed that even though the example is demonstrated with the Gumbel distribution, the method applies also to other distribution. The application of the method with the Weibull distribution to other practical data has drawn the same conclusions.

6 Acknowledgement

Dr. J.D. Sørensen is gratefully acknowledged for the valuable comments on the paper.

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8 Appendix: Estimation of reliability index β

The followings explain the procedure for the calculation of β.

Eq (10) gives the distribution function of the maximum significant wave height within the structure lifetime as the function of the distribution function of significant wave height F_X, sample intensity λ and the structure lifetime L.

$$F_{X^1}(x^1) = exp\left[\lambda \, L \, (\, F_X(x^1) \, - \, 1)\right]$$

which can be rewritten as

$$F_X(x^1) = 1 + \frac{\ln F_{X^1}(x^1)}{\lambda L} \qquad (21)$$

Insert the Gumbel distribution

$$F_X(x^1) = exp\left(-exp\left(-(\frac{x^1 - B}{A})\right)\right)$$

into eq (21) and solve for x^1 we obtain

$$x^1 = A \left[-\ln\left(-\ln\left(1 + \frac{\ln F_{X^1}(x^1)}{\lambda L} \right)\right)\right] + B \tag{22}$$

X^1 can be converted to the standard normal distributed random variable U^1 by

$$\Phi(u_1) = F_{X^1}(x^1) \tag{23}$$

where Φ is the distribution function of the standard normal distributed random variable. Inserting eq (23) into eq (22) is obtained

$$x^1 = A \left[-\ln\left(-\ln\left(1 + \frac{\ln \Phi(u_1)}{\lambda L} \right)\right)\right] + B \tag{24}$$

The failure function is defined by

$$\begin{aligned} g(u_1, a, b) &= x_0 - X^1|_{A,B} \\ &= x_0 - A \left[-\ln\left(-\ln\left(1 + \frac{\ln \Phi(u_1)}{\lambda L} \right)\right)\right] - B \end{aligned} \tag{25}$$

The normal random variables A and B are converted into the standard normal distributed random variables U_2 and U_3 respectively

$$\frac{A - \mu_A}{\sigma_A} = u_2 \qquad \frac{B - \mu_B}{\sigma_B} = u_3 \tag{26}$$

Insert eq (26) into eq (25) is obtained

$$\begin{aligned} g(u_1, u_2, u_3) = x_0 - (\mu_A + \sigma_A\, u_2) &\left[-\ln\left(-\ln\left(1 + \frac{\ln \Phi(u_1)}{\lambda L} \right)\right)\right] \\ &- (\mu_B + \sigma_B\, u_3) \end{aligned} \tag{27}$$

The differentiations of the failure function are

$$\begin{aligned} a_1 &= \frac{\partial g}{\partial u_1} = \frac{(\mu_A + \sigma_A\, u_2)\, \phi(u_1)}{\ln\left(1 + \frac{\ln\Phi(u_1)}{\lambda L}\right)\left(1 + \frac{\ln\Phi(u_1)}{\lambda L}\right)\, \lambda\, L\, \Phi(u_1)} \\ a_2 &= \frac{\partial g}{\partial u_2} = -\sigma_A \left[-\ln\left(-\ln\left(1 + \frac{\ln \Phi(u_1)}{\lambda L} \right)\right)\right] \\ a_3 &= \frac{\partial g}{\partial u_3} = -\sigma_B \end{aligned} \tag{28}$$

where ϕ is the density function of the standard normal distribution.

The iterative procedure for calculation of β is

1) Select trial values : $u^* = (u_1^*, u_2^*, u_3^*)$.

2) Insert u^* into eq (28) and get (a_1, a_2, a_3).

3) Determine a better estimate of u^* by

$$u_i^* = a_i \frac{\sum_{i=1}^{3} (a_i u_i^*) - g|_{u^*}}{\sum_{i=1}^{3} a_i^2}$$

4) Repeat steps 2) and 3) to achieve convergence.

5) Calculate β by

$$\beta = \left(\sum_{i=1}^{3} (u_i^*)^2 \right)^{\frac{1}{2}}$$

9 Notation

A, B	Gumbel distribution parameters.
C	Variational coefficient (standard deviation over mean) of measured/hindcast extreme data due to measurement/hindcast error.
$F(x)$	Cumulative distribution function of X, $F(x) = \text{Prob}(X \leq x)$.
$F_{X^1}(x)$	Cumulative distribution function of X^1, $F_{X^1}(x) = \text{Prob}(X^1 \leq x)$.
g	Failure function.
L	Structure lifetime (in years).
N	Number of observations in a period of t.
p	Encounter probability, i.e. exceedence probability of the design wave height within a structure lifetime.
T	Return period (in years).
t	Number of years of observation of X.
U_i	Standard normal distributed random variable
u_i	Realization of U_i.
X	Significant wave height, which is a random variable.
X^1	Maximum significant wave height within a structure lifetime, which is a random variable.
x	Realization of X or X^1.
x^T	Return value of X corresponding to a return period of T.
λ	Sample intensity, $\lambda = N/t$.
β	Hasofer and Lind reliability index.
μ_A, μ_B	Mean values of A and B respectively.
σ_A, σ_B	Standard deviations of A and B respectively.
Φ	Standard normal distribution.
ϕ	Standard normal density function.