Beyond Multiplexing Gain in Large MIMO Systems

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Abstract—Given the common technical assumptions in the literature on MIMO channel modeling, we derive generic results for MIMO systems in the large system limit LSL. Consider a \( \phi T \times T \) MIMO system with \( T \) tending to infinity. By increasing the antenna ratio \( \phi \) when \( \phi \geq 1 \), the amount of capacity increase per receive antenna converges to the binary entropy function of the antenna ratio \( 1/\phi \) at high SNR. We also show this "binary entropy increase" for \( \phi < 1 \). Furthermore, we define the deviation of the effective capacity growth from the traditionally assumed linear growth (multiplexing gain). Even when the channel entries are i.i.d. the deviation from the linear growth is significant. We also find an additive property of the deviation for a concatenated MIMO system. Finally, we quantify the deviation of the large antenna ratio \( 1/\phi \) when the channel entries have a limiting singular value distribution and being free of \( \beta \). Then we have

\[
\mathcal{I} \left( \gamma; \mu H \right) - \mathcal{I} \left( \gamma; \mu P_\beta H \right) = H(\beta); \quad \phi \geq 1
\]

\[
\mathcal{I} \left( \gamma; \mu H \right) - \mathcal{I} \left( \gamma; \mu P_\beta H \right) = H(\beta); \quad \phi < 1.
\]

In other words, in the LSL, when \( \phi \geq 1 \) (\( \phi < 1 \)) increasing the antenna ratio \( \phi (1/\phi) \) for a given number of transmit (receive) antennas, the amount of capacity increase per receive (transmit) antenna converges to the BEF of the antenna ratio \( \beta \) at high SNR.

IV. Deviation from the Linear Capacity Growth

It is commonly admitted that at high SNR, with the number of receive antenna kept constant the capacity of a MIMO system grows linearly with the number of transmit antennas\(^1\), see e.g. [1]. This statement is obvious when the channel matrix has orthogonal columns. However, when the matrix has i.i.d. entries for instance, a significant cross-talk arises due to the lack of orthogonality of its columns and cross-talk is a quite non-linear phenomenon. The example in Section V shows that in this case (i.i.d. entries) the deviation from the linear growth is significant. In this section, we investigate the behavior of the deviation from the linear growth.

Let the channel matrix \( H \in \mathbb{C}^{R \times R} \) have a limiting singular value distribution and be asymptotically free of an \( R \times R \) projector \( P_\beta \) as \( R \to \infty \). Then we introduce the deviation of the linear capacity growth in the LSL as

\[
\Delta \mathcal{L}_H P_\beta \triangleq \mathcal{I} \left( \gamma; \mu H P_\beta \right) - \beta \mathcal{I} \left( \gamma; \mu H \right).
\]

Notice that \( \mathcal{I} \left( \gamma; \mu H \right) \) in (3) corresponds to the growth rate of the multiplexing gain \( \beta \) (per receive antenna).

Result 2 Consider an almost surely full-rank random matrix \( H = X Y \) with \( X \in \mathbb{C}^{R \times R} \) and \( Y \in \mathbb{C}^{R \times R} \). Let the matrices \( X, Y \) and the \( R \times R \) projector \( P_\beta \) have a LED each and be asymptotically free of each other as \( R \to \infty \). Then we have

\[
\Delta \mathcal{L}_H P_\beta = \Delta \mathcal{L}_X P_\beta + \Delta \mathcal{L}_Y P_\beta.
\]

\(^1\)For the sake of brevity, throughout the paper we sacrifice the specification of the probability measures for the finite size matrices.

\(^2\)For the sake of brevity, in this section we assume the number of transmit antenna less than the number of receive antennas. The generalization is straightforward.
V. EXAMPLE

We consider the random matrix

\[ H = \prod_{n=1}^{N} A_n \]  

where the \( R \times R \) matrices \( A_n \), \( n = 1, \ldots, N \), have i.i.d entries with zero mean and variance \( 1/R \). Then as \( R \) tends to infinity the deviation from the linear capacity growth reads

\[ \Delta \mathcal{L}_{HP} = N (H(\beta) + \beta \log_2 \beta) . \]  

Indeed, the entries of the product of two matrices with i.i.d. entries are not i.i.d. anymore, but correlated. Furthermore, as \( N \) in (4) increases, so does the correlation between the entries of \( H \), implying that the cross-talk increases as well.

VI. THE NON-HIGH SNR DEVIATION

Calculation of the high SNR lower bound for a given channel model is often analytically tractable. However the high SNR lower bound in itself is a crude approximation of the capacity. On the other hand, calculation of the deviation of the high SNR lower bound from the exact capacity, i.e. \( \mathcal{I} \) in (2), is usually, though not always, analytically intractable. We derive in this section an analytical approximation of this deviation that can then be used in combination with the high SNR lower bound to obtain an approximation of the capacity. To this end we introduce the parameters

\[ m \triangleq \int xd\hat{\mu}_H(x); \quad \hat{m} \triangleq \int x^{-1} d\hat{\mu}_H(x) \]

where \( \hat{m} \) is known in the literature as the harmonic mean of the distribution \( \hat{\mu}_H \). We further note that, due to Lemma 2&4 in [2], we have \( \hat{m} < m \) as \( \hat{\mu}_H \) is not a Dirac measure.

**RESULT 3** Let \( \mu_H \) in (1) be not a Bernoulli distribution and have a fixed mean. Define \( \lambda = \frac{m}{m-\hat{m}} \) and \( \gamma' = (m-\hat{m})\gamma \). Furthermore let \( \mu_{\text{MP}} \) be the Marĉenko-Pastur distribution with the rate parameter \( \lambda \). Then we have

\[ \mathcal{I}_- (\gamma; \mu_H) \approx \alpha \mathcal{I}_- (\gamma'; \mu_{\text{MP}}) \]  

such that

\[ \mathcal{I}_- (\gamma; \mu_{\text{MP}}) = \mathcal{I}_- (\gamma; \mu_{\text{MP}}) - \log_2 \gamma - \lambda H \left( \frac{1}{\lambda} \right) + 1 . \]  

The term \( \mathcal{I}(\gamma; \mu_{\text{MP}}) \) in (8) is the well-known large system capacity expression for i.i.d. zero-mean fading coefficients. We refer to [3, Eq. (9)] for a closed form expression of it. In addition, further analytical expositions by using some facts presented in [2] shows that under some technical conditions (not reported here for the sake of the space), (7) gives a lower bound of \( \mathcal{I}_- (\gamma; \mu_H) \). Finally we define the approximation of the capacity in (2) as

\[ \mathcal{I}_NC (\gamma; \mu_H) \triangleq \mathcal{I}_- (\gamma; \mu_H) + \alpha \mathcal{I}_- (\gamma'; \mu_{\text{MP}}) \]

with \( \mathcal{I}_- (\gamma'; \mu_{\text{MP}}) \) given in (7).

A. Analytical and Numerical Comparison

To validate the analytical results, we consider the concatenated channel model with the parameter setting \( \rho_0 = 1/2, \rho_1 = 2 \) so that \( m = 1, \hat{m} = 0.338 \).

**REFERENCES**


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**Figure 1.** Approximation of the capacity of the concatenated scattering channels with the parameter setting \( \rho_0 = 1/2, \rho_1 = 2 \) so that \( m = 1, \hat{m} = 0.338 \).