Multiple Model Adaptive Control Using Dual Youla-Kucera Factorisation

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Abstract: We propose a multi-model adaptive control scheme for uncertain linear plants based on the concept of model unfalsification. The approach relies on examining the ability of a pre-computed set of plant-controller candidates and choosing the one that is best able to reproduce observed in- and output signal samples. The ability to reproduce observations is measured as an easily computable signal norm. Compared to other related approaches, our procedure is designed to be able to handle significant measurement noise and closed-loop correlations between output measurements and control signals.

Keywords: Robust Adaptive Control; Switched Systems; Identification for Robust Control

1. INTRODUCTION

As evidenced by the significant research interest generated over the last two decades, multi-model adaptive control (see e.g., Narendra et al. (1995), Narendra and Xiang (2000) and Kuipers and Ioannou (2010), among many others) have significant appealing properties in practical control settings, where some nominal model knowledge is available, but uncertainties, nonlinearities and time variations potentially might ruin an otherwise nice linear time-invariant (LTI) design.

In its simplest form, multi-model adaptive control involves a supervisor that switches among one of a finite number of controllers as more is learnt about the plant, until one of the controllers is finally selected and remains unchanged. Several research results on multi-model adaptive control have been reported in the past. Anderson et al. (2001) investigates indirect adaptive control of linear plants and employs the Vinnicombe nu-gap metric (Vinnicombe (2001)) to determine whether or not it is safe to switch to a new controller; however, one difficulty with this approach is that it not trivial to compute the safe to switch to a new controller; however, one difficulty with this approach is that it is not trivial to compute the desired metric online from data. Dehghani et al. (2004) extends this work to so-called Windsurfer control (Lee et al. (1993, 1995)); in this concept, the initial control is by design chosen to be simple, conservative and very robust to uncertainties and disturbances. Then, as more and more data becomes available, the fidelity of model can be increased and the controller can gradually be tuned more aggressively as a consequence. Windsurfer control is thus related to multi-model adaptive control, but differs in the sense that candidate controllers are typically not computed a priori.

The multi-model adaptive control method recently presented in Baldi et al. (2011) extends previous work on adaptive control using model unfalsification-based control (Safonov and Tsao (1997); Battistelli et al. (2008)) to multi-input-multi-output systems. Roughly speaking, model unfalsification-based control revolves around the basic premise that if a model and controller is unable to reproduce the observed behaviour of an actual closed-loop system, then the controller and/or plant model must be an inappropriate representation of the actual system and should therefore be replaced by a better candidate. Baldi et al. (2011) uses coprime factorisations of the controllers and plant models to find so-called virtual reference signals, by which to test the ability of candidates to reproduce the observations. The method clearly looks promising for high signal-to-noise ratios, but it is not quite clear how it will fare if the input/output data is contaminated with significant noise and the time in which data sets can be collected is limited (e.g., due to shaky stability margins).

The contribution of the present paper is to propose a reformulation of the method in Baldi et al. (2011) that eliminates this potential noise correlation problem in closed loop by exploiting a key feature of the dual Youla-Kucera parameterisation Youla et al. (1976); Kucera (1976) and choose the controller that provides the plant-controller pair with “smallest” model deviation in the form of an easily computable signal norm. In this respect, we borrow from the so-called Hansen scheme (Hansen et al. (1989); Tay et al. (1989); Anderson (1998); Ansay et al. (1999)).

The outline of the rest of the paper is as follows. Section 2 first provides some background on the Youla-Kucera factorisation. Then, in Section 3 we first outline the proposed switching strategy, which is based on manipulation of so-called virtual references, and then describe in details how this approach can be combined with open-loop-like identification to provide a novel switching criterion in Section 4. Next, Section 5 provides an illustrative example, and finally we conclude with some remarks in Section 6.

2. PRELIMINARIES

We start out by describing the basic notation and the Youla-Kucera factorisation, which will be used extensively in the sequel.
2.1 Problem setup and notation

We consider control loops of the form

\[ y_k = G(z)u_k \quad (1) \]
\[ u_k = K(z)y_k \quad (2) \]

where \( G(z) \) is a linear, possibly time-varying, discrete-time plant mapping input signals \( u_k \in \mathbb{R}^m \) to output signals \( y_k \in \mathbb{R}^p, k \in \mathbb{Z}_+ \) is the sample number and \( z \) is the time shift operator. \( K(z) \) is a controller, which at each sample \( k \) maps output measurements to control signals, such that the closed-loop behaviour achieves some specified performance.

Let \( \mathbb{S}^n \) denote the linear subspace of \( \ell_2 \) consisting of all real-valued \( n \)-dimensional sequences of finite 2-norm. For any element \( x = \{ x_k \}, k = 0,1,2,\ldots \) in \( \mathbb{S} \), we will define its truncation at sample \( \kappa \) as \( \bar{x}^\kappa = \{ x_0, x_1, \ldots, x_\kappa, 0, 0, \ldots \} \). The truncated \( \ell_2 \)-norm of \( x \) is then defined as

\[ \| \bar{x}^\kappa \|_2 = \sqrt{\sum_{k=0}^\kappa x_k^T x_k} \]

For notational convenience, we will in general not write the \( z \)- and \( k \)-dependencies in the following, as long as the meaning can be understood from the context. Capital letters denote systems, while small letters denote signals. Furthermore, the plant-controller interconnection (1)–(2) will be written as \( [G, K] \).

2.2 Model-controller parameterisation

First off, it is well known that the plant and controller

\[ y_k = G(z)u_k \]
\[ u_k = K(z)y_k \]

where \( G(z) \) is a linear, possibly time-varying, discrete-time plant mapping input signals \( u_k \in \mathbb{R}^m \) to output signals \( y_k \in \mathbb{R}^p, k \in \mathbb{Z}_+ \) is the sample number and \( z \) is the time shift operator. \( K(z) \) is a controller, which at each sample \( k \) maps output measurements to control signals, such that the closed-loop behaviour achieves some specified performance.

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Given a plant and one stabilising controller, it is now possible to parameterise all stabilising controllers as follows, see e.g. Anderson (1998):

**Lemma 1.** Let a plant \( G = NM^{-1}, \) with \( N \) and \( M \) coprime and stable, be stabilised by a controller (in positive feedback loop) \( K = UV^{-1} \), with \( U \) and \( V \) coprime and stable. Then the set of all plants stabilised by \( K \) is given as

\[ \mathbf{G} = \{ G(S) = (N + SV)(M + SU)^{-1} \} \]

\[ = \{ G(S) = (\hat{M} + \hat{U}S)^{-1}(\hat{N} + \hat{V}S) \} \quad (5) \]

where \( S \) is any stable system of appropriate dimensions.

**Fig. 1. Youla-Kucera parameterisation of all controllers stabilising the plant \( G = \hat{M}^{-1}\hat{N} \).**

**Lemma 2.** Let a plant \( G = NM^{-1}, \) with \( N \) and \( M \) coprime and stable, be stabilised by a controller (in positive feedback loop) \( K = UV^{-1} \), with \( U \) and \( V \) coprime and stable. Then the set of all plants stabilised by \( K \) is given as

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\[ \mathbf{S} \] in Lemma 1 is known as a dual Youla-Kucera parameter. This situation is depicted in Figure 2.

**Fig. 2. Dual Youla-Kucera parameterisation of all plants stabilised by the controller \( K = V^{-1}U \).**

Now consider the closed-loop interconnection of the plant and controller in Figures 1 and 2, i.e., setting \( u \) and \( y \) in Fig. 1 equal to \( u \) and \( y \) in Fig. 2. Then we observe that the following relations must hold:

\[ w = \hat{V}u - \hat{U}y = \hat{V}V^{-1}(\xi + \hat{U}y) - \hat{U}y = \xi \]

and

\[ x = \tilde{M}y - \tilde{N}u = \tilde{M}\tilde{M}^{-1}(\zeta + \tilde{N}u) - \tilde{N}u = \zeta \]

The closed loop \( \{G(0), K(0)\} \) is stable by construction. Furthermore, we see that since the matrix \( \begin{bmatrix} \bar{M} & \bar{V} \end{bmatrix} \) is invertible over \( \mathcal{RH}_\infty \), the stability of the interconnection \( \{G(S), K(S)\} \) is, in fact, equivalent to the stability of the simpler loop \( \{S, Q\} \).

3. MULTI-MODEL ADAPTIVE CONTROL SCHEME

The goal of multi-model adaptive control is to handle uncertainties and disturbances by switching between different possible controllers that match a given observed input-output behaviour of the plant in some sense.
3.1 Controller switching

To make this more precise, we now extend the class of control loops under consideration slightly as follows:

\[ y_k = G_\theta u_k \quad (7) \]
\[ u_k = K_{\sigma_k} y_k \quad (8) \]

where \( G_\theta \) is a linear plant similar to (1), but affected by some disturbance, uncertainty or time variation represented by some unknown, possibly time varying parameter \( \theta \).

Let \( \mathcal{K} = \{ K_0, K_1, \ldots, K_N \} \) denote a set of \( a \text{ priori} \) designed linear controllers. Each candidate controller \( K_i \in \mathcal{K} \) is designed for a particular value of \( \theta \), say, \( \theta_i \), providing specified stability and performance for \( [G_{\theta_i}, K_i] \). \( \sigma_k \in \mathcal{N} = \{0, 1, \ldots, N\} \) is a switching sequence that determines which of the controllers is active at sample \( k \). The switching sequence is generated by some sort of supervisor, whose task it is to pick the “best suited” controller in some appropriate sense. This essentially means finding a set of switching rules that achieve good performance in the presence of uncertainties and noise.

To this end, at each sample step \( k \), the supervisor evaluates a family of test functionals \( J_i : \mathbb{S}^n \times \mathbb{S}^p \rightarrow \mathbb{R}, i \in \mathcal{N} \), where \( J_i = J_i(\tilde{u}^k, \tilde{y}^k) \) loosely speaking quantifies how well the \( i \)th potential plant-controller pair \( [G_{\theta_i}, K_i] \) suits the data up to the current sample.

In particular, one first chooses \( \sigma_0 \in \mathcal{N} \). At every \( k \) the supervisor then computes the least index \( i \in \mathcal{N} \) such that \( J_i \leq J_i, i \in \mathcal{N} \), wherein the subsequent switching index \( \sigma_{k+1} \) is given by

\[ \sigma_{k+1} = \begin{cases} \sigma_k & \text{if } J_{\sigma_k} < J_{i_\ast} + h \\ i_\ast & \text{otherwise} \end{cases} \quad (9) \]

where \( h > 0 \) is a hysteresis constant.

Assuming the above switching strategy is employed to switch controllers in the plant-controller interconnection (7)–(8), the so-called Hysteresis Switching Logic lemma Morse et al. (1992) then states that as far as \( (7)–(8) \), the so-called Hysteresis Switching Logic lemma Morse et al. (1992) then states that as far as \( G_\theta \) is constant, \( \{\sigma_k\} \) admits a limit and there exist \( i \) such that \( J_i \) is bounded, then, for any initial condition and reference sequence, switching will cease in finite time. That is, the supervisor will eventually settle upon a stabilising controller in the set \( \mathcal{K} \), if such a controller exists.

3.2 Introduction of virtual reference

This idea is now combined with the concept of model falsification, in which a candidate model is tested against plant measurements and rejected in case it fails to reproduce the observations. Here, instead of attempting to reproduce output values as a function of inputs, we look at the input-output data produced by the plant and controller currently in the loop, and select the plant-controller pair \( [G_{\theta_i}, K_i] \) that best matches the observed behaviour.

Assume now that an external reference sequence \( r \in \mathbb{S}^p \) is introduced as shown in Figure 3. Let

\[ \psi^k_i = \begin{bmatrix} \bar{K}_i(\bar{y}^k, \bar{r}^k) \\ [G_{\theta_i}, K_i](\hat{y}^k, \hat{r}^k) \end{bmatrix} \quad (10) \]

Fig. 3. Introduction of reference in control loop with \( K = \bar{V}^{-1} \bar{U} \)
denote the collection of in- and output samples (behavioural data) that would have been generated up to sample \( k \) if the \( i \)th controller had been in closed loop with the plant \( G_{\theta_i} \), while

\[ \tilde{\varphi}^k = \begin{bmatrix} \bar{u}^k \\ \bar{y}^k \end{bmatrix} \quad (11) \]

\[ J_i(\tilde{u}^k, \tilde{y}^k) = \max_{\nu \neq 0} \| \tilde{\psi}^k_i - \tilde{\varphi}^k \|_2 \quad (12) \]

Unfortunately, even if we were able to apply some specific \( r \), simply applying the same reference sequence to all plant-controller pairs would not allow us to determine whether or not different \( r \)'s would have been able to generate the observed behaviours for other plant-controller pairs. To avoid this problem, we introduce a virtual reference for each candidate plant-controller pair as known from model falsification Battistelli et al. (2008), and use that for evaluating \( J_i \). Let \( \bar{V}^{-1}, \bar{U} \) and \( r \) in Figure 3 be replaced by \( \bar{V}_i^{-1}, \bar{U}_i \) and the virtual reference \( \bar{r}_i \), respectively. Then we compute

\[ \bar{r}_i = \bar{V}_i u - \bar{U}_i y \quad (13) \]

for all \( i \in \mathcal{N} \), replace \( \tilde{r}^k \) by \( \bar{r}^k \) in (10), use the resulting \( \psi^k_i \) to evaluate (12) for each plant-controller pair, and pick the controller corresponding to the pair that best matches the observed behaviour.

3.3 Noisy data

As shown in Baldi et al. (2011), the above approach works well as long as the measurements are largely uncorrupted by noise. However, it is a commonly known problem in closed-loop identification that noise may be fed back through a controller operating in closed loop with the plant and make the input correlated with the output samples.

To illustrate this, consider the situation in Figure 4. Here, the output measurements are contaminated by the additive noise signal \( \nu_y \). From the block diagram,

\[ u = \bar{V}_i^{-1} \left( r + \bar{U}_i (\nu_y + G_{\theta_i} u) \right) = \bar{V}_i^{-1} \left( r + \bar{U}_i \nu_y \right) \]

which results in (13) becoming unbiased by \( \nu_y \) in the best case where the “correct” \( i \) has been selected. For the other plant-controller pairs, however, \( G_{\theta,i} \neq \bar{M}_j^{-1} \bar{N}_j \), which means that the noise is not canceled out, causing a mismatch between the true and virtual references.
4. DUAL YOUULA-KUCERA SWITCHING

In order to avoid the potential bias problems pointed out in the previous section, we propose an alternative strategy that makes use of the open-loop-like features of the dual Youla-Kucera parameterisation discussed in Section 2.2.

Let $y$ be affected by noise as in Figure 4, i.e., $y = G_\theta u + \nu_y$, and let $\hat{G}_\theta$ be parameterised as in Figure 2; then, we have the expressions

$$\nu_{\zeta,i} = \zeta_i = S_i w_i$$

$$w_i = \hat{V}_i u - \hat{U}_i (y - \nu_y)$$

$$\hat{M}_i (y - \nu_y) - \hat{N}_i u = \zeta_i$$

for the $i$th candidate loop. Now, let us define the signals $\hat{w}_i = \hat{V}_i u - \hat{U}_i y$ and $\hat{\zeta}_i = \hat{M}_i y - \hat{N}_i u$ and relocate the output noise from the measurement output to the output of the dual Youla-Kucera parameter, as shown in Figure 5.

Using (14)–(16), we can compute the relocated noise as

$$\nu_{\zeta,i} = \zeta_i + \hat{M}_i \nu_y - S_i (w_i + \hat{U}_i \nu_y)$$

$$\hat{\zeta}_i = \hat{\nu}_{\zeta,i} + \hat{M}_i \nu_y - S_i (w_i + \hat{U}_i \nu_y)$$

$$\hat{\zeta}_i = \hat{M}_i y - \hat{N}_i u$$

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i.e., $\nu_{\zeta,i}$ is independent of $\hat{\zeta}_i$. Furthermore, from the block diagram it is immediately seen that

$$w_i = \hat{r}_i = -\hat{V}_i u + \hat{U}_i y$$

$$\zeta_i = \hat{M}_i y - \hat{N}_i u$$

and $\zeta_i = S_i w_i + \nu_{\zeta,i}$.

At this point we make an additional observation: If the candidate matches exactly with the true plant-controller loop, i.e., $[G_\theta, K_i] = [\hat{G}_\theta, \hat{K}_i]$, then we would have $S_i = 0$ and hence $\zeta_i = \nu_{\zeta,i}$ for any $w_i$ given $u$ and $y$.

As a consequence, we may in principle reject any plant-controller pair that yields an estimate of $\zeta_i$ that exceeds the standard deviation of $\nu_{\zeta,i}$. Alternatively, in the absence of that information, we may simply choose the plant-controller pair that yields the smallest signal norm of $\zeta_i$.

The latter can be summed up in the following straightforward procedure:

**Procedure 1.** Let $G_\theta$ be an uncertain plant parameterised by $\theta$. Assume a set of stabilising controllers $K$ has been designed for appropriate values of $\theta$, say, $\theta_i$ for $i \in \mathcal{N}$. Choose $\sigma_0 \in \mathcal{N}$ as the index of the controller corresponding to the ‘nominal’ plant, i.e., $\theta = \theta_0 = 0$. Fix $h > 0$.

For every $k$ greater than the plant order:

1. Apply controller $K_{\sigma_k}$
2. Obtain measurements $\hat{u}^k$ and $\hat{y}^k$
3. For every $i \in \mathcal{N}$ compute
   $$\tilde{\zeta}_i^k = \hat{M}_i \hat{y}^k - \hat{N}_i \hat{u}^k$$
   $$J_i = \| \tilde{\zeta}_i^k \|_2^2$$
4. Find $i_* = \arg\min_{i \in \mathcal{N}} J_i$
5. Evaluate whether or not to switch controller:
   $$\sigma_{k+1} = \begin{cases} \sigma_k & \text{if } J_{\sigma_k} < J_{i_*} + h \\ i_* & \text{otherwise} \end{cases}$$
6. Wait for next sample

In practice, $\zeta_i = S_i w_i + \nu_{\zeta,i}$ will contain both model errors and noise. An underlying assumption is that the noise component will be approximately the same for all the models, and that the difference in signal norms will reflect the different distances of the models to the real plant.

Note also that if stability is of prime concern, one could also consider identifying the $S_i$ parameters from $w_i$ and $\zeta_i$ and pick the plant-controller pair that yields the greatest stability margin.

5. ILLUSTRATIVE EXAMPLE

We consider the uncertain scalar system

$$x_{k+1} = A_\theta x_k + B u_k, \quad y_k = C x_k + n_k$$

with $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [1 \ 0 \ 0]$ and

$$A_\theta = \begin{bmatrix} 0.5 & 0.6 & 0.3 \\ 0.4 & 0.6 + 0.3\theta & 0 \\ -0.2 + 0.1\theta & 0 & 0.9 + 0.1\theta \end{bmatrix}$$

where $\theta \in [0:1]$ is an unknown parameter, and $n$ white Gaussian noise with variance $\rho^2$. For each of the five points $\theta_1 = 0$, $\theta_2 = 0.25$, $\theta_3 = 0.5$, $\theta_4 = 0.75$ and $\theta_5 = 1$, we design an LQR controller $K_i$:

$$x_{c,k+1} = (A_{\theta_i} + L_i C + B F_i) x_{c,k} - L_i y_k, \quad u_k = F_i x_{c,k}$$

(18)
where

\[
L_1 = \begin{bmatrix} -0.782 \\ -0.551 \\ -0.195 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -0.816 \\ -0.596 \\ -0.264 \end{bmatrix}, \quad L_3 = \begin{bmatrix} -0.853 \\ -0.648 \\ -0.343 \end{bmatrix}, \quad L_4 = \begin{bmatrix} -0.893 \\ -0.703 \\ -0.443 \end{bmatrix}, \quad L_5 = \begin{bmatrix} -0.94 \\ -0.735 \\ -0.625 \end{bmatrix}
\]

and

\[
F_1 = \begin{bmatrix} -0.506 & -0.954 & -1.1 \end{bmatrix}, \\
F_2 = \begin{bmatrix} -0.717 & -1.3 & -1.17 \end{bmatrix}, \\
F_3 = \begin{bmatrix} -0.988 & -1.79 & -1.25 \end{bmatrix}, \\
F_4 = \begin{bmatrix} -1.32 & -2.44 & -1.34 \end{bmatrix}, \\
F_5 = \begin{bmatrix} -1.71 & -3.28 & -1.44 \end{bmatrix}.
\]

The factorisation of a controller-model pair is found as

\[
\begin{bmatrix} \dot{V}_i - \dot{U}_i \\ -N_i \end{bmatrix} = \begin{bmatrix} A_{\theta_i} + L_i C & -B \quad L_i \\ F_i & I \\ C & 0 \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{20}
\]

Figs. 6 and 7 illustrate the performance of the switching scheme. The true reference is chosen as a series of random numbers, and the noise and unknown parameter are arbitrarily chosen as \( \rho^2 = 10^{-6} \) and \( \theta = 0.75 \). The factorisation of the five controllers is shown in Figure 6, with the dashed lines representing the method in Section 4 and the solid lines representing Baldi et al. (2011). The dots show the optimal selection. Both methods still perform very well.

Now the noise is increased to \( \rho^2 = 10^{-6} \). For each value of \( \theta \), the controller selection as values between 1 and 5, we can take mean and deviation of the 40 tests. Ideally the mean should be the same as in the previous series and the deviation should be zero.

Figure 9 shows the means plus/minus one standard deviation. The method in Baldi et al. (2011) is shown by the dashed lines, the method in Section 4 the solid lines. The dots show the optimal selection. Both methods still perform very well.

However, when the noise level is increased to \( \rho^2 = 10^{-4} \), the method in Baldi et al. (2011) not only gets a significant variance of the selection, it also shows a strong bias towards the controllers with lower numbers. The method presented here also shows some deviations, but on average the correct controller is selected.

6. DISCUSSION

In this paper, we proposed a novel scheme for adaptive control of linear systems subject to significant uncertainties and/or time variations. The approach relies on examining...
the ability of a pre-computed set of plant-controller candidates and choosing the one that is best able to reproduce observed in- and output signal samples. The ability to reproduce observations is measured as an easily computable signal norm. Compared to other related approaches, our procedure is designed to be able to handle significant measurement noise and closed-loop correlations between output measurements and control signals. However, it must be pointed out that simulation studies indicate that as far as the signal-noise is high or the closed-loop operation permits long data sequences to be obtained, our approach does not appear to provide significantly better results than the method presented in Baldi et al. (2011).

Like most adaptive control algorithms, the method suggested here is initially formulated under the assumption that the plant is time-invariant. However, there is an underlying requirement that the adaptive controller has the capability to track time variations in the plant, which are generally relatively slow compared to the input-output dynamics. To do so, it may be advantageous to introduce exponential forgetting in the evaluation of $J_i$.

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